Congruences for hook lengths of integer partitions

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(travail en commun avec David Wahiche)

Outline

Integer partitions

A partition of a non-negative integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

- The integer $n = |\lambda|$ is the weight of λ.
- The integers λ_k are the parts of λ.
- The integer ℓ is the length of λ.
- We let $\mathcal{P}(n)$ be the set of partitions of n.

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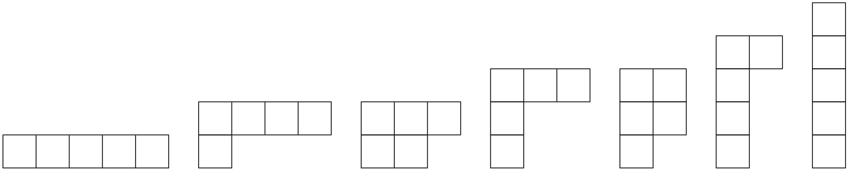
Example. The set $\mathcal{P}(5)$ of partitions of 5 is made of

$$(5) \quad (4,1) \quad (3,2) \quad (3,1,1) \quad (2,2,1) \quad (2,1,1,1) \quad (1,1,1,1,1)$$

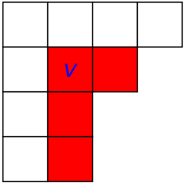
The corresponding Ferrers diagrams are



Hook lengths of a partition λ

For each cell v in the Ferrers diagram of λ, its hook length h_v is the number of cells u such that either $u = v$, or u lies strictly below (resp. to the right of) v in the same column (resp. row).

$\lambda = (4, 3, 2, 2)$



$h_v = 4$

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$\lambda = (4, 3, 2, 2)$

7	6	3	1
5	4	1	
3	2		
2	1		

$\mathcal{H}(\lambda) = \{\{7, 6, 3, 1, 5, 4, 1, 3, 2, 2, 1\}\}$

$\mathcal{H}(\lambda)$ is the multi-set of hook lengths of λ.

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$\mathcal{H}_3(\lambda) = \{\{6, 3, 3\}\}$

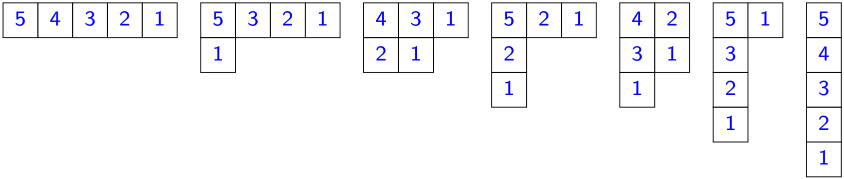
$\mathcal{H}(\lambda)$ is the multi-set of hook lengths of λ. For a fixed integer $t \geq 2$,

$$\mathcal{H}_t(\lambda) = \{h \in \mathcal{H}(\lambda) \,|\, h \equiv 0 \mod t\}.$$

λ is a t-core if $\mathcal{H}_t(\lambda) = \emptyset$, this set of partitions is denoted $\mathcal{P}_{(t)}$.

An observation

Recall the partitions of 5 filled with their hook lengths :



t	2	3	4	5	≥ 6
# hooks of length t among $\mathcal{P}(5)$	8	6	4	5	0
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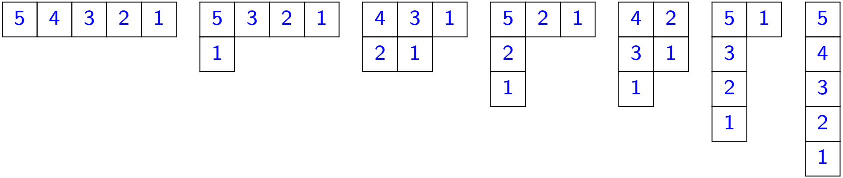

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Is this true for other values of n ? Is there an explanation ?

A combinatorial explanation

Theorem (Bacher–Manivel, 2001, Bessenrodt, 1998)

For all integers $n \geq 0$ and $t \geq 2$, the total number $a_t(n)$ of hooks of length t among all partitions of n is t times the total number of occurrences of the part t among all partitions of n.

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This is equivalent to the formula

$$a_t(n) = t \sum_{j \geq 1} |\mathcal{P}(n - jt)|,$$

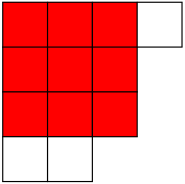
and a consequence is

$$a_t(n) \equiv 0 \pmod{t}.$$

Conjugation of partitions

Durfee square (here the size is $d = 3$)

$\lambda = (4, 3, 3, 2)$



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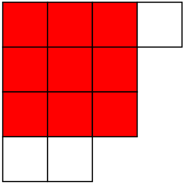
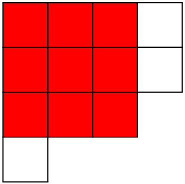


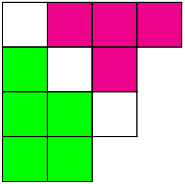
Conjugate

$$\lambda' = (4, 4, 3, 1)$$



Frobenius notation for partitions

$\lambda = (4, 3, 3, 2)$



$$\mathrm{Frob}(\lambda) = \begin{pmatrix} \lambda_1 - 1 & \lambda_2 - 2 & \ldots & \lambda_d - d \\ \lambda'_1 - 1 & \lambda'_2 - 2 & \ldots & \lambda'_d - d \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}$$

So

$$\mathrm{Frob}(\lambda') = \begin{pmatrix} 3 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

A congruence for hook lengths of self-conjugate partitions

$\lambda \in \mathcal{SC}$ if $\lambda = \lambda'$. Equivalently, $\mathrm{Frob}(\lambda) = \begin{pmatrix} a_1 & \dots & a_d \\ a_1 & \dots & a_d \end{pmatrix}$.

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Conjecture (Ballantine–Burson–Craig–Folsom–Wen, 2023)

For any integer $n \geq 0$ and any even integer $t \geq 2$, the total number $a_t^*(n)$ of hooks of length t among all self-conjugate partitions of n satisfies :

$$a_t^*(n) \equiv 0 \mod t.$$

Theorem (Amdeberhan–Andrews–Ono–Singh, 2024)

The above conjecture is true.

Moreover, it appears that no such congruence occurs for odd t.

Outline

Historical context

For a fixed integer $t \geq 2$, the Littlewood decomposition is a bijection $\Phi_t : \lambda \in \mathcal{P} \mapsto (\omega, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$ such that

$$\mathcal{H}_t(\lambda) = t \bigcup_{i=0}^{t-1} \mathcal{H}\left(\nu^{(i)}\right) \quad \text{and} \quad |\lambda| = |\omega| + t \sum_{i=0}^{t-1} \left|\nu^{(i)}\right|$$

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In 1940, Littlewood proves that the Schur function s_λ, evaluated at the tn variables $\xi^j x_i$, $0 \leq j \leq t-1$, $1 \leq i \leq n$, vanishes if the t-core ω is non-empty, and otherwise it factorizes as a product of Schur functions indexed by the partitions forming the t-quotient $\underline{\nu}$, each evaluated at x_i^t.

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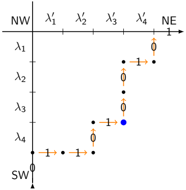
Finally Φ_t yields the generating function of t-cores :

$$\sum_{\omega \in \mathcal{P}_{(t)}} q^{|\omega|} = \prod_{n \geq 1} \frac{(1-q^{nt})^t}{1-q^n}.$$

Description using bi-infinite words



Along the border of $\lambda = (4, 3, 3, 2)$:
0 if vertical, 1 if horizontal
$s(\lambda) = \cdots 0000110100101111 \cdots$

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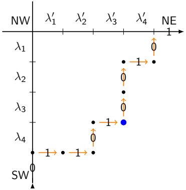

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Not unique !

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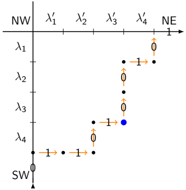
with

$$\#\{i \leq -1, c_i = 1\} = \#\{i \geq 0, c_i = 0\}.$$



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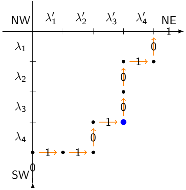

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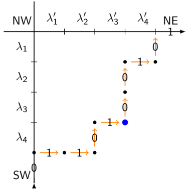

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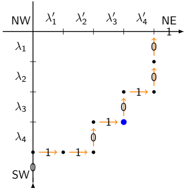
In Frobenius notation (Lecouvey–Wahiche, Brunat–Nath) :

$\{i \ge 0, c_i = 0\} = \{a_j, 1 \le j \le d\}$ and $\{i > 0, c_{-i} = 1\} = \{b_j + 1, 1 \le j \le d\}$

An example : $\lambda = (4, 4, 3, 2) \in \mathcal{SC}$ and $t = 4$



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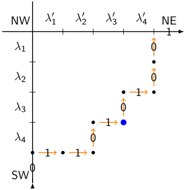

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The t subsequences $(c_{j+it})_i$ give $\underline{\nu}$

$$\begin{aligned} s\left(\nu^{(0)}\right) &= \cdots 001|011 \cdots \\ s\left(\nu^{(1)}\right) &= \cdots 001|111 \cdots \\ s\left(\nu^{(2)}\right) &= \cdots 000|011 \cdots \\ s\left(\nu^{(3)}\right) &= \cdots 001|011 \cdots \end{aligned} \longmapsto \begin{aligned} s\left(w_0\right) &= \cdots 000|111 \cdots \\ s\left(w_1\right) &= \cdots 001|111 \cdots \\ s\left(w_2\right) &= \cdots 000|011 \cdots \\ s\left(w_3\right) &= \cdots 000|111 \cdots \end{aligned}$$

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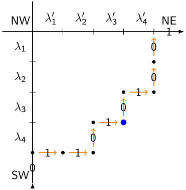

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$$s(\omega) = \cdots 00000100|11011111 \cdots \to \omega = (3,1,1) \in \mathcal{SC}_{(4)} \subset \mathcal{P}_{(4)}$$

$$\underline{\nu} = \left(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \nu^{(3)}\right) = ((1), \emptyset, \emptyset, (1))$$

Outline

Consequences of an addition theorem for partitions

Notation : $(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$, for $n \in \mathbb{N} \cup \{\infty\}$.

Theorem (Han–Ji, 2011)

Set $t \in \mathbb{N}^*$, ρ a function defined on $\mathbb{N}$, and $g_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \sum_{h \in \mathcal{H}(\lambda)} \rho(th)$.

Then we have

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \sum_{h \in \mathcal{H}_t(\lambda)} \rho(h) = t\, g_t(q^t) \frac{(q^t;q^t)_\infty}{(q;q)_\infty}.$$

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With $\rho(h) = \delta_{h,t}$, setting $n_t(\lambda)$ the number of hooks of length t in λ,

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Extracting coefficients : $a_t(n) = t \sum_{j \geq 1} |\mathcal{P}(n-jt)| \equiv 0 \pmod{t}$.

What about self-conjugate partitions ?

Littlewood for t even : $\lambda \in \mathcal{SC} \mapsto (\omega, \underline{\nu}) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{t/2}$.

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By Bessenrodt (1991), Brunat–Gramain (2010), Bernal (2019) for t prime and Wahiche (2022) for all odd t, we have

$$\mathrm{BG}_t := \{\lambda \in \mathcal{SC} \mid \forall i \in \{1, \ldots, d\}, t \nmid h_{(i,i)}\} = \{\lambda \in \mathcal{SC} \mid \mu = \emptyset \text{ above}\}.$$

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Using Wahiche's addition theorem for BG_t (2022) :

$$a_t^*(n) = \sum_{\lambda \in \mathrm{BG}_t(n)} n_t(\lambda) = (t-1) \sum_{j \geq 1} |\mathrm{BG}_t(n - 2jt)| \equiv 0 \pmod{(t-1)}.$$

Outline

z-asymmetric partitions

Following Ayyer–Kumari (2022), Albion (2024), Erickson–Hunziker (2026)

$$\mathrm{ASC}_z := \left\{ \lambda \in \mathcal{P}, \mathrm{Frob}(\lambda) = \begin{pmatrix} a_1 + z & a_2 + z & \dots & a_d + z \\ a_1 & a_2 & \dots & a_d \end{pmatrix} \right\}$$

z-asymmetric partitions

Following Ayyer–Kumari (2022), Albion (2024), Erickson–Hunziker (2026)

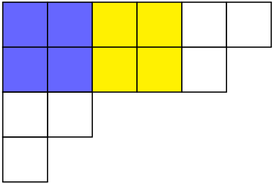
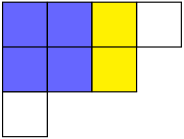
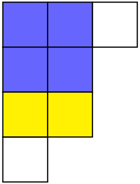
$$\mathrm{ASC}_z := \left\{ \lambda \in \mathcal{P}, \mathrm{Frob}(\lambda) = \begin{pmatrix} a_1 + z & a_2 + z & \dots & a_d + z \\ a_1 & a_2 & \dots & a_d \end{pmatrix} \right\}$$

Ayyer–Kumari generalize Littlewood's factorizations to the characters of the classical groups $SO(2n+1, \mathbb{C})$, $O(2n, \mathbb{C})$, and $Sp(2n, \mathbb{C})$, showing that the twisted characters are non-zero if and only if the t-core of the associated partition is respectively in

- ASC_0 : the set of self-conjugate partitions,
- ASC_1 : the set of doubled distinct partitions,
- ASC_{-1} : the set of their conjugates.

Generating function

We have $(6,5,2,1) \in \mathrm{ASC}_2$, $(4,3,1) \in \mathrm{ASC}_1 = \mathrm{DD}$, $(3,2,2,1) \in \mathrm{ASC}_{-1}$:

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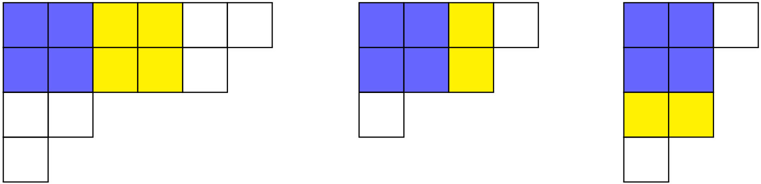

The generating function is

$$\sum_{\lambda \in \mathrm{ASC}_z} q^{|\lambda|} = \sum_{d \geq 0} \frac{q^{d^2+d|z|}}{(q^2;q^2)_d} = (-q^{1+|z|};q^2)_\infty .$$

Behaviour under the Littlewood decomposition

Albion (2024) extends the factorizations of Ayyer–Kumari to the universal characters, defined by Koike–Terada (1987), for the aforementioned groups. He uses a characterization of the Littlewood decomposition applied to partitions in ASC_z, generalizing work of Garvan–Kim–Stanton for $z=0$ and $z=1$.

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We need, for $0 \leq z \leq t-1$, to define $\mathrm{BG}_{z,t}$ as

$$\{\lambda \in \mathrm{ASC}_z, t \nmid (a_j + k), (2a_j + z + 1)/t \notin 2\mathbb{N} + 1, 1 \leq j \leq d, 1 \leq k \leq z\}.$$

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Then (J.–Wahiche, 2025), $\lambda \in \mathrm{BG}_{z,t} \mapsto (\omega, \underline{\nu})$ such that :

- $\omega \in \mathrm{ASC}_z$,
- $\nu^{(0)} = \cdots = \nu^{(z-1)} = \emptyset$ and if $t + z - 1$ is even, $\nu^{((t+z-1)/2)} = \emptyset$,
- $\nu^{(r)} = \left(\nu^{(t+z-r-1)}\right)'$ for all $r \in \{z, \ldots, t-1\}$.

An addition-multiplication theorem

Theorem (J.–Wahiche, 2025)

For integers $0 \leq z \leq t-1$ and functions ρ_1, ρ_2 defined on $\mathbb{N}$, set

$$f_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th)^2$$

and

$$g_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th)^2 \sum_{h \in \mathcal{H}(\lambda)} \rho_2(th).$$

Then we have

$$\sum_{\lambda \in \mathrm{BG}_{z,t}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h) = 2\lfloor (t-z)/2 \rfloor g_t(x^2 q^{2t})$$
$$\times (f_t(x^2 q^{2t}))^{\lfloor \frac{t-z}{2} \rfloor - 1} \prod_{i=0}^{\lfloor (t-z)/2 \rfloor - 1} (-q^{2i+z+1}, -q^{2t-2i-z-1}, q^{2t}; q^{2t})_\infty.$$

First consequences

We have the generating function

$$\sum_{\lambda \in \mathrm{BG}_{z,t}} q^{|\lambda|} y^{n_t(\lambda)} = ((1-y^2)q^{2t}; q^{2t})_\infty^{\lfloor \frac{t-z}{2} \rfloor} \times \prod_{i=0}^{\lfloor (t-z)/2 \rfloor - 1} (-q^{2i+z+1}, -q^{2t-2i-z-1}; q^{2t})_\infty,$$

where $n_t(\lambda)$ is the number of hooks of length t in λ.

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If $a_{z,t}(n)$ denotes the number of hooks of length t among all partitions of n in $\mathrm{BG}_{z,t}$, then

$$a_{z,t}(n) = 2\lfloor (t-z)/2 \rfloor \sum_{j \geq 1} |\mathrm{BG}_{z,t}(n-2tj)| \equiv 0 \pmod{2\lfloor (t-z)/2 \rfloor}.$$

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Remark : no hope for a general similar congruence for the whole set ASC_z for any $t-z$.

Doubled distinct partitions

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$$\mathsf{BG}_{1,t} = \{\lambda \in \mathsf{ASC}_1 = \mathcal{DD} \mid \forall i \in \{1, \ldots, d\}, t \nmid h_{(i,i)}(\lambda)\},$$

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$$((1-y^2)q^{2t};q^{2t})_\infty^{\lfloor \frac{t-1}{2} \rfloor}(-q^2;q^2)_\infty \times \begin{cases} (-q^{2t};q^{2t})_\infty^{-1} & \text{for } t \text{ odd} \\ (-q^t;q^t)_\infty^{-1} & \text{for } t \text{ even.} \end{cases}$$

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Finally

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A new modular Nekrasov–Okounkov formula

Corollary (J.–Wahiche, 2025)

For all integers $0 \leq z \leq t-1$, and any complex number u, we have

$$\sum_{\lambda \in \mathrm{BG}_{z,t}} q^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(1 - \frac{u}{h^2}\right)^{1/2} = (q^{2t}; q^{2t})_\infty^{\lfloor \frac{t-z}{2} \rfloor u/t^2} \times \prod_{i=0}^{\lfloor (t-z)/2 \rfloor - 1} (-q^{2i+z+1}, -q^{2t-2i-z-1}; q^{2t})_\infty .$$

We use the Nekrasov–Okounkov formula

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{u}{h^2}\right) = (q;q)_\infty^{u-1}.$$