Around fully commutative elements in finite and affine Coxeter groups

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Collaborations with R. Biagioli, M. Bousquet-Melou and P. Nadeau

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Coxeter groups

(W, S) Coxeter group W given by Coxeter matrix $(m_{st})_{s,t\in S}$

Relations: $$\begin{cases} s^2 = 1 \\ \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}} \end{cases}$$ → Braid relations

$m_{st} = 2$: commutation relation

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Matsumoto property (1964): Given two reduced decompositions of w, there is a sequence of braid relations which can be applied to transform one into the other

FC elements

Full commutativity is a strenghtening of Matsumoto's property

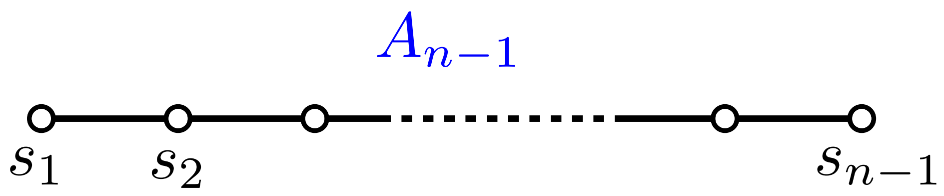
An element w is **fully commutative** if given two reduced decompositions of w, there is a sequence of commutation relations which can be applied to transform one into the other

Equivalently, w is fully commutative if its reduced decompositions form only one commutation class

Type A_{n-1} $\rightarrow$ The symmetric group S_n

Consider $S = \{s_1, \ldots, s_{n-1}\}$, with relations $s_i^2 = 1$ and

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i, \quad |j-i| > 1 \end{cases}$$



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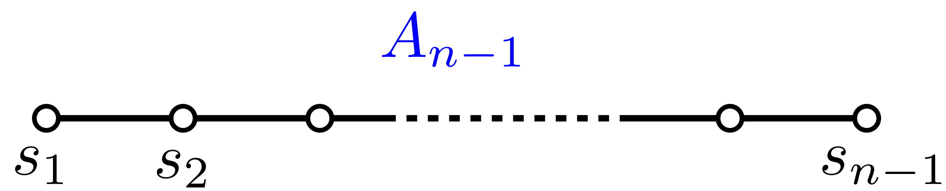

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Theorem [Billey-Jockush-Stanley (1993)]

w is fully commutative $\Leftrightarrow$ $\vartheta(w)$ is 321-avoiding

One can use this to show that FC elements in type A_{n-1} are counted by Catalan numbers, i.e., $|S_n^{FC}| = \dfrac{1}{n+1}\dbinom{2n}{n}$

Previous work

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- [Green–Losonczy (2001), Shi (2003), ...] connect FC elements to Kazhdan-Lusztig polynomials
- [Barcucci et al (2001)] enumerate in type A with respect to the Coxeter length using pattern-avoidance
- [Hanusa–Jones (2010)] enumerate in type $\widetilde{A}$ with respect to the Coxeter length, using affine permutations

Outline

We enumerate FC elements and involutions according to the Coxeter length for any finite or affine Coxeter group W

$$W^{FC}(q) := \sum_{w \text{ is FC}} q^{\ell(w)} \text{ and } \bar{W}^{FC}(q) := \sum_{w \text{ is FC involution}} q^{\ell(w)}$$

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Main results: we compute $W^{FC}(q)$ and $\bar{W}^{FC}(q)$ for any finite or affine W. When W is affine, the coefficients of the series form ultimately periodic sequences

I will focus on types A and $\widetilde{A}$, corresponding to the finite and affine symmetric groups. The idea is to encode the FC elements in these cases by certain lattice paths

Characterization of FC elements

Proposition[Stembridge] A reduced word represents a FC element if and only if no element of its commutation class contains a factor $\underbrace{sts\cdots}_{m_{st}}$ for a $m_{st} \geq 3$

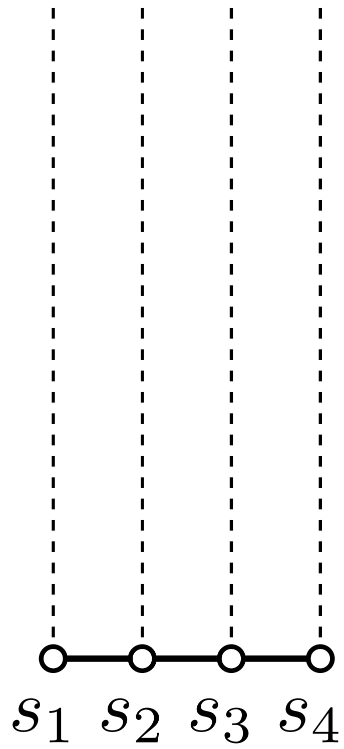
How to see if a commutation class verifies the above property ?
$\Rightarrow$ use the theory of heaps, which are posets encoding commutation classes

Example of heaps in $A_4 (= S_5)$

Heap of a word = poset H labeled by generators s_i of W

Linear extensions of H $\Leftrightarrow$ words of the commutation class

$s_1 s_3 s_4 s_1 s_2 s_3$



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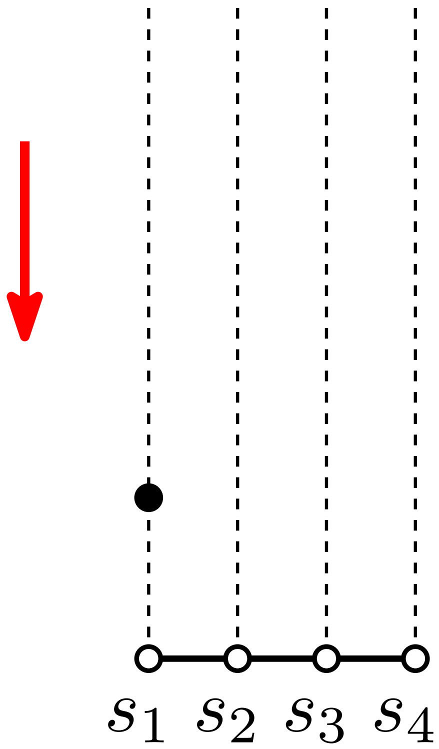

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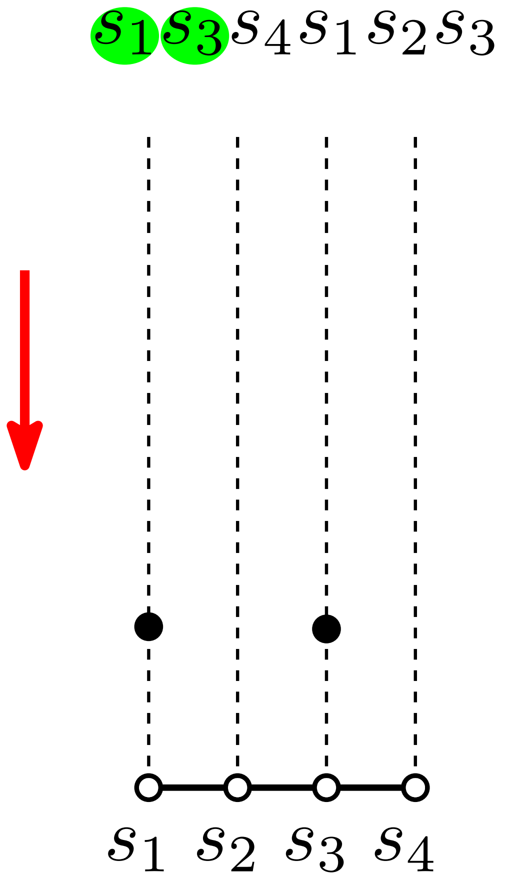

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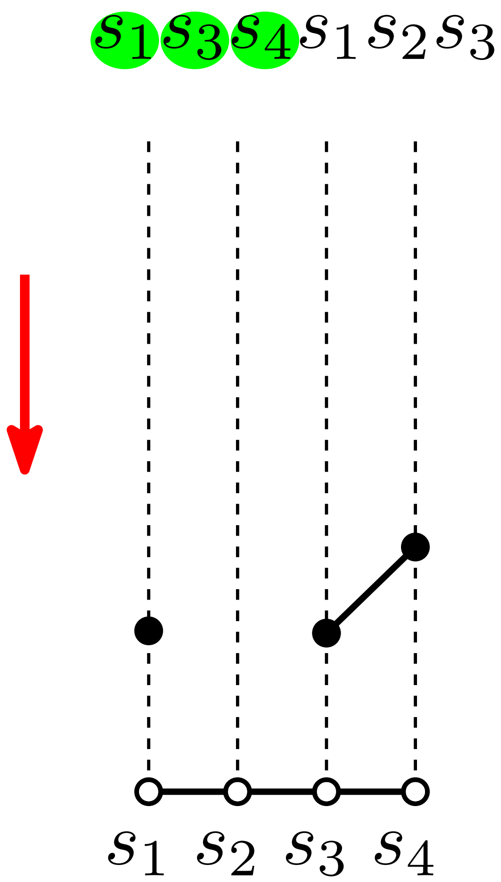

Vertex stays above if corresponding generators do not commute.

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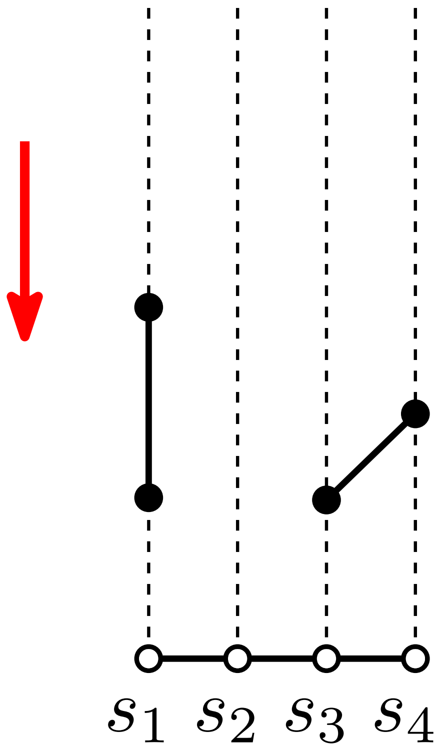

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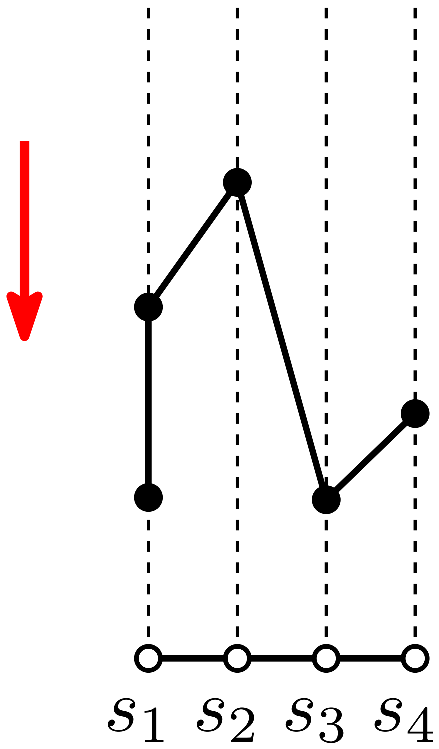

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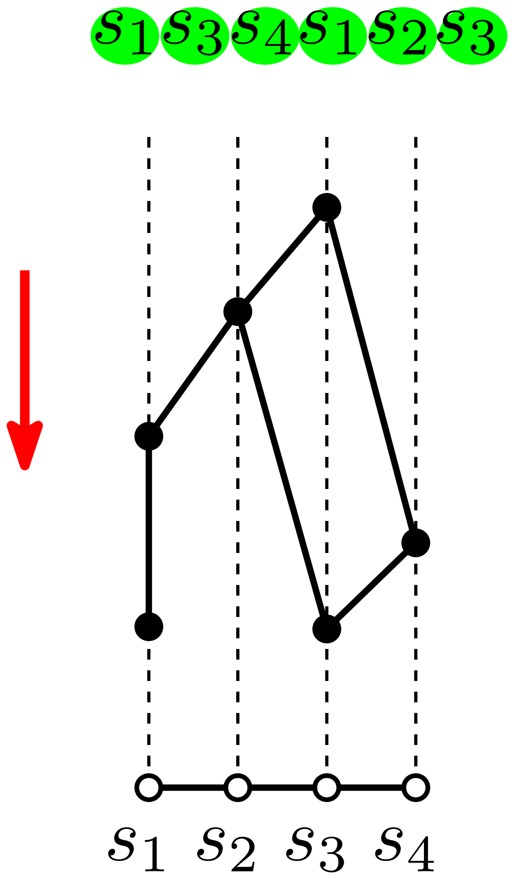

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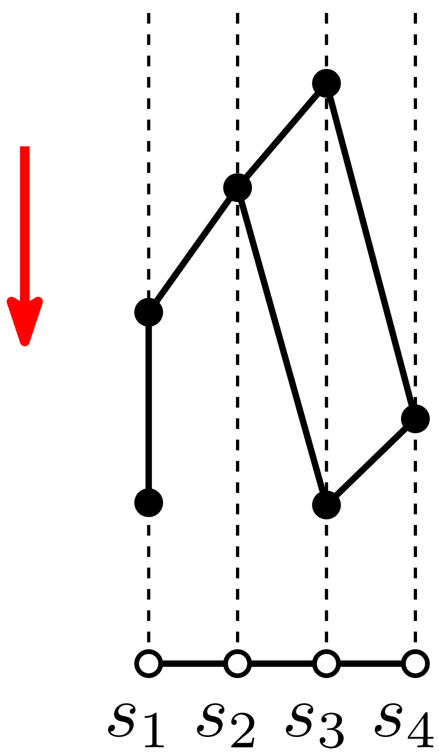

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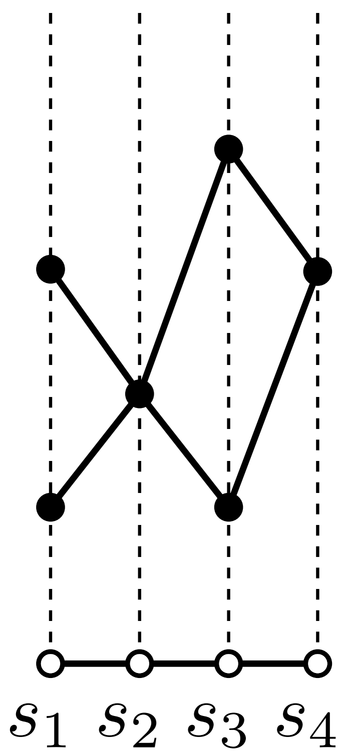

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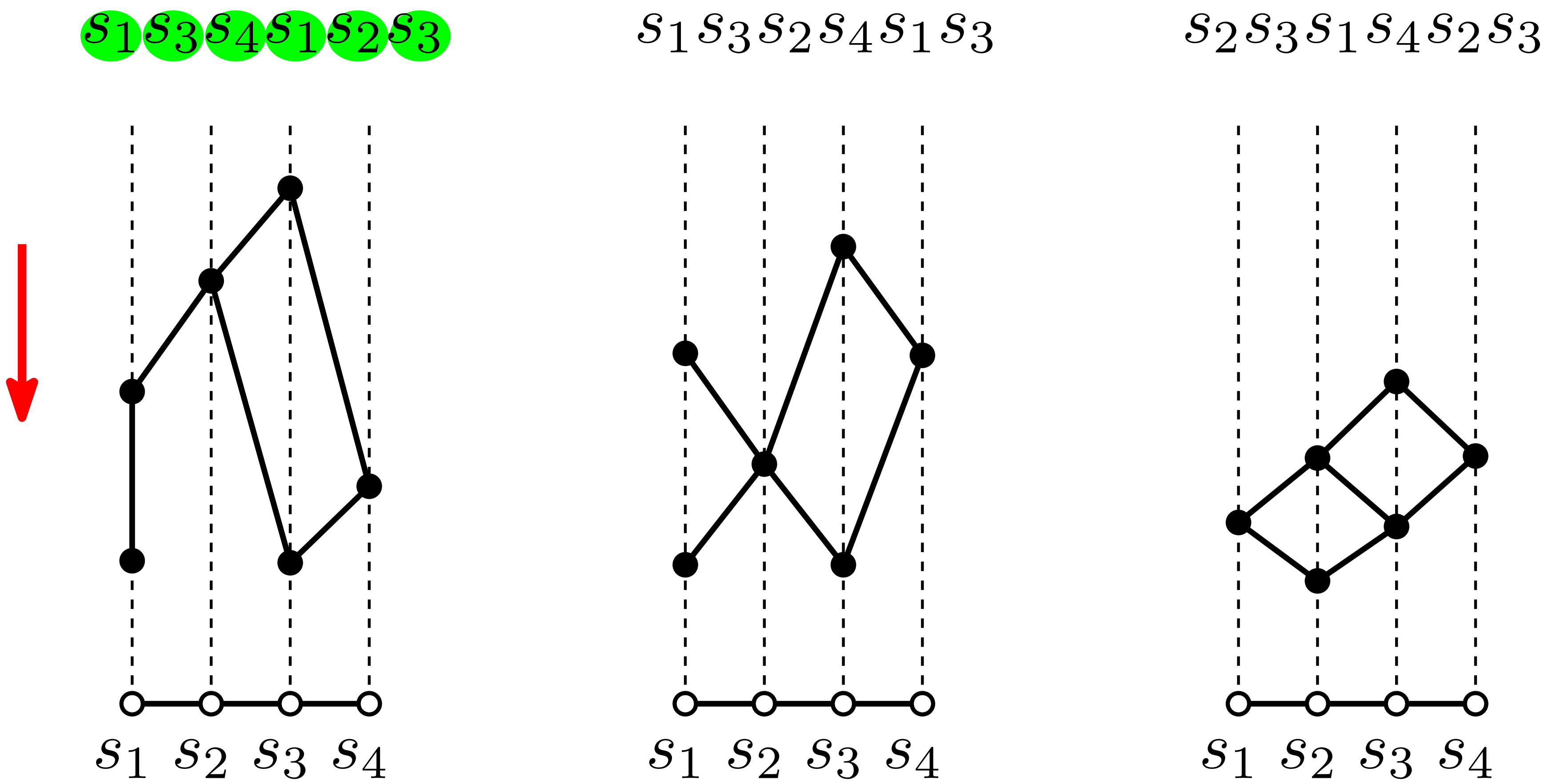

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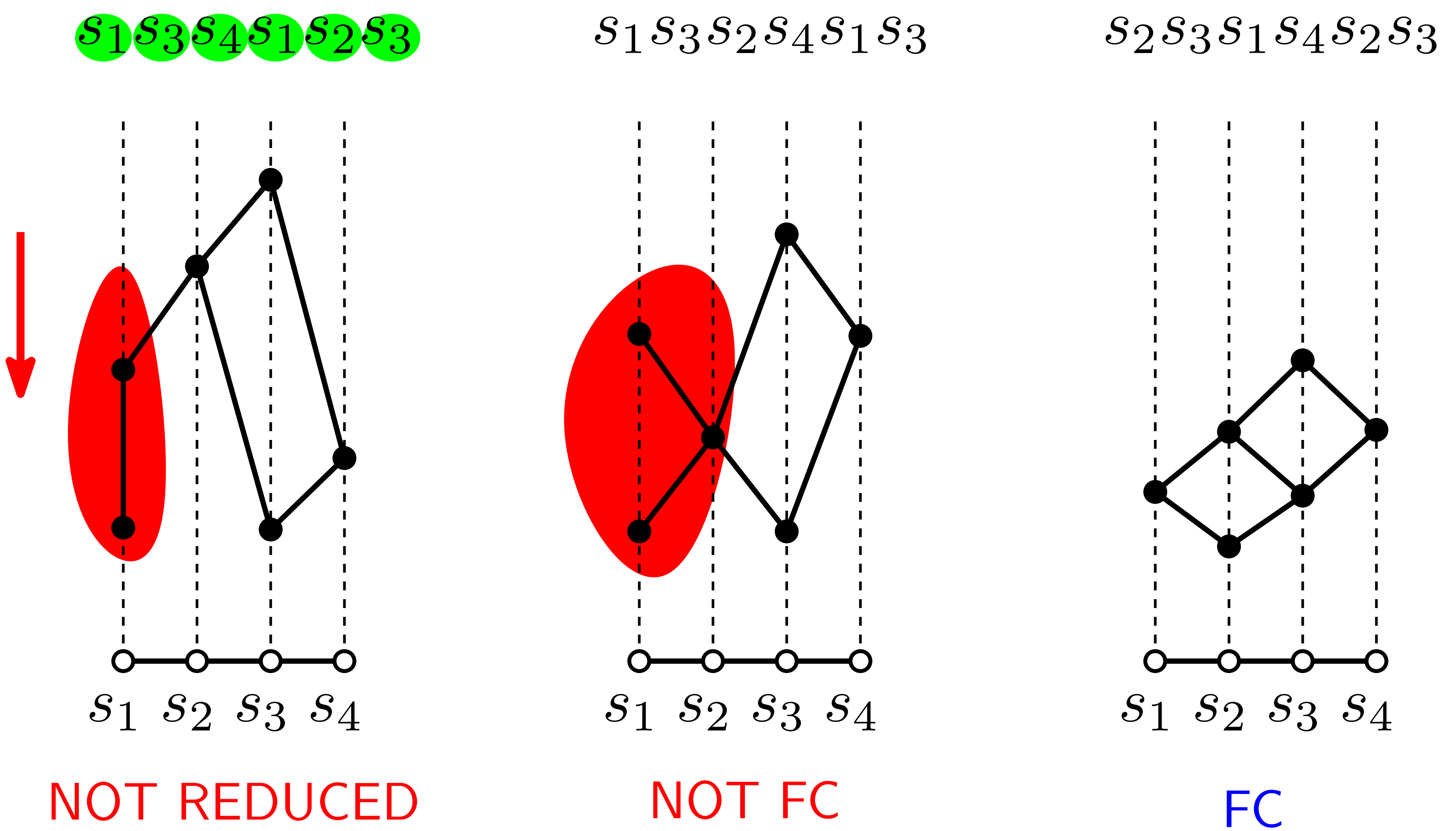
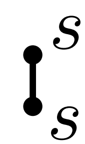
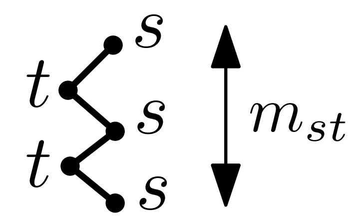


Characterization of FC heaps

FC element w $\longleftrightarrow$ Heap H avoiding 

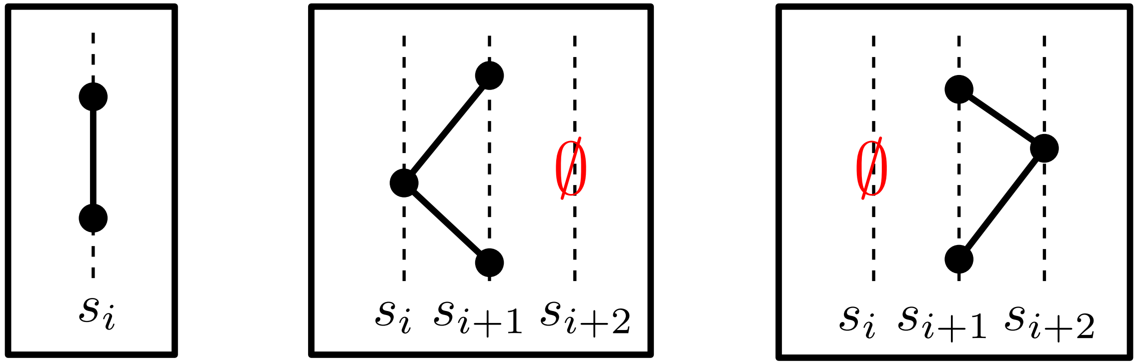
and 

Length $\ell(w)$ $\longleftrightarrow$ Number of elements $|H|$

In type A and $\widetilde{A}$: FC heaps above are particularly simple

Type A

FC heaps avoid precisely



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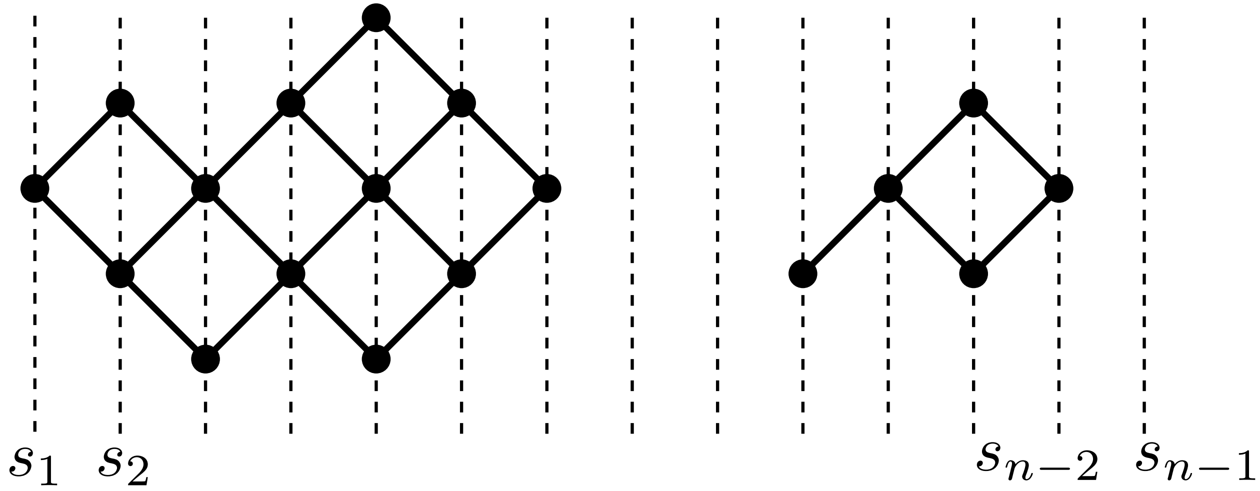

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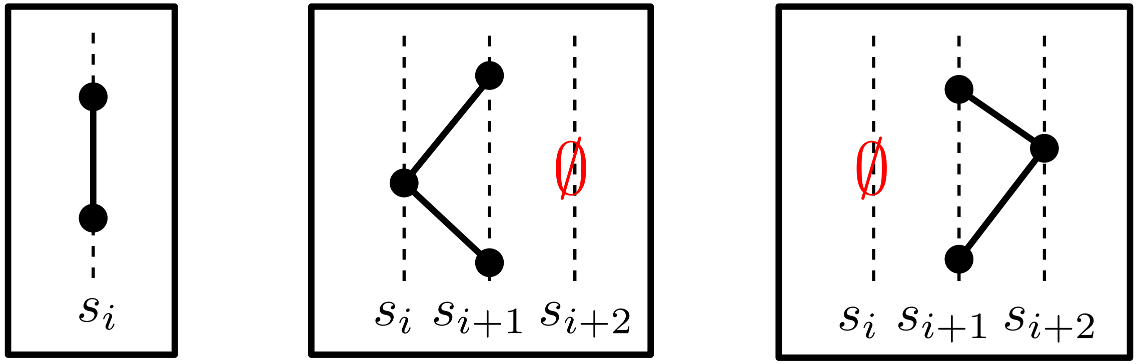

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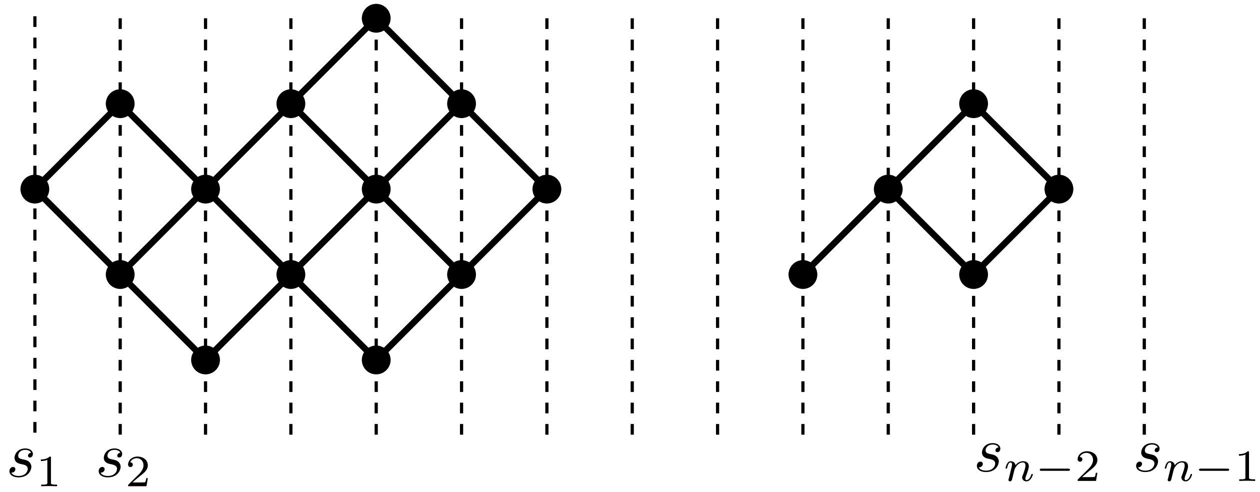
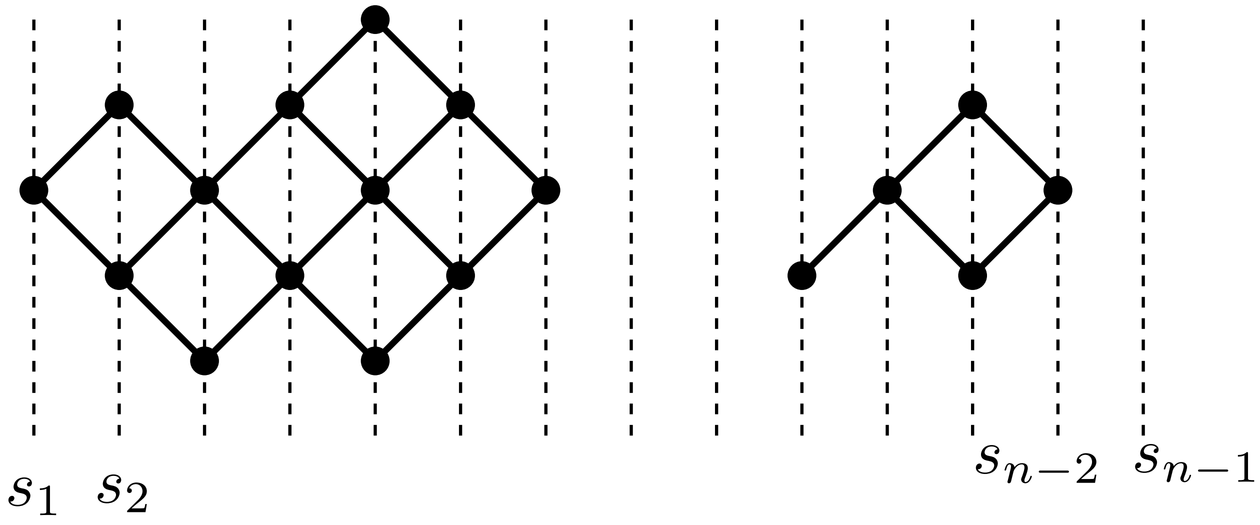


Proposition FC Heaps of type A are characterized by:
(a) At most one occurrence of s_1 (*resp.* s_{n-1})
(b) $\forall i$, elements with labels s_i, s_{i+1} form an alternating chain

Type A: bijection with Motzkin-type paths

FC Heap



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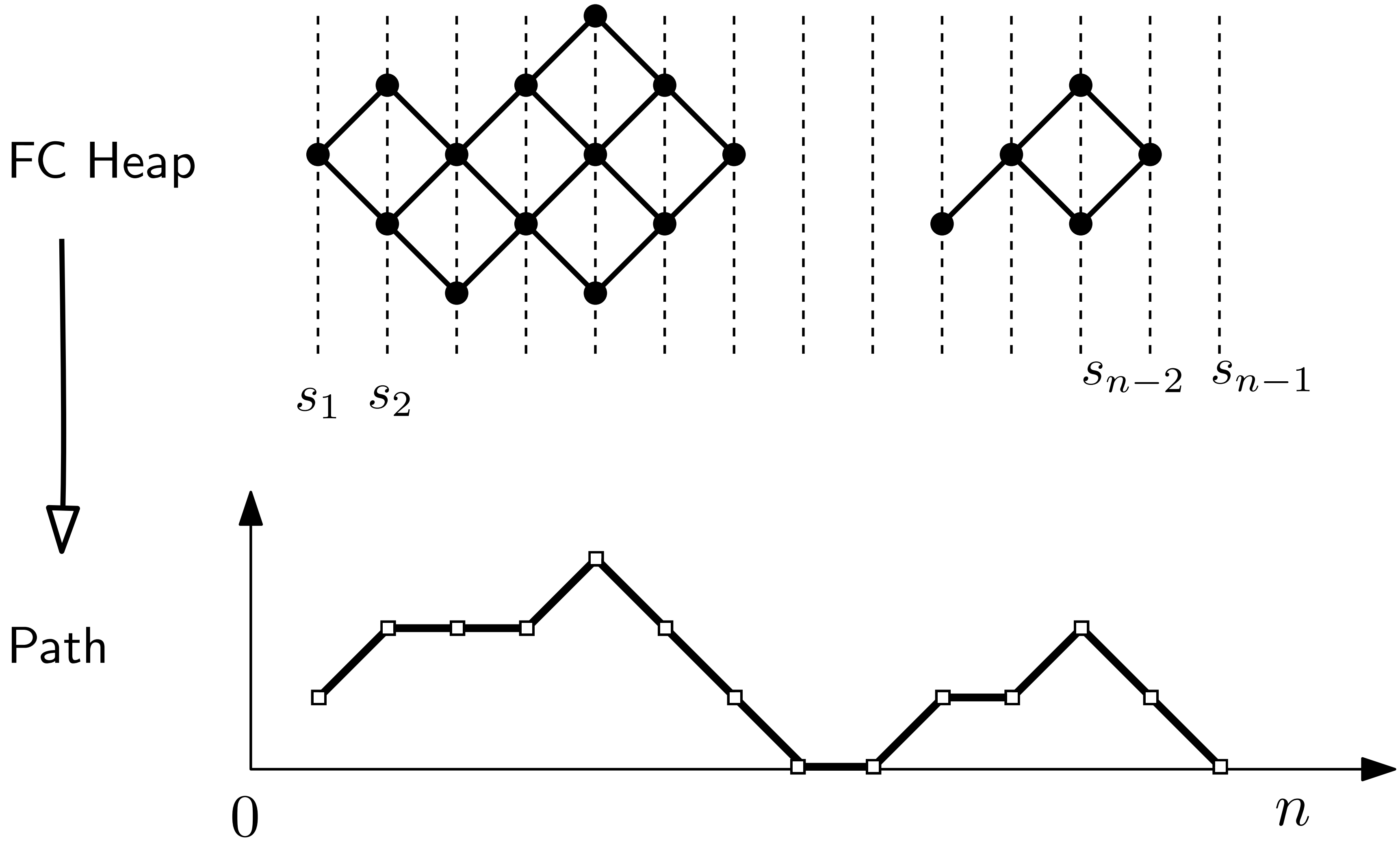

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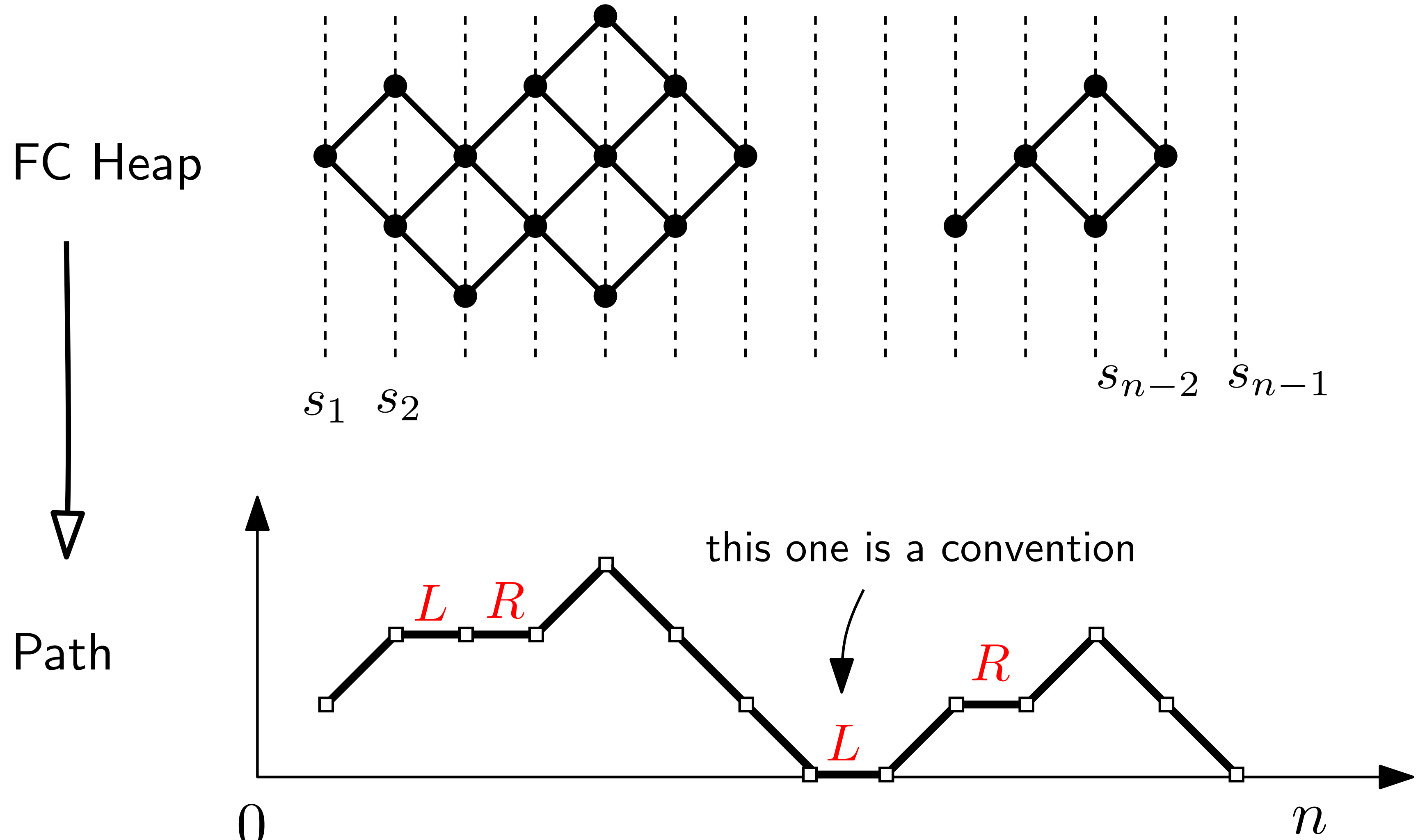
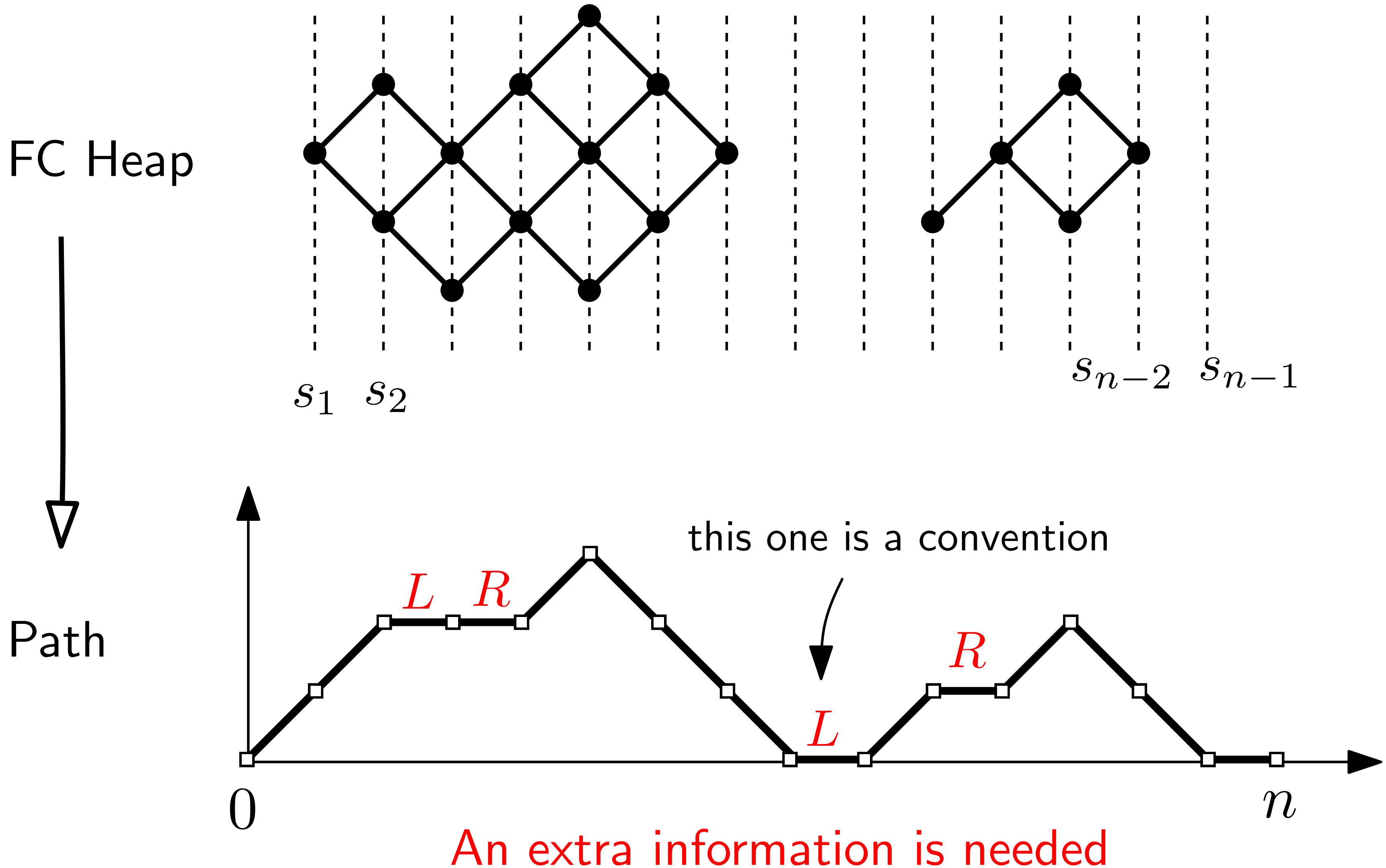


An extra information is needed

Type A: bijection with Motzkin-type paths



To finish, add initial and final steps to the path

Type A: bijection with Motzkin paths

Theorem: this is a bijection between FC heaps of type A_{n-1} and Motzkin paths of length n with horizontal steps at height $h > 0$ (*resp.* $h = 0$) labeled L or R (*resp.* labeled L)

Size of the heap ⇔ **Area** of the path

(Sum of the heights of all vertices)

We have: $A^{FC}(x) := \sum_{n \geq 1} A_{n-1}^{FC}(q) x^n = M^*(x) - 1$

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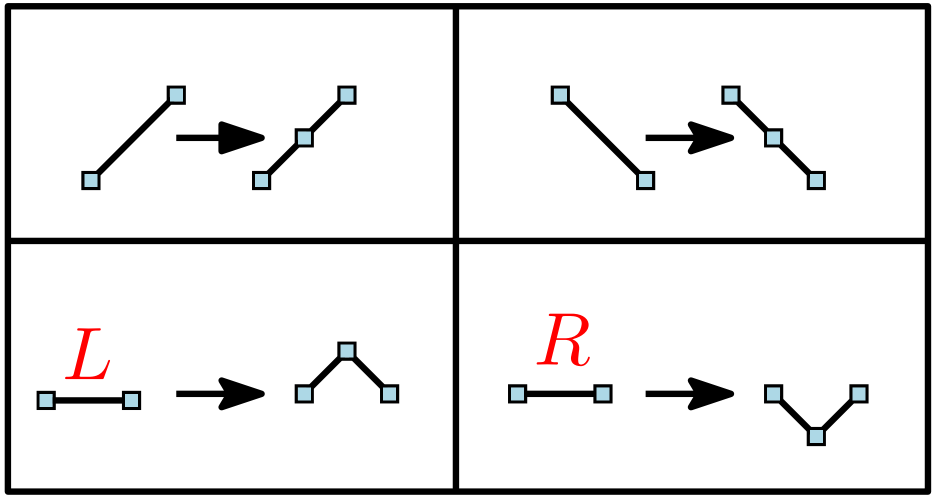
Remark



transforms these paths into Dyck paths $\Rightarrow$ Catalan numbers

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FC involutions in $\bar{W}$ are FC elements whose commutation class is palindromic: it includes the mirror images of its members

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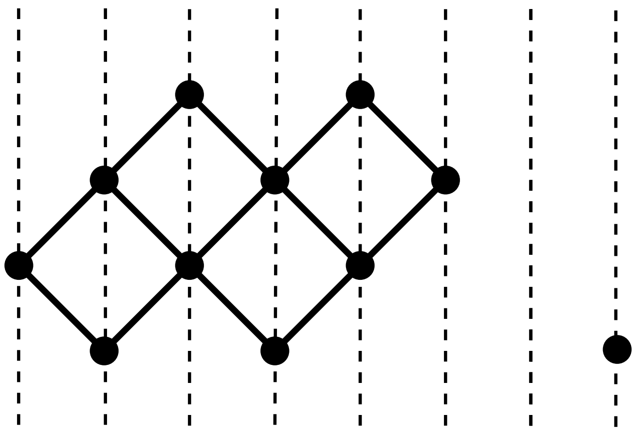
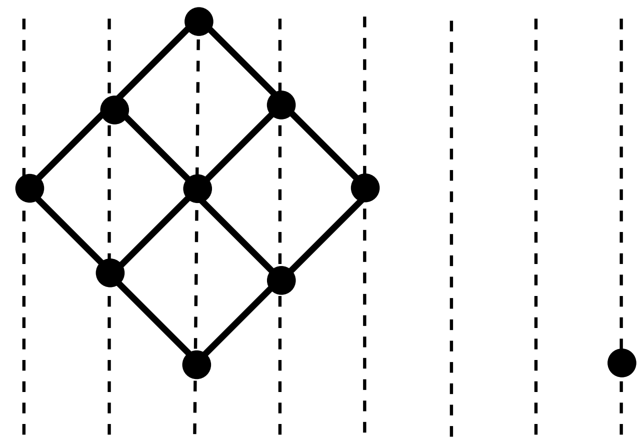


$s_2 s_4 s_8 s_1 s_3 s_5 s_2 s_4 s_6 s_3 s_5$

Not palindromic



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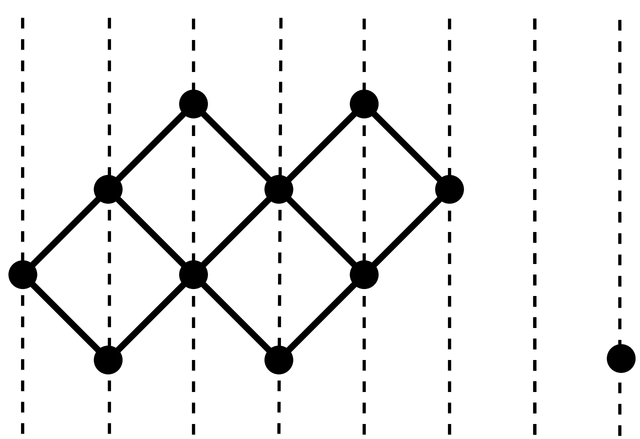

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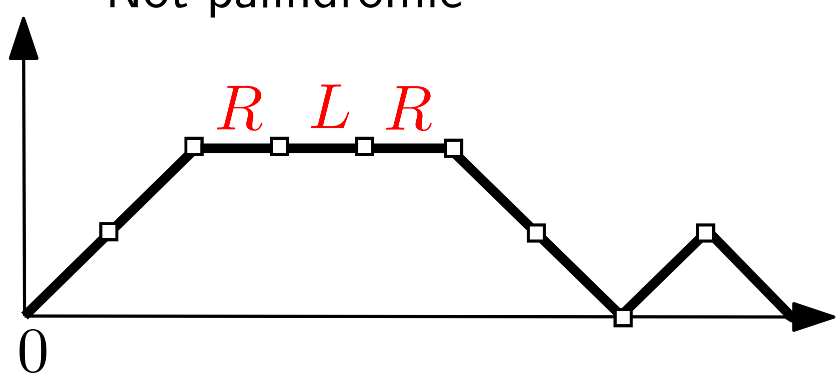
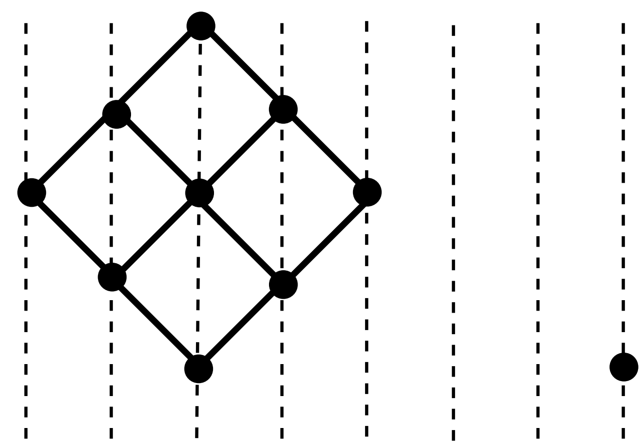


Motzkin path

$M^*(x) - 1$



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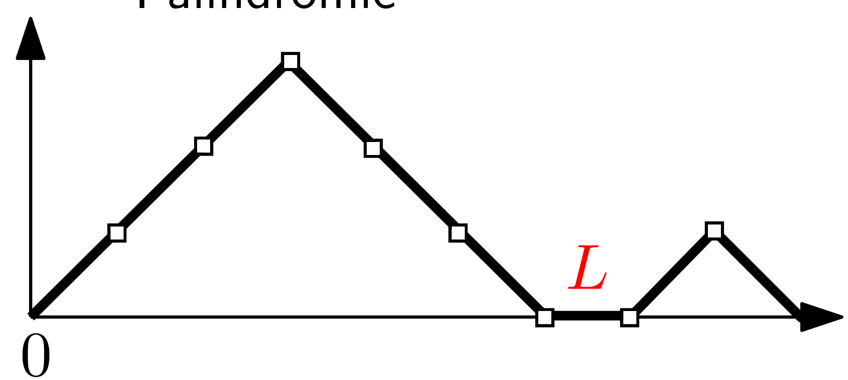
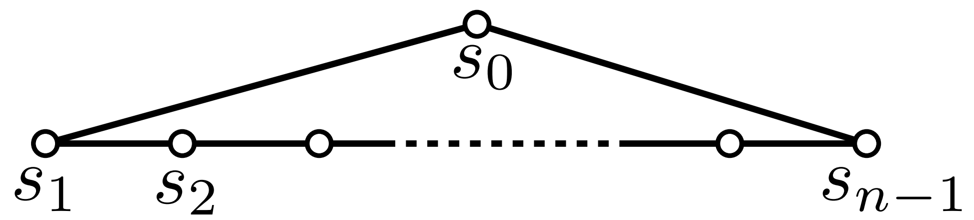


Dyck paths and h steps

$\frac{Cat(x)}{1 - xCat(x)} - 1$

Affine types

$\widetilde{A}_{n-1}$



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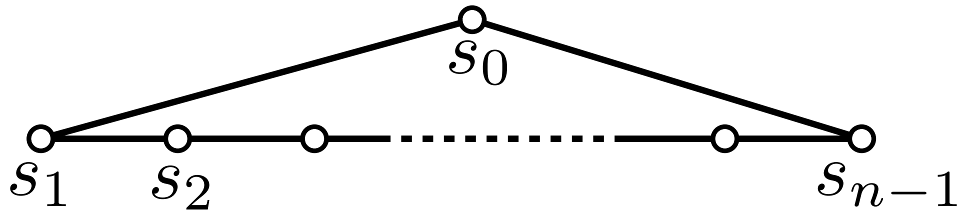


Theorem [Green (2001)] FC elements of type $\widetilde{A}_{n-1}$ correspond to 321-avoiding affine permutations

Hanusa–Jones used this to compute $\widetilde{A}_{n-1}^{FC}(q)$ and derived a complicated expression for this infinite series

Theorem [Hanusa-Jones (2010)] The coefficients of $\widetilde{A}_{n-1}^{FC}(q)$ are ultimately periodic of period dividing n

Generating functions

They computed the generating functions $f_n(q) = \widetilde{A}_{n-1}^{FC}(q)$; here are the first ones

$$f_3(q) = 1 + 3q + \mathbf{6q^2} + \mathbf{6q^3} + \mathbf{6q^4} + \cdots$$

$$f_4(q) = 1 + 4q + 10q^2 + \mathbf{16q^3} + \mathbf{18q^4} + \mathbf{16q^5} + \mathbf{18q^6} + \cdots$$

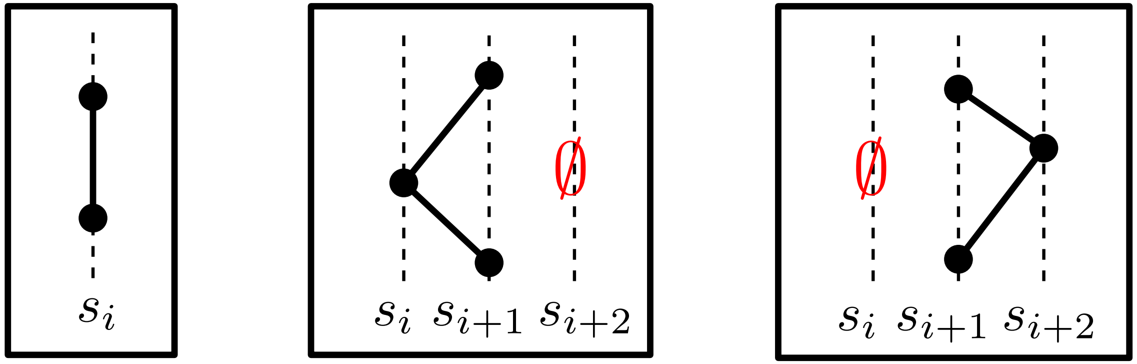
$$f_5(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + \mathbf{50q^5} + \mathbf{50q^6} + \mathbf{50q^7} + \mathbf{50q^8} + \mathbf{50q^9} + \cdots$$

$$f_6(q) = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + \mathbf{150q^7} + \mathbf{156q^8} + \mathbf{152q^9} + \mathbf{156q^{10}} + \mathbf{150q^{11}} + \mathbf{158q^{12}} + \mathbf{150q^{13}} + \mathbf{156q^{14}} + \mathbf{152q^{15}} + \mathbf{156q^{16}} + \mathbf{150q^{17}} + \mathbf{158q^{18}} + \cdots$$

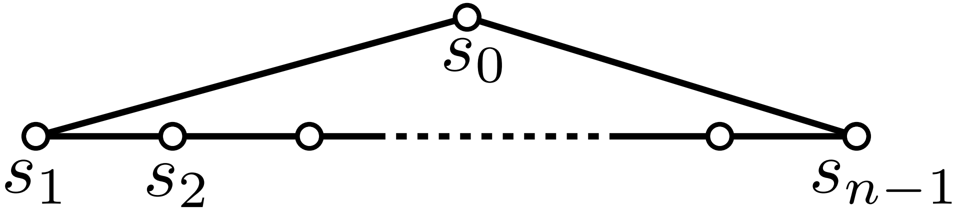
FC elements in type $\widetilde{A}$

FC heaps satisfy the same local conditions as in finite type A

$\rightarrow$ The heaps must avoid



Difference: the cyclic shape of the Coxeter diagram



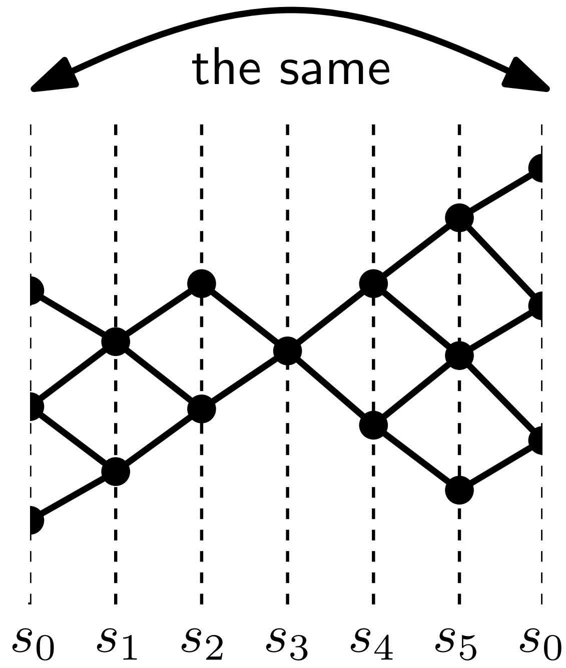
$\rightarrow$ The labels above must be taken with index modulo n; the heaps must be thought of as "drawn on a cylinder"

Heaps become Motzkin-type paths

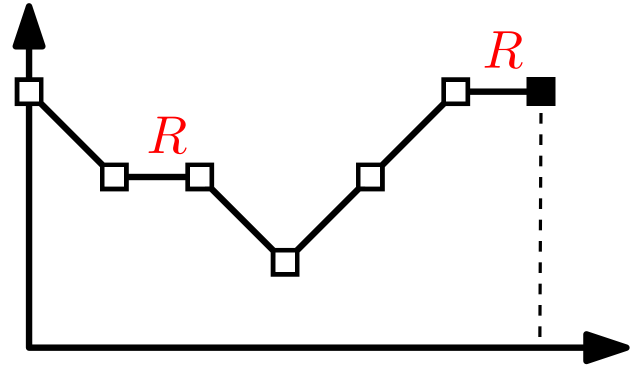
We can form a path as before from a heap: because of the cyclic diagram, our paths will start and end at the same height

Example:

Heap



Path



The area does not take into account the final height

Bijection

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Indeed such paths can clearly not correspond to FC elements

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Corollary $\widetilde{A}_{n-1}^{FC}(q) = \mathcal{O}_n^*(q) - \dfrac{2q^n}{1-q^n}$

Periodicity revisited

For a large enough degree, the series $\mathcal{O}_n^*(q)$ has periodic coefficients with period n: just shift the path up by 1 unit

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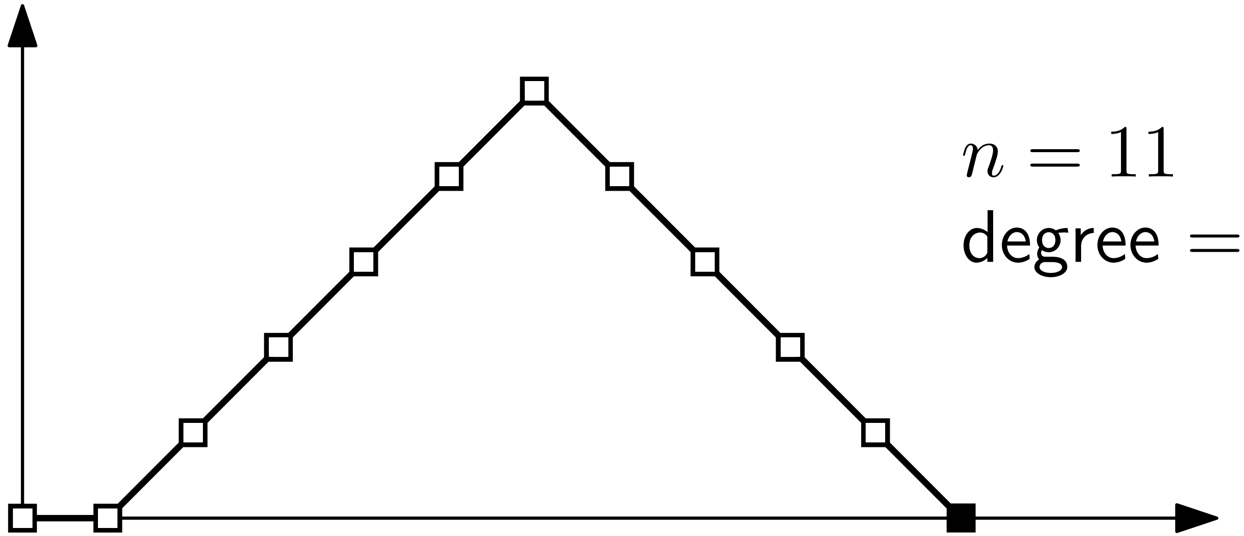
"Large enough" ? As soon as the degree k is such that no path with area k can have a horizontal step at height $h = 0$
$\rightarrow k = 1 + \lceil (n-1)/2 \rceil \lfloor (n-1)/2 \rfloor$ is optimal

This proves the conjecture of Hanusa and Jones



$n = 11$
degree $= 25 = 5^2$

Functional equations

To compute $\mathcal{O}_n^*(q)$, decompose the walks according to whether they touch the x-axis or not

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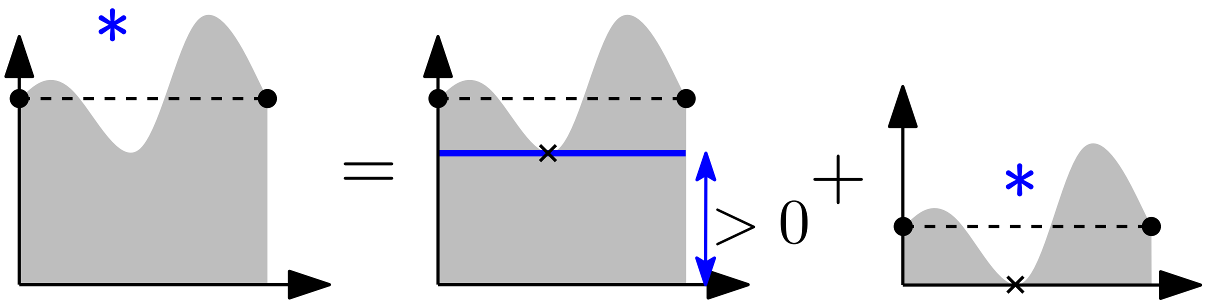

$$\mathcal{O}_n^*(q) = \frac{q^n \check{\mathcal{O}}_n(q)}{1 - q^n} + \check{\mathcal{O}}_n^*(q)$$

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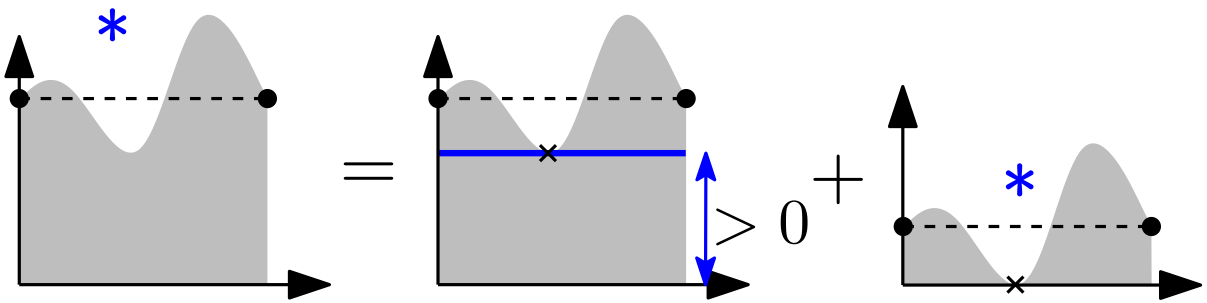
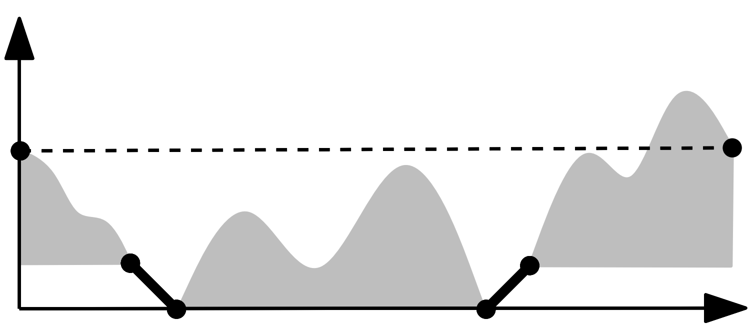


$$\mathcal{O}_n^*(q) = \frac{q^n \check{\mathcal{O}}_n(q)}{1 - q^n} + \check{\mathcal{O}}_n^*(q)$$

Next $\check{\mathcal{O}}_n(q)$ and $\check{\mathcal{O}}_n^*(q)$ can be computed through



$$\check{\mathcal{O}}(x) = M(x)\left(1 + qx^2 \frac{\partial(xM)}{\partial x}(xq)\right)$$

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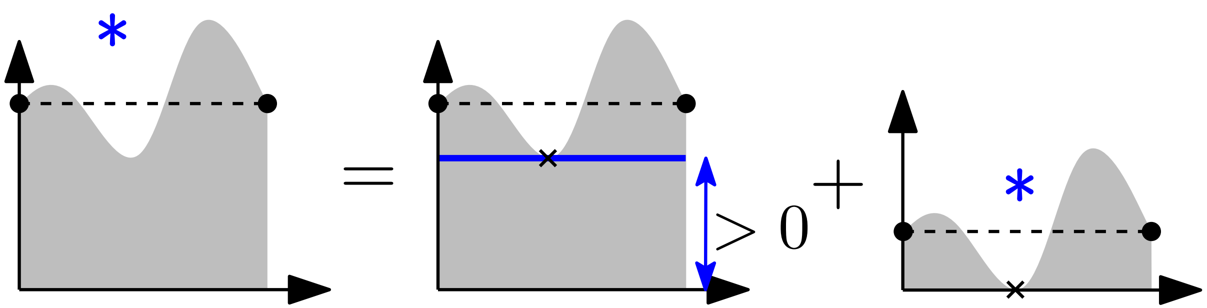

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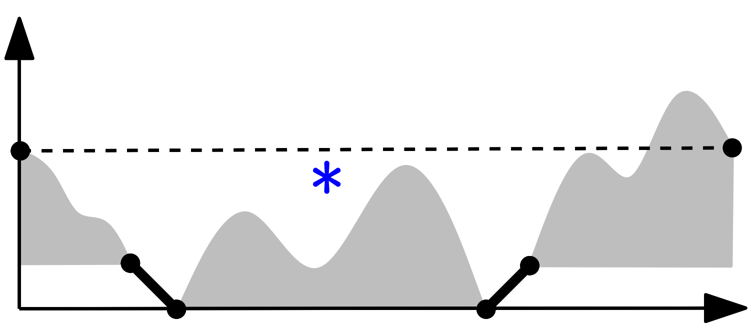
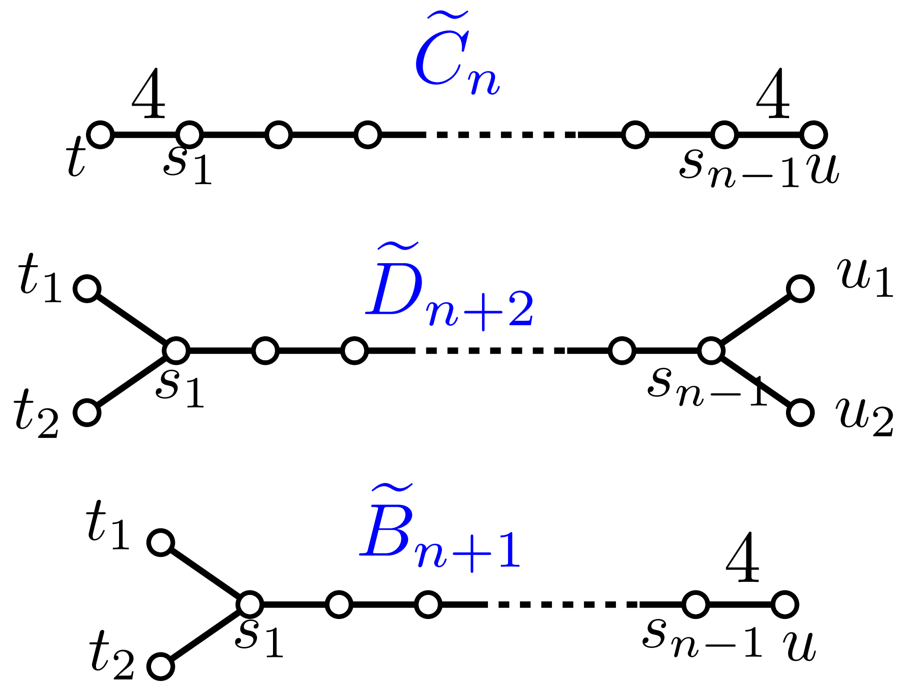


$$\check{\mathcal{O}}^*(x) = M^*(x)\left(1 + qx^2 \frac{\partial(xM)}{\partial x}(xq)\right)$$

Other affine types

There are 3 classical types



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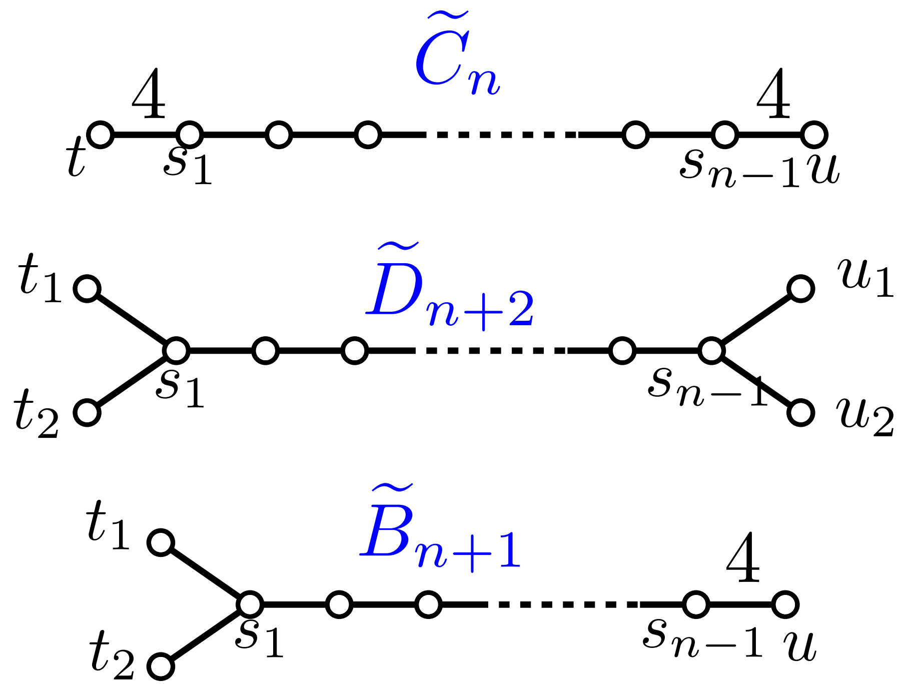
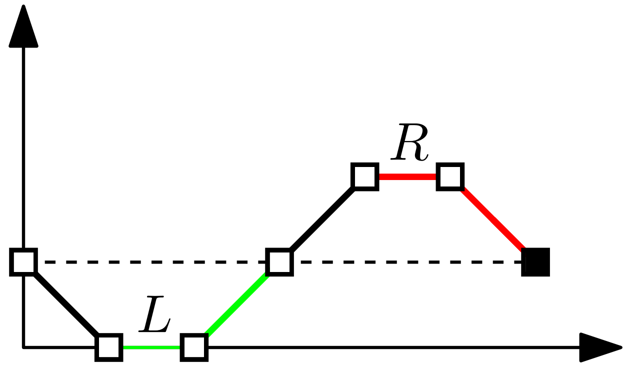


Theorem: for each irreducible affine group W, the sequence of coefficients of $W^{FC}(q)$ is ultimately periodic, with period dividing the following values:

Affine Type	$\widetilde{A}_{n-1}$	$\widetilde{C}_n$	$\widetilde{B}_{n+1}$	$\widetilde{D}_{n+2}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{G}_2$	$\widetilde{F}_4, \widetilde{E}_8$
Periodicity	n	$n+1$	$(n+1)(2n+1)$	$n+1$	4	9	5	1

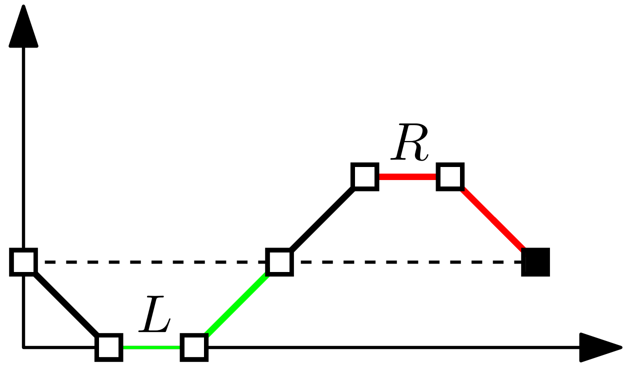
Moreover, we have the same kind of table for FC involutions

Getting formulas: Pyramids of monomers and dimers



A walk in $\mathcal{O}^*$ for $\widetilde{A}^{FC}$

Getting formulas: Pyramids of monomers and dimers



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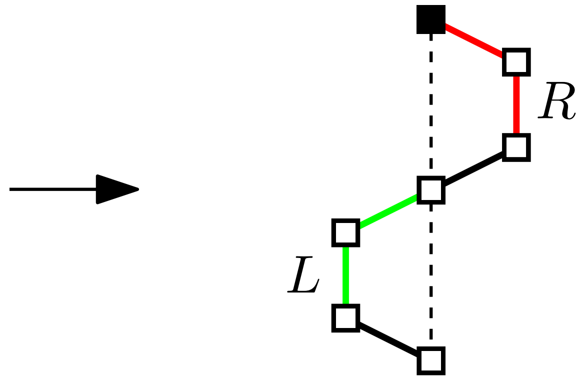
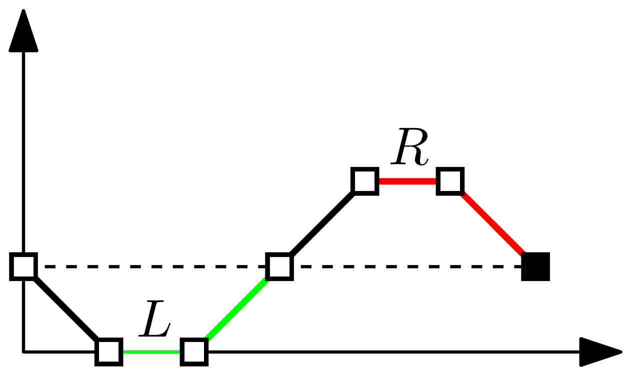


Getting formulas: Pyramids of monomers and dimers



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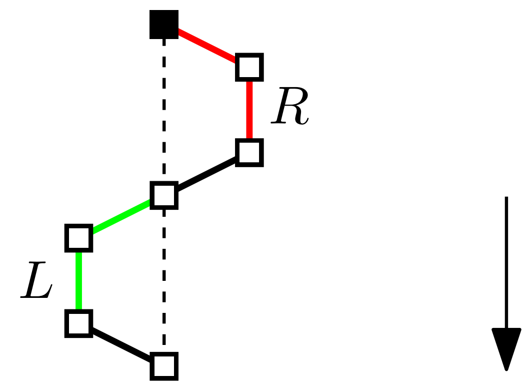
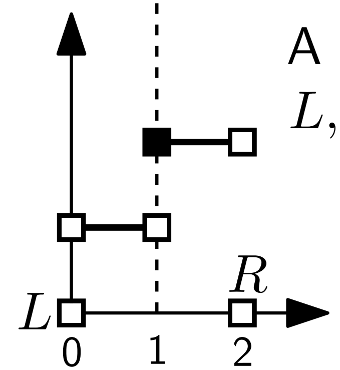
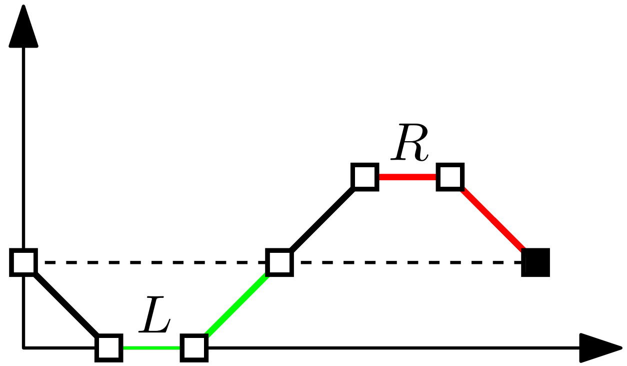




A *marked* pyramid (P_m) of L, R-monomers* and dimers

$(*)$ only L at $i = 0$

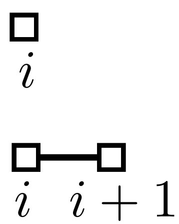
Getting formulas: Pyramids of monomers and dimers

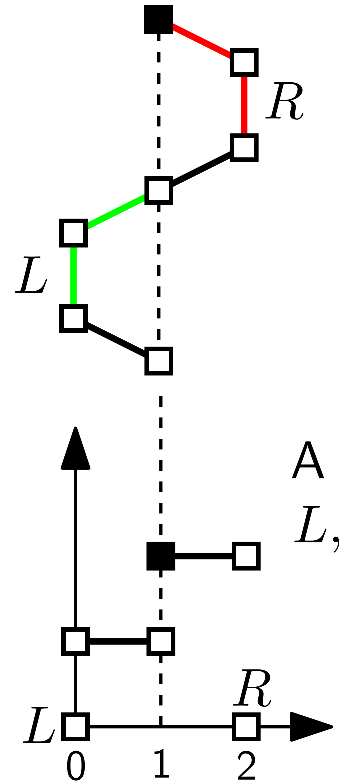


A walk in $\mathcal{O}^*$ for $\widetilde{A}^{FC}$

Monomers: weight xq^i

Dimers: weight x^2q^{2i+1}





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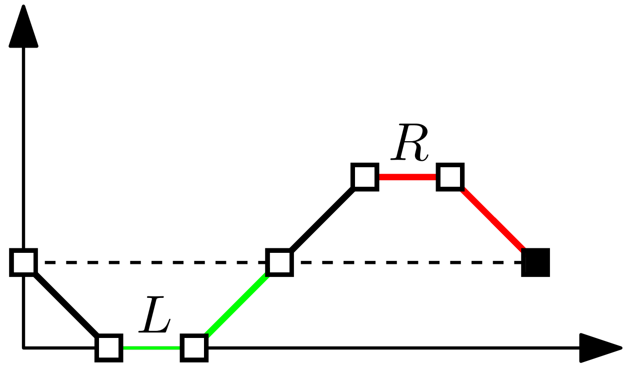
Getting formulas: Pyramids of monomers and dimers



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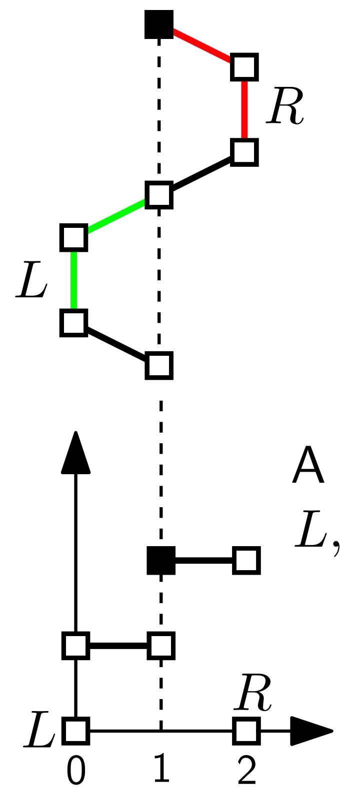

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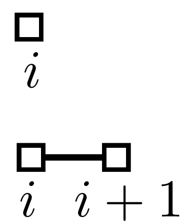
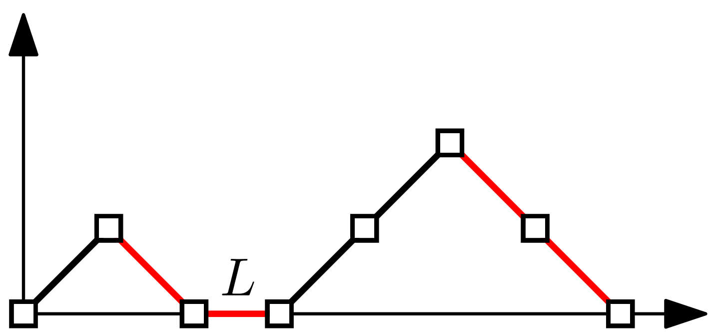
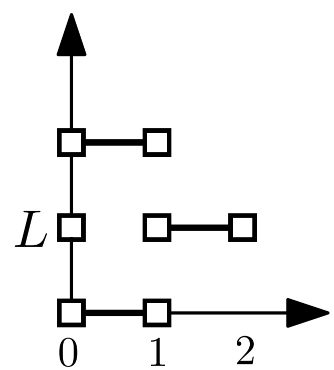




A walk in M^* for A^{FC}



A *semi* pyramid (SP) of L, R-monomers* and dimers

Getting formulas: Viennot's theory of (marked) heaps

For a (marked) heap $\mathcal{E}$, the weight is

$$v(\mathcal{E}) := \prod_{\text{monomers } i} xq^i \prod_{\text{dimers } [i;i+1]} x^2 q^{2i+1} \longrightarrow$$

GF $E(x)$

$SP(x)$, $P_m(x)$

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By definition the GF for marked heaps is $xE'(x)$

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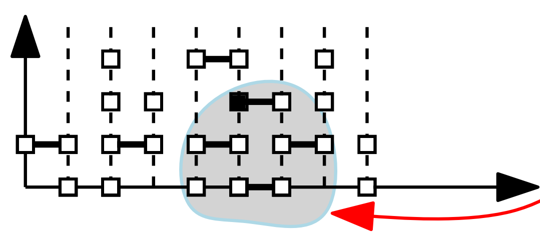
Proposition[Viennot, 1985] We have $xE'(x) = P_m(x) \times E(x)$



Getting formulas: Viennot's theory of (marked) heaps

For a (marked) heap $\mathcal{E}$, the weight is

$$v(\mathcal{E}) := \prod_{\text{monomers } i} xq^i \prod_{\text{dimers } [i;i+1]} x^2 q^{2i+1} \longrightarrow \begin{matrix} \text{GF } E(x) \\ SP(x),\ P_m(x) \end{matrix}$$

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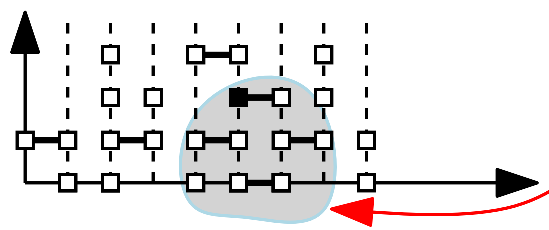


"Push down" the marked object

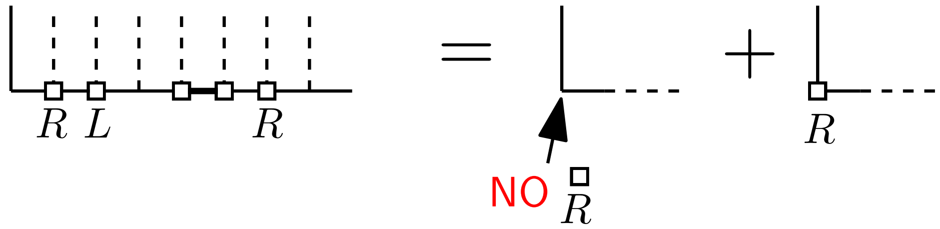
Proposition[Bousquet-Melou, Viennot, 1992] We have

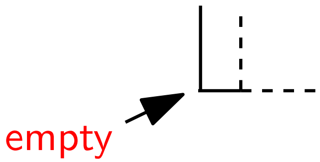
$$E(x) = \frac{1}{T(x)} \text{ and } SP(x) = \frac{T^c(x)}{T(x)}$$

where $T(x)$ (resp. $T^c(x)$) is the signed GF for trivial heaps (resp. not touching 0)

Getting formulas: compute signed GF for trivial heaps

Note that: $T(x) = T^*(x) - xT(xq)$ and $T^c(x) = T(xq)$





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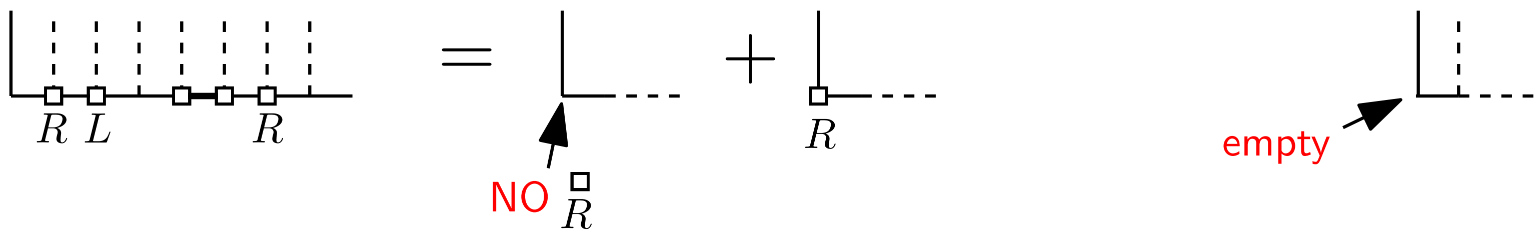
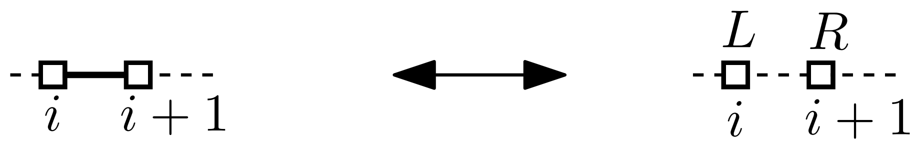


To compute $T(x)$: sign-reversing, weight-preserving involution



$\mathcal{M}$:= set of infinite words on $\{0, L, R\}$ avoiding the factor LR and ending with an infinite number of letters 0

$T(x) = \sum_{m \in \mathcal{M}} (-x)^k q^l$ where m has k letters L, R having l as sum of indices

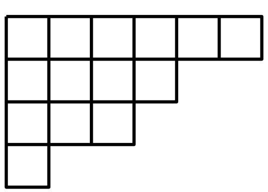
Getting formulas: a bijection with integer partitions

Proposition There is a bijection between the elements $m \in \mathcal{M}$ and pairs of integer partitions (λ, μ) with distinct nonnegative parts, such that the weight of m is $x^{\ell(\lambda)+\ell(\mu)}q^{|\lambda|+|\mu|+\ell(\lambda)\ell(\mu)}$

$$m = ROLOLLORRLORR \qquad x^k q^l = x^9 q^{58}$$

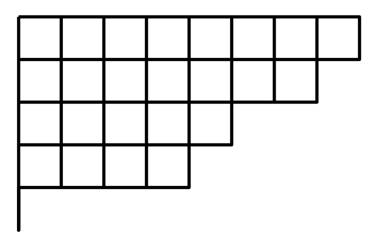
$$\downarrow$$

$$(\pi_L(m) = OLOLLOLO \quad , \quad \pi_R(m) = ROOORRORR)$$



$$\mu_L = (6,4,3,1)$$

$$|\mu_L| = 14 \quad \ell(\mu_L) = 4$$



$$\mu_R = (8,7,5,4,0)$$

$$|\mu_R| = 24 \quad \ell(\mu_R) = 5$$

Explicit results

$$H(x) := \sum_{n\geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n (x;q)_n} \qquad J(x) := \sum_{n\geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n (xq;q)_n}$$

We have $T(x) = (x;q)_\infty H(x)$ and $T^*(x) = (xq;q)_\infty J(x)$

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We have $T(x) = (x;q)_\infty H(x)$ and $T^*(x) = (xq;q)_\infty J(x)$

Theorem: [Barcucci et al. (2001)]
We have $A^{FC}(x) = \dfrac{H(xq)}{J(x)} - 1$

Theorem: We have $\widetilde{A}^{FC}(x) = -x\dfrac{J'(x)}{J(x)} - \sum_{n\geq 1} \dfrac{x^n q^n}{1-q^n}$

What about involutions?

Our walks have no horizontal step at height > 0. Therefore all monomers can only appear at abscissa 0 with label L

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$$U(x) := \sum_{n \geq 0} \frac{(-x^2)^n q^{n(2n-1)}}{(q^2;q^2)_n}$$

Theorem: We have $\bar{A}^{FC}(x) = \dfrac{U(xq)}{U(x) - xU(xq)} - 1$

Theorem: We have $\widetilde{\bar{A}}^{FC}(x) = -x \dfrac{U'(x) - qxU'(xq)}{U(x) - xU(xq)}$