Andrews–Gordon identities and commutative algebra

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(avec P. Afsharijoo, J. Dousse et H. Mourtada)

Partitions and Rogers–Ramanujan identities

A partition of a nonnegative integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. The integers λ_k are the parts.

Example. Partitions of 4 : (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$.

Theorem (Rogers–Ramanujan, MacMahon, 1916)

Let n be a nonnegative integer and set $i \in \{1;2\}$. Denote by $T_{2,i}(n)$ the number of partitions of n such that the difference between two consecutive parts is at least 2 and the part 1 appears at most $i-1$ times. Let $E_{2,i}(n)$ be the number of partitions of n into parts not congruent to $0, \pm i \mod 5$. Then we have $T_{2,i}(n) = E_{2,i}(n)$.

Appear in combinatorics (Andrews, Bressoud, Warnaar,...), statistical mechanics (Baxter,...), number theory (Ono, Zagier,...), representation theory (Lepowski, Milne, Wilson,...), algebraic geometry (Mourtada,...),...

Generating series

If $P(n)$ is the number of partitions of n, then (Euler, 1750) :

$$\sum_{n\geq 0} P(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}$$

Theorem (Rogers–Ramanujan identites, analytic version)

We have

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q)\cdots(1-q^k)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(1-q)\cdots(1-q^k)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

A graded algebra of polynomials with countable number of variables

Algebra of polynomials : $\mathcal{R} = K[x_i, i \geq 1]$ over K of characteristic 0.

Graduation by assigning to x_i the weight i.
Then set $R_0 := K$ and R_n the K-vector space with a basis given by the monomials $x_{i_1} \cdots x_{i_\ell}$ (we can assume that $i_1 \geq i_2 \geq \cdots \geq i_\ell > 0$) such that $i_1 + \cdots + i_\ell = n$. Then

$$\mathcal{R} = \oplus_{n \geq 0} R_n$$

Trivial bijection between monomials of weight n and partitions of n :

$$x_{i_1} \cdots x_{i_\ell} \mapsto \lambda = (i_1, \ldots, i_\ell)$$

Therefore the Hilbert–Poincaré series of $\mathcal{R}$ is

$$HP_{\mathcal{R}}(q) := \sum_{n \geq 0} \dim_K R_n \, q^n = \sum_{n \geq 0} p(n) q^n$$

Algebraic interpretation of the Rogers–Ramanujan identities

In 2011, Bruschek–Mourtada–Schepers studied the ring $\mathcal{R}/[x_1^2]$ of global sections of the space of arcs centered at a fat point.
Here $[x_1^2] = (x_1^2, 2x_1x_2, 2x_2^2 + 2x_1x_3, \ldots)$ is the differential ideal generated by $\{D^k(x_1^2), k \geq 1\}$ where $D(x_j) := x_{j+1}$.

They proved that the leading ideal of $J := (x_1, [x_1^2])$ with respect to the "weighted reverse lexicographic order"(ideal generated by the leading monomials of all the elements in J) is $J_0 = (x_1, x_k^2, x_kx_{k+1}; k \geq 1)$.

Remark : J_0 is in general NOT generated by the leading monomials of a system of generators of J. But a system of generators of J whose leading monomials generate J_0 is called a Gröbner basis.

$$HP_{\mathcal{R}/J}(q) = HP_{\mathcal{R}/J_0}(q) = \sum_{n \geq 0} T_{2,1}(n) q^n = \sum_{k \geq 0}^{\infty} \frac{q^{k^2+k}}{(1-q)\cdots(1-q^k)}$$

Generalization : Gordon's identities

Theorem (Gordon, 1961)

Let r, i be integers with $r \geq 2$, $1 \leq i \leq r$. Denote by $\mathcal{T}_{r,i}$ the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ where $\lambda_j - \lambda_{j+r-1} \geq 2$ for all j, and the part 1 appears at most $i-1$ times. Let $\mathcal{E}_{r,i}$ be the set of partitions into parts not congruent to $0, \pm i \mod (2r+1)$.
Let n be a nonnegative integer, and let $T_{r,i}(n)$ (respectively $E_{r,i}(n)$) denote the number of partitions of n which belong to $\mathcal{T}_{r,i}$ (respectively $\mathcal{E}_{r,i}$). Then we have $T_{r,i}(n) = E_{r,i}(n)$.

The Rogers–Ramanujan identities correspond to the cases $r = i = 2$ and $r = i + 1 = 2$, respectively.

Bruschek–Mourtada–Schepers (2011) : define $J := (x_1^i, [x_1^r])$, then its leading ideal with respect to the weighted reverse lexicographic order is

$$J_{r,i} = (x_1^i, x_k^{r-s} x_{k+1}^s; k \geq 1; s = 0, \ldots, r-1)$$

A conjecture arising from commutative algebra

Afsharijoo (2019) : predicted the leading ideal $I_{r,i}$ of $J = (x_1^i, [x_1^r])$ with respect to the weighted lexicographic order. Even for $r = 2$ a Gröbner basis is not differentially finite (Afsharijoo–Mourtada, 2020).
For $\lambda := (\lambda_1, \dots, \lambda_\ell, 0, \dots)$, define $N_{r,i}(\lambda) := |\{m \mid p_{i,m}(\lambda) \neq 0\}|$, with

$$p_{i,m}(\lambda) := \begin{cases} \lambda_\ell & \text{if } m = 1, \\ \lambda_{\ell - \sum_{j=1}^{m-1} p_{i,j}(\lambda)} & \text{if } 2 \leq m \leq i, \\ \lambda_{\ell + m - i - \sum_{j=1}^{m-1} p_{i,j}(\lambda)} & \text{if } i < m \leq r-1. \end{cases}$$

Conjecture (Afsharijoo, 2019)

Set $r \geq 2$, $1 \leq i \leq r$ and $\mathcal{C}_{r,i}$ the set of partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$ such that at most $i-1$ of the parts are equal to 1 and either $N_{r,i}(\lambda) < r-1$, or $N_{r,i}(\lambda) = r-1$ and $\ell \leq \sum_{j=1}^{r-1} p_{i,j}(\lambda) - (r-i)$. Let n be a nonnegative integer, and denote by $C_{r,i}(n)$ the number of partitions of n which belong to $\mathcal{C}_{r,i}$. Then we have $C_{r,i}(n) = T_{r,i}(n) = E_{r,i}(n)$.

Combinatorial results

Theorem (ADJM, 2022)

The above conjecture is true.

We define later two new sets of partitions $\mathcal{B}_{r,i}$ and $\mathcal{D}_{r,i}$ related to the classical Durfee dissection used by Andrews to define his set $\mathcal{A}_{r,i}$.

Theorem (ADJM, 2022)

Set $r \geq 2$, $1 \leq i \leq r$. Let n be a nonnegative integer. Then we have

$$A_{r,i}(n) = B_{r,i}(n) = C_{r,i}(n) = D_{r,i}(n) = T_{r,i}(n) = E_{r,i}(n)$$

Actually it is almost immediate that $\mathcal{B}_{r,i} = \mathcal{C}_{r,i}$. We then prove (both combinatorially and algebraically) that $\mathcal{B}_{r,i} = \mathcal{D}_{r,i}$. Finally we show that $\mathcal{D}_{r,r-i}$ and $\mathcal{E}_{r,r-i}$ have the same generating series.

Theorem (ADJM, 2022)

Explicit bijection between partitions of n in $\mathcal{D}_{r,r-1}$ and in $\mathcal{A}_{r,r-1}$.

Analytic results

Recall $(a)_\infty := \prod_{j\geq 0}(1-aq^j)$ and for any $k \in \mathbb{Z}$

$$(a)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty} \quad \text{and} \quad (a_1,\dots,a_m)_k := (a_1)_k \cdots (a_m)_k$$

Theorem (Andrews–Gordon identities, Andrews, 1974)

Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. We have

$$\sum_{n_1\geq\cdots\geq n_{r-1}\geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2+n_i+\cdots+n_{r-1}}}{(q)_{n_1-n_2}\dots(q)_{n_{r-2}-n_{r-1}}(q)_{n_{r-1}}} = \frac{(q^{2r+1},q^i,q^{2r-i+1};q^{2r+1})_\infty}{(q)_\infty}$$

Theorem (Bressoud, 1980, ADJM, 2021)

Let $r \geq 2$ and $0 \leq i \leq r-1$ be two integers. We have

$$\sum_{n_1\geq\cdots\geq n_{r-1}\geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2-n_1-\cdots-n_i}(1-q^{n_i})}{(q)_{n_1-n_2}\dots(q)_{n_{r-1}}} = \frac{(q^{2r+1},q^{r-i},q^{r+i+1};q^{2r+1})_\infty}{(q)_\infty}$$

q-binomial coefficients

The q-binomial coefficient is defined as :

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_k (q)_{n-k}}$$

Note that $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$ or $k > n$.

It is the generating function for partitions with largest part $\leq k$ and number of parts $\leq n - k$, or equivalently partitions whose Young diagram fits inside a $k \times (n-k)$ rectangle.

Andrews–Gordon identities can be rewritten as :

$$\sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{r-1}^2 + n_i + \cdots + n_{r-1}}}{(q)_{n_1}} \begin{bmatrix} n_1 \\ n_1 - n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{r-2} \\ n_{r-2} - n_{r-1} \end{bmatrix}_q = \sum_{n \geq 0} E_{r,i}(n) q^n$$

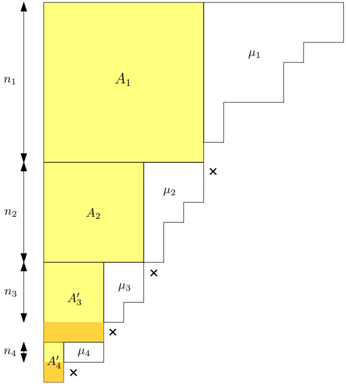
Andrews' vertical $(i-1)$-Durfee dissection and $\mathcal{A}_{r,i}$

Durfee square : largest $A = n \times n$ fitting in the top-left corner of λ.

Vertical Durfee rectangle : largest rectangle $A' = (n-1) \times n$.

Repeat the process until the row below a square/rectangle is empty.

Vertical $(i-1)$-Durfee dissection : the first $i-1$ are squares and all the following ones are rectangles.



$\mathcal{A}_{r,i}$: partitions λ such that in their vertical $(i-1)$-Durfee dissection, all vertical Durfee rectangles below A'_{r-1} are empty, and such that the last row of each non-empty Durfee rectangle is actually a part of λ.

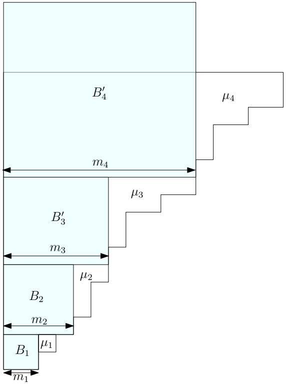
A $(i-1)$-bottom dissection and $\mathcal{B}_{r,i}$ equivalent to $\mathcal{C}_{r,i}$

Bottom square of $\lambda = (\lambda_1, \dots, \lambda_\ell)$: $B = \lambda_\ell \times \lambda_\ell$.

Bottom rectangle : horizontal rectangle $B' = \lambda_\ell \times (\lambda_\ell - 1)$.

Repeat the process until the row above a square/rectangle is empty.

$(i-1)$-bottom dissection : the first $i-1$ are bottom squares and all the following ones are rectangles.



$\mathcal{B}_{r,i}$: partitions λ such that in their $(i-1)$-bottom dissection, all bottom rectangles above B'_{r-1} are empty.

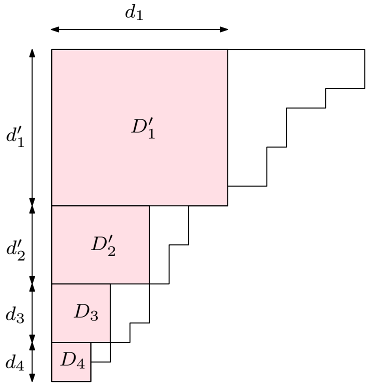
By definition $\mathcal{C}_{r,i} = \mathcal{B}_{r,i}$

A new horizontal $(r-i)$-Durfee dissection and $\mathcal{D}_{r,i}$

Horizontal Durfee rectangle of λ : largest $D' = k \times (k-1)$ fitting in the top-left corner of λ.

Repeat the process until the row above a square/rectangle is empty.

$(r-i)$-Durfee dissection : the first $r-i$ are horizontal rectangles and all the following ones are squares.

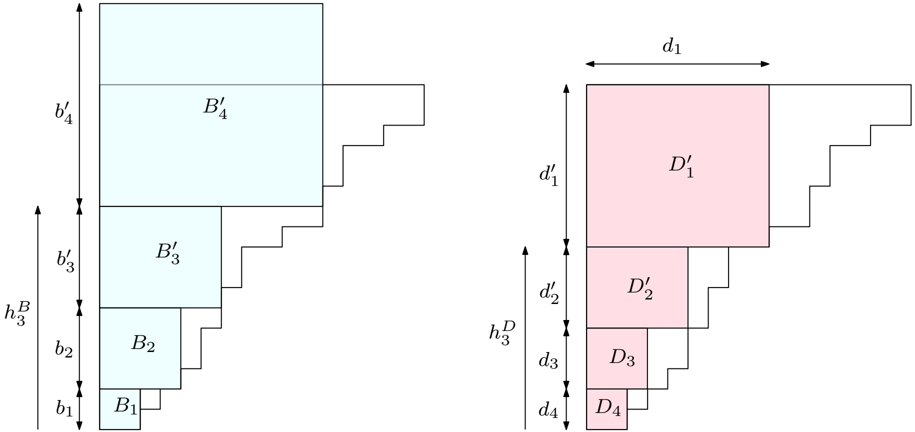


$\mathcal{D}_{r,i}$: partitions such that in their $(r-i)$-Durfee dissection, all Durfee squares below D_{r-1} are empty.

$\mathcal{D}_{r,i}$ is actually $\mathcal{B}_{r,i}$

Theorem (ADJM, 2022)

Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. We have $\mathcal{B}_{r,i} = \mathcal{D}_{r,i}$.



Sketch of proof

$\mu^j :=$ partition to the right of the j-th Durfee square/rectangle.
h_j^B (resp. h_j^D) $:=$ height of the top of the j-th bottom (resp. Durfee) square/rectangle starting from the bottom.

Proposition

Set λ having exactly $r-1$ non-empty Durfee squares. Consider also the bottom dissection with only squares. Then for all $1 \leq j \leq r-1$, we have $h_j^D \leq h_j^B < h_{j+1}^D$. Moreover $h_j^D = h_j^B$ iff $\mu^{r-1}, \ldots, \mu^{r-j}$ have strictly less than $d_{r-1}, \ldots, d_{r-j}$ parts, respectively.

We derive $\mathcal{D}_{r,r} \setminus \mathcal{D}_{r-1,r-1} = \mathcal{B}_{r,r} \setminus \mathcal{B}_{r-1,r-1}$ and by

$$\mathcal{D}_{r,r} = (\mathcal{D}_{r,r} \setminus \mathcal{D}_{r-1,r-1}) \sqcup \mathcal{D}_{r-1,r-1}$$

conclude that $\mathcal{D}_{r,r} = \mathcal{B}_{r,r}$.
Similarly for the case $\mathcal{D}_{r,1} = \mathcal{B}_{r,1}$ and we use these for the general case $\mathcal{D}_{r,i} = \mathcal{B}_{r,i}$.

Generating series

Simplifying q-binomial coefficients :

$$\sum_{\lambda \in \mathcal{D}_{r,i}} q^{|\lambda|} = \sum_{d_1 \geq \cdots \geq d_{r-1} \geq 0} \frac{q^{d_1^2 + \cdots + d_{r-1}^2 - d_1 - \cdots - d_{r-i}}}{(q)_{d_1 - d_2} \ldots (q)_{d_{r-2} - d_{r-1}} (q)_{d_{r-1}}} (1 - q^{d_{r-i}})$$

Equivalently

$$\sum_{\lambda \in \mathcal{D}_{r,r-i}} q^{|\lambda|} = \sum_{d_1 \geq \cdots \geq d_{r-1} \geq 0} \frac{q^{d_1^2 + \cdots + d_{r-1}^2 - d_1 - \cdots - d_i}}{(q)_{d_1 - d_2} \ldots (q)_{d_{r-2} - d_{r-1}} (q)_{d_{r-1}}} (1 - q^{d_i})$$

Therefore it remains to prove that the R.H.S. is the generating function for partitions in Andrews–Gordon theorem (with $i \to r - i$) : the only way we found was showing it is the one of $\mathcal{E}_{r,r-i}$, that is

$$\frac{(q^{2r+1}, q^{r-i}, q^{r+i+1}; q^{2r+1})_\infty}{(q)_\infty}$$

A formula of Bressoud

For all integers $r > 0$ and $0 \leq i \leq r-1$, we have (Bressoud, 1980) :

$$S_i(q) := \sum_{n_1 \geq \cdots \geq n_{r-1} \geq 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2-n_1-\cdots-n_i}}{(q)_{n_1-n_2}\dots(q)_{n_{r-2}-n_{r-1}}(q)_{n_{r-1}}}$$

$$= \sum_{k=0}^{i} \frac{(q^{2r+1}, q^{r-i+k}, q^{r+i-k+1}; q^{2r+1})_\infty}{(q)_\infty}$$

Therefore $S_0(q)$ is the GF of $\mathcal{D}_{r,r}$ and for $i > 0$:

$$S_i(q) - S_{i-1}(q) = \frac{(q^{2r+1}, q^{r-i}, q^{r+i+1}; q^{2r+1})_\infty}{(q)_\infty}$$

A direct bijection between $\mathcal{A}_{r,r-1}$ and $\mathcal{D}_{r,r-1}$

Fix a positive integer n. We give a bijection T between some set $\mathcal{A}''_r(n)$ and $\mathcal{A}_{r,r-1}(n)$. Then we prove that actually $\mathcal{A}''_r(n) = \mathcal{D}_{r,r-1}(n)$.

Our bijection T leaves $\mathcal{A}_{r-1,r-1}(n)$ unchanged, where

$$\mathcal{A}''_r(n) := \mathcal{A}_{r-1,r-1}(n) \sqcup \mathcal{A}'_r(n)$$

and

$$\mathcal{A}_{r,r-1}(n) = \mathcal{A}_{r-1,r-1}(n) \sqcup \mathcal{F}_{r-1}(n)$$