Relations de dualité pour les séries hypergéométriques basiques

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Functional equations and special functions:
From combinatorics to model theory

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Gauss' hypergeometric function

Gauss (end of 19th) : for $a, b, c, z \in \mathbb{C}$ with $|z| < 1$ and $c \notin \mathbb{Z}^-$, consider

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{k \geq 0} \frac{(a)_k (b)_k}{k!(c)_k} z^k $$

$(a)_k := a(a+1)\cdots(a+k-1)$ is the Pocchammer symbol (note $(1)_k = k!$)

Solution around 0 of the hypergeometric equation :

$$ z(z-1)y'' + ((a+b+1)z - c)y' + aby = 0 \qquad (1) $$

Second order Fuchsian equation with singularities $0, 1, \infty$

Setting $\theta := z\dfrac{d}{dz}$, rewrite (1) as $z(\theta+a)(\theta+b)y = \theta(\theta+c-1)y$ and setting $y = z^{1-c}Y$ one gets

$$ z(\theta+a+1-c)(\theta+b+1-c)Y = \theta(\theta-c+1)Y $$

Other solution : $z^{1-c} \times {}_2F_1\left(\begin{matrix} a+1-c, b+1-c \\ 2-c \end{matrix}; z\right)$

These two functions form a basis of solution for (1) if $c \notin \mathbb{Z}$

Generalization to order r

Thomae (end of 19th) : take $b_r = 1$, then

$$ {}_rF_{r-1}\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{r-1} \end{matrix}; z\right) := \sum_{k\geq 0} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_r)_k} z^k $$

is solution around 0 of

$$ (\theta + b_1 - 1)\cdots(\theta + b_r - 1)f = z(\theta + a_1)\cdots(\theta + a_r)f $$

Basis of solutions if the b_i (including $b_r = 1$) are distinct modulo $\mathbb{Z}$:

$$ f_i(z) := z^{1-b_i}\, {}_rF_{r-1}\left(\begin{matrix} a_1 + 1 - b_i, \dots, a_r + 1 - b_i \\ b_1 + 1 - b_i, \dots, \vee, \dots, b_r + 1 - b_i \end{matrix}; z\right), \qquad 1 \leq i \leq r, $$

where $\vee$ denotes deletion of the term with index i.

Note that in this notation $f_r(z)$ is the hypergeometric function we started with.

Beukers–Heckman (1989) : irreducibility, rigidity and monodromy of the hypergeometric equation

Heine's basic hypergeometric series

Gauss, Heine (end of 19th) : for $a, b, c, q, z \in \mathbb{C}$ with $|q|, |z| < 1$ and $c \neq q^{\alpha}$, $\alpha \in \mathbb{Z}^{-}$

$$ {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; q; z \right] := \sum_{k \geq 0} \frac{(a;q)_k (b;q)_k}{(q;q)_k (c;q)_k} z^k $$

$(a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1})$ is the q-Pocchammer symbol

Note that if $a, b, c \to q^a, q^b, q^c$ and $q \to 1$, then ${}_2\phi_1 \to {}_2F_1$

Also note $\dfrac{f(z) - f(qz)}{1-q} \to \theta f(z) = zf'(z)$ when $q \to 1$

Define the dilatation operator $\Delta f(z) := f(qz)$

Jackson (1910) : ${}_2\phi_1$ solution of

$$ z(1 - a\Delta)(1 - b\Delta)y = (1 - \Delta)(1 - c\Delta/q)y \tag{2} $$

Note that if $a, b, c \to q^a, q^b, q^c$, divide by $(1-q)^2$ and $q \to 1$, then (2) $\to$ (1)

$$ z(\theta + a)(\theta + b)y = \theta(\theta + c - 1)y $$

A basis of solutions

Recall (2) $z(1 - a\Delta)(1 - b\Delta)y = (1 - \Delta)(1 - c\Delta/q)y$

Setting $\gamma := \log c/\log q$ (i.e. $q^\gamma = c$) and $y = z^{1-\gamma}Y$, (2) becomes

$$z(1 - aq\Delta/c)(1 - bq\Delta/c)Y = (1 - \Delta)(1 - q\Delta/c)Y$$

Other solution for (2) around 0 : $z^{1-\gamma} \times {}_2\phi_1\left[\begin{matrix} aq/c, bq/c \\ q^2/c \end{matrix}; q; z\right]$

We have a basis of solution for (2) if $c \notin q^{\mathbb{Z}}$

Generalization to order r

Jackson (1910) : take $b_r = q$, then

$$ {}_r\phi_{r-1}\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{r-1} \end{matrix}; q, z\right] := \sum_{k\geq 0} \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_r;q)_k} z^k $$

is solution around 0 of

$$ (1 - b_1\Delta/q)\cdots(1 - b_r\Delta/q)f = z(1 - a_1\Delta)\cdots(1 - a_r\Delta)f $$

Basis of solutions if for $i \neq j$, the b_i/b_j (including $b_r = q$) are not in $q^{\mathbb{Z}}$:

$$ f_i(q;z) := z^{1-\beta_i}\, {}_r\phi_{r-1}\left[\begin{matrix} qa_1/b_i, \dots, qa_r/b_i \\ qb_1/b_i, \dots, \vee, \dots, qb_r/b_i \end{matrix}; q, z\right], \qquad 1 \leq i \leq r $$

where $\beta_i := \log b_i / \log q$

In this notation $f_r(q;z)$ is the hypergeometric function we started with.

Roques (2011, 2014) : irreducibility and rigidity of the basic hypergeometric equation

Some definitions

Let K be a field of characteristic zero and $\Delta : K \to K$ a fixed isomorphism. We denote by $K_0 := \{a \in K | \Delta(a) = a\}$ the subfield of constants.

Definition

A K-vector space M is called a Δ-module (over K) if there is a bijective map $\nabla : M \to M$ such that
(i) $\nabla(m_1 + m_2) = \nabla(m_1) + \nabla(m_2)$ for all $m_1, m_2 \in M$,
(ii) $\nabla(fm) = \Delta(f)\nabla(m)$ for all $f \in K$ and $m \in M$.

We denote ∇ by Δ again.

Definition

Let M, M' be Δ-modules over K. A (bijective) K-linear map $\varphi : M \to M'$ is called a Δ-(iso)morphism if

$$\Delta \circ \varphi = \varphi \circ \Delta$$

The tensor product $M \otimes M'$ has a Δ-module structure via

$$\Delta(m \otimes m') := \Delta(m) \otimes \Delta(m')$$

The dual vector space M^* has a Δ-module structure via

$$\Delta(m^*)(m) := \Delta(m^*(\Delta^{-1}(m)))$$

A characterization

Proposition

Let M, N be Δ-modules of finite rank r. Let $m_1, \ldots, m_r$ be a basis of M and $m_1^*, \ldots, m_r^*$ its dual basis in M^*.
Then the Δ-morphisms $M^* \to N$ are in one-to-one correspondence with the tensors $\Omega \in N \otimes M$ such that $\Delta(\Omega) = \Omega$.
Moreover, φ is a Δ-isomorphism if and only if Ω is non-degenerate, i.e. it can not be written $\sum_{i=1}^{s} n_i \otimes m_i$ with $m_i \in M, n_i \in N$ and $s < r$.

Proof.
If $\Delta(m_i) = \sum_{j=1}^{r} A_{ij} m_j$, then $\Delta(m_i^*) = \sum_{j=1}^{r} B_{ij} m_j^*$, where $(B_{ij})_{1 \leq i,j \leq r}$ is the transposed inverse of $(A_{ij})_{1 \leq i,j \leq r}$.

To a Δ-morphism $\varphi : M^* \to N$, associate $\Omega := \sum_{i=1}^{r} \varphi(m_i^*) \otimes m_i$.

To a tensor $\Omega = \sum_{i=1}^{r} n_i \otimes m_i$ with $\Delta(\Omega) = \Omega$, associate the K-linear map generated by $m_i^* \mapsto n_i$.

The tensor $\sum_{i=1}^{r} n_i \otimes m_i$ is non-degenerate if and only if $n_1, \ldots, n_r$ are linearly independent. But this is equivalent to φ being an isomorphism.

The Casoratian

Let $\mathcal{K}$ be a field extension of K and suppose Δ extends to an isomorphism $\Delta : \mathcal{K} \to \mathcal{K}$.

Suppose also that the field of fixed elements under Δ is still K_0.

Let $h_1, \ldots, h_r \in \mathcal{K}$. Define the Casoratian matrix by

$$W(h_1, \ldots, h_r) := \begin{pmatrix} h_1 & h_2 & \ldots & h_r \\ \Delta(h_1) & \Delta(h_2) & \ldots & \Delta(h_r) \\ \vdots & & & \vdots \\ \Delta^{r-1}(h_1) & \Delta^{r-1}(h_2) & \ldots & \Delta^{r-1}(h_r) \end{pmatrix}$$

Lemma

We have that $\det(W(h_1, \ldots, h_r)) \neq 0$ if and only if $h_1, \ldots, h_r$ are linearly independent over K_0.

Module associated to a rth order operator

Consider the skew ring $K[\Delta, \Delta^{-1}]$ and an operator $L \in K[\Delta, \Delta^{-1}]$ of rank r :

$$L := A_r \Delta^r + A_{r-1}\Delta^{r-1} + \cdots + A_1 \Delta + A_0, \qquad \text{with} \quad A_0, A_r \neq 0$$

Let $(L) := \{\mu L | \mu \in K[\Delta, \Delta^{-1}]\}$ be the left ideal generated by L

Then $K[\Delta, \Delta^{-1}]/(L)$ is again a Δ-module, the module associated to the operator L. The action of Δ is given by left composition with Δ.

Theorem (Beukers–J, 2014)

The dual of $K[\Delta, \Delta^{-1}]/(L)$ *is* Δ*-isomorphic to* $K[\Delta, \Delta^{-1}]/(L^*)$ *where*

$$L^* := \Delta^{r-1}(A_0)\Delta^r + \Delta^{r-2}(A_1)\Delta^{r-1} + \cdots + A_{r-1}\Delta + \Delta^{-1}(A_r)$$

Proof. One has to find a non-degenerate $\Omega \in (K[\Delta, \Delta^{-1}]/(L^*)) \otimes (K[\Delta, \Delta^{-1}]/(L))$ satisfying $\Delta(\Omega) = \Omega$

The case $r = 2$

For

$$L = A\Delta^2 + B\Delta + C \quad \text{and} \quad L^* = \Delta(C)\Delta^2 + B\Delta + \Delta^{-1}(A)$$

take

$$\Omega := C(\Delta \otimes 1) - \Delta^{-1}(A)(1 \otimes \Delta)$$

Then

$$\begin{aligned} \Delta(\Omega) &= \Delta(C)(\Delta^2 \otimes \Delta) - A(\Delta \otimes \Delta^2) \\ &= (-B\Delta - \Delta^{-1}(A)) \otimes \Delta + \Delta \otimes (B\Delta + C) \\ &= -\Delta^{-1}(A)(1 \otimes \Delta) + C(\Delta \otimes 1) \\ &= \Omega \end{aligned}$$

If $f, g \in \mathcal{K}$ satisfy $L(f) = L^*(g) = 0$, then we have

$$\begin{aligned} \Omega(g, f) &:= C\Delta(g)f - \Delta^{-1}(A)g\Delta(f) \\ &= (g, \Delta(g)) \begin{pmatrix} 0 & -\Delta^{-1}(A) \\ C & 0 \end{pmatrix} \begin{pmatrix} f \\ \Delta(f) \end{pmatrix} \in K_0 \quad \text{constant} \end{aligned}$$

Consequences for general r

Corollary

Suppose that $f, g \in \mathcal{K}$ satisfy the equations $L(f) = 0$ and $L^*(g) = 0$. Then $\Omega(g, f) \in K_0$, the subfield of elements of $\mathcal{K}$ fixed under Δ, where $\Omega(g, f)$ is

$$(g, \dots, \Delta^{r-1}(g)) \begin{pmatrix} 0 & 0 & \cdots & 0 & -\Delta^{-1}(A_r) \\ A_0 & A_1 & \cdots & A_{r-2} & 0 \\ 0 & \Delta(A_0) & \cdots & \Delta(A_{r-3}) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \Delta^{r-3}(A_1) & 0 \\ 0 & 0 & \cdots & \Delta^{r-2}(A_0) & 0 \end{pmatrix} \begin{pmatrix} f \\ \Delta(f) \\ \vdots \\ \Delta^{r-1}(f) \end{pmatrix}$$

Denote the middle matrix by $\Psi \in M_r(K)$

Corollary

If $f_1, \dots, f_r$ basis of solutions of $L(f) = 0$ and $g_1, \dots, g_r$ basis of solutions of $L^*(g) = 0$, denote by $C \in M_r(K_0)$ the matrix $(\Omega(g_i, f_j))_{1 \le i,j \le r}$. Then

$$W(f_1, \dots, f_r) C^{-1} W(g_1, \dots, g_r)^t = \psi^{-1}$$

Proof. In $W(g_1, \dots, g_r)^t \Psi W(f_1, \dots, f_r)^t = C$, the l-h-s is invertible

Dual of the basic hypergeometric equation

Recall the basic hypergeometric equation

$$(1 - b_1\Delta/q)\cdots(1 - b_r\Delta/q)f = z(1 - a_1\Delta)\cdots(1 - a_r\Delta)f$$

where we have the default parameter $b_r = q$.

By our theorem, the dual equation reads

$$(\Delta - b_1/q)\cdots(\Delta - b_r/q)g = (\Delta - a_1)\cdots(\Delta - a_r)(z/q)g$$

After rearranging factors we obtain

$$(1 - q\Delta/b_1)\cdots(1 - q\Delta/b_r)g = \frac{a_1\cdots a_r}{b_1\cdots b_{r-1}}q^{r-2}z(1 - q\Delta/a_1)\cdots(1 - q\Delta/a_r)g$$

So the dual equation is again a basic hypergeometric equation with parameters

$$a_i' = q/a_i, b_i' = q^2/b_i \quad \text{and} \quad z \to a_1\cdots a_r q^{r-2}z/(b_1\cdots b_{r-1})$$

Basis of solutions

Suppose that none of the ratios b_i/b_j with $i \neq j$ is an integer power of q.

Recall that a basis of solutions of the basic hypergeometric equation reads

$$f_i(q;z) = z^{1-\beta_i}\, {}_r\phi_{r-1}\left[\begin{matrix} qa_1/b_i, \ldots, qa_r/b_i \\ qb_1/b_i, \ldots, \vee, \ldots, qb_r/b_i \end{matrix}; q, z\right], \qquad 1 \leq i \leq r$$

Therefore a basis of solutions for the dual equation is given by

$$g_i(q;z) := z^{\beta_i - 1}\, {}_r\phi_{r-1}\left[\begin{matrix} b_i/a_1, \ldots, b_i/a_r \\ qb_i/b_1, \ldots, \vee, \ldots, qb_i/b_r \end{matrix}; q; \frac{a_1 \ldots a_r z q^{r-2}}{b_1 \ldots b_{r-1}}\right]$$

The ground field is now the field of rational functions $K = H_q(z)$ where H_q is the field $\mathbb{Q}$ extended with q and the a_i, b_j.

For the field $\mathcal{K}$ containing the solutions of the difference equation we can take the field $H_q((z))$ of Laurent series with coefficients in H_q extended with the functions $z^{1-\beta_i}$, where $\beta_i = \log(b_i)/\log(q)$.

Duality relations

Proposition

For the basic hypergeometric equation and its dual, and the previous basis of solutions, the matrix C is diagonal, with

$$C_{ii} = \frac{1}{qb_i^{r-2}} \prod_{\substack{j=1 \\ j\neq i}}^{r} (b_i - b_j)$$

Theorem (Beukers–J, 2014)

Let $(f_i)_{1\leq i\leq r}$ and $(g_j)_{1\leq j\leq r}$ be the basis of solutions of the q-hypergeometric equation and the dual equation. Let H_q be the field generated over $\mathbb{Q}$ by the a_i, b_j and q. Then, with C_{ii} as defined above

$$\sum_{i=1}^{r} \frac{1}{C_{ii}} \Delta^k(f_i)\Delta^l(g_i) = (\Psi^{-1})_{kl} \in H_q(z), \quad \text{for} \quad 0 \leq k, l \leq r-1$$

Moreover these rational fractions can be explicitly computed. In particular

$$\sum_{i=1}^{r} \frac{1}{C_{ii}} \Delta^k(f_i) g_i = 0, \quad \text{for} \quad k = 0, 1, \ldots, r-2$$

Special cases

For $k = l = 0, r = 2$ we obtain the relation (special case of Heine's transformation)

$$
{}_2\phi_1\left[\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; q, z\right] {}_2\phi_1\left[\begin{matrix} q/a_1, q/a_2 \\ q^2/b_1 \end{matrix}; q; \frac{a_1 a_2 z}{b_1}\right]
= {}_2\phi_1\left[\begin{matrix} qa_1/b_1, qa_2/b_1 \\ q^2/b_1 \end{matrix}; q, z\right] {}_2\phi_1\left[\begin{matrix} b_1/a_1, b_1/a_2 \\ b_1 \end{matrix}; q; \frac{a_1 a_2 z}{b_1}\right],
$$

where we explicitly set $b_2 = q$.

Bailey (1933) : $k = l = 0, r = 3$ by using contour integration techniques

Sears (1951) and Shukla (1957) : $k = r - 2, l = 0$ and general r

For $r = 2$, we have

$$
\Psi^{-1} = \begin{pmatrix} 0 & \frac{1}{1-z} \\ \frac{-q^2}{b_1 b_2 - a_1 a_2 qz} & 0 \end{pmatrix},
$$

and for $r = 3$,

$$
\Psi^{-1} = \begin{pmatrix} 0 & \frac{1}{1-z} & \frac{(b_1+b_2+b_3)-qz(a_1+a_2+a_3)}{(1-z)(1-qz)} \\ 0 & 0 & \frac{1}{1-qz} \\ \frac{q^3}{b_1 b_2 b_3 - a_1 a_2 a_3 q^2 z} & 0 & 0 \end{pmatrix}
$$