Cyclotomic valuation of q-Pochhammer symbols and q-Integrality of basic hypergeometric series

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q-integers and q-factorials in combinatorics

Set q a formal parameter. Define $[0]_q := 0$ and $[n]_q := 1 + q + \cdots + q^{n-1}$, $n > 0$. Therefore the following extend n and $n!$, respectively :

$$[n]_q = \frac{1-q^n}{1-q} \quad \text{and} \quad [n]!_q := \prod_{i=1}^{n} \frac{1-q^i}{1-q}$$

Classical combinatorial set $\mathcal{P}_n$: integer partitions λ of weight $|\lambda|$, with largest part $\leq n$ and length $\leq n$. Then

$$\#\mathcal{P}_n = \binom{2n}{n} \quad \text{and} \quad \begin{bmatrix} 2n \\ n \end{bmatrix}_q := \frac{[2n]!_q}{[n]!_q^2} = \sum_{\lambda \in \mathcal{P}_n} q^{|\lambda|} \in \mathbb{N}[q]$$

This q-binomial is also the number of vector subspaces of dimension n in a vector space of dimension $2n$ over a finite field $\mathbb{F}_q$.

q-integers and cyclotomic polynomials

For a positive integer b, recall the b-th cyclotomic polynomial :

$$\phi_b(q) := \prod_{\substack{1 \le k \le b \\ (k,b)=1}} (q - e^{2ik\pi/b}) \in \mathbb{Z}[q]$$

The role played by prime numbers for integers is now played by cyclotomic polynomials :

$$[n]_q = \frac{1-q^n}{1-q} = \prod_{b \ge 2,\, b|n} \phi_b(q) \Longrightarrow n = \prod_{b \ge 2,\, b|n} \phi_b(1)$$

Recall $\phi_b(1) = 1$ if b is divisible by at least two distinct primes, while $\phi_{p^\ell}(1) = p$ for p prime and $\ell > 0$. "Finer" arithmetics for q-analogs :

$$v_p(n) = \sum_{\ell \ge 1} v_{\phi_{p^\ell}}([n]_q)$$

Factorial ratios

Famous class of sequences in combinatorics, number theory, mathematical physics, or geometry :

$$Q_{e,f}(n) := \frac{(e_1 n)! \cdots (e_v n)!}{(f_1 n)! \cdots (f_w n)!}, \quad n \geq 0$$

where $e := (e_1, \dots, e_v) \in \mathbb{Z}_{>0}^v$ and $f := (f_1, \dots, f_w) \in \mathbb{Z}_{>0}^w$.

Using Landau step functions :

$$\Delta_{e,f}(x) := \sum_{i=1}^{v} \lfloor e_i x \rfloor - \sum_{j=1}^{w} \lfloor f_j x \rfloor$$

their p-adic valuations :

$$v_p\left(Q_{e,f}(n)\right) = \sum_{\ell \geq 1} \Delta_{e,f}\left(n/p^\ell\right)$$

generalize the Legendre formula $v_p(n!) = \sum_{\ell \geq 1} \left\lfloor n/p^\ell \right\rfloor$

Arithmetic properties of factorial ratios

Assume $\sum_i e_i = \sum_j f_j$.

(i) Landau (1900), Bober (2009) : integrality.

$$\forall n \geq 0,\ Q_{e,f}(n) \in \mathbb{Z} \iff \forall x \in [0,1],\ \Delta_{e,f}(x) \geq 0$$

(ii) Rodriguez-Villegas (2007), Beukers–Heckman (1989) : algebricity.

$$\sum_{n=0}^{\infty} Q_{e,f}(n) x^n \text{ is algebraic over } \mathbb{Q}(x) \iff \forall x \in [0,1],\ \Delta_{e,f}(x) \in \{0,1\}$$

Example. For $\Delta_{e,f}(x) = \lfloor 30x \rfloor + \lfloor x \rfloor - \lfloor 15x \rfloor - \lfloor 10x \rfloor - \lfloor 6x \rfloor \in \{0,1\}$, the following quotient is integral with an algebraic generating series (R-V : degree 483 840) :

$$\frac{(30n)!n!}{(15n)!(10n)!(6n)!}$$

q-factorial ratios

Recall

$$[n]!_q := \prod_{i=1}^{n} \frac{1-q^i}{1-q} = \prod_{b\geq 2} \phi_b(q)^{\lfloor n/b \rfloor}$$

Thus Warnaar–Zudilin (2011), ABDJ (2017) :

$$Q_{e,f}(q;n) := \frac{[e_1 n]!_q \cdots [e_v n]!_q}{[f_1 n]!_q \cdots [f_w n]!_q} = \prod_{b\geq 2} \phi_b(q)^{\Delta_{e,f}(n/b)}$$

and assuming $\sum_i e_i = \sum_j f_j$:

$$\forall n \geq 0,\ Q_{e,f}(q;n) \in \mathbb{Z}[q] \Longleftrightarrow \forall x \in [0,1],\ \Delta_{e,f}(x) \geq 0$$

Example. $\Delta_{(2),(1,1)}(x) = \lfloor 2x \rfloor - 2\lfloor x \rfloor \geq 0$ on $[0,1]$.

Dwork map and Christol valuations of rising factorials

Pochhammer symbol : for $\alpha \in \mathbb{Q}$, set $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$, so that $(1)_n = n!$.

Dwork maps (1973) : for a prime p satisfying $v_p(\alpha) \geq 0$, there exists a unique rational number $D_p(\alpha)$ whose denominator is not divisible by p and such that $pD_p(\alpha) - \alpha \in \{0, \dots, p-1\}$.

Christol (1986), Delaygue–Rivoal–Roques (2017) :

$$v_p((\alpha)_n) = \sum_{\ell \geq 1} \left\lfloor \frac{n - \lfloor 1-\alpha \rfloor}{p^\ell} - D_p^\ell(\alpha) + 1 \right\rfloor$$

Example. We have $D_5(1/3) = 2/3$, so that

$$v_5((1/3)_1) = v_5((1/3)_2) = v_5((1/3)_3) = 0 \text{ and } v_5((1/3)_4) = 1, \dots$$

When $\alpha = 1$, we have $D_p(1) = 1$ giving the Legendre formula.

Generalized hypergeometric terms

For $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_v)$ and $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_w)$ with coordinates in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$:

$$Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n) := \frac{(\alpha_1)_n \cdots (\alpha_v)_n}{(\beta_1)_n \cdots (\beta_w)_n} \in \mathbb{Q}\,, \qquad n \geq 0$$

Generalize $Q_{e,f}(n)$ up to $\mathbb{Q}^n$, as

$$(dn)! = d^{dn} \left(\frac{1}{d}\right)_n \cdots \left(\frac{d-1}{d}\right)_n (1)_n$$

Set $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ the lcm of the denominators of all α_i, β_j.

Christol (1986) : step functions $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, \cdot)$, for all $a \in \{1, \ldots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, which replace the Landau functions $\Delta_{e,f}$.

Christol step functions

Set $\langle x \rangle := \{x\}$ if $x \notin \mathbb{Z}$, 1 else.

Christol order on $\mathbb{R}$: $x \preceq y \iff (\langle x \rangle < \langle y \rangle \text{ or } (\langle x \rangle = \langle y \rangle \text{ and } x \geq y))$

Christol step functions defined for $a \in \{1, \ldots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$:

$$\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) := \#\{i \in \{1, \ldots, v\} \,:\, a\alpha_i \preceq x\} - \#\{j \in \{1, \ldots, w\} \,:\, a\beta_j \preceq x\}$$

Example. For $\boldsymbol{\alpha} = (1/9, 4/9, 5/9)$ and $\boldsymbol{\beta} = (1/3, 1, 1)$, we have $d_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 9$ and $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1, x) \geq 0$, $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(2, x) \geq 0$ as their jumps are respectively given by

$$\frac{1}{9} \preceq \frac{1}{3} \preceq \frac{4}{9} \preceq \frac{5}{9} \preceq 1 \preceq 1 \text{ and } \frac{10}{9} \preceq \frac{2}{9} \preceq \frac{2}{3} \preceq \frac{8}{9} \preceq 2 \preceq 2$$

Christol (1986) : N-integrality instead of integrality.
Delaygue–Rivoal–Roques (2017), Beukers–Heckman (1989) : interlacing criterion in terms of the step functions $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, \cdot)$.

N-integrality of generalized hypergeometric sequences

The sequence $(R(n))_{n\geq 0}$ is N-integral if there exists an integer $N \neq 0$ such that $N^n R(n) \in \mathbb{Z}$ for all $n \geq 1$.

Theorem (Christol, 1986)

Let $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_v)$ and $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_w)$ be two vectors with coordinates in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then the two following assertions are equivalent.

(a) The hypergeometric sequence $(Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n))_{n\geq 0}$ is N-integral.

(b) For all $x \in \mathbb{R}$ and $a \in \{1, \ldots, d_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ coprime to $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) \geq 0$.

Classical example by Christol, for $\boldsymbol{\alpha} = (1/9, 4/9, 5/9)$ and $\boldsymbol{\beta} = (1/3, 1, 1)$:

$$Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}(n) = \frac{(1/9)_n(4/9)_n(5/9)_n}{(1/3)_n(1)_n^2}$$

Then $d_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 9$ and for the 6 values $a \in \{1, \ldots, 9\}$ coprime to 9, we have $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) \geq 0$ for all $x \in \mathbb{R}$. Therefore it is N-integral.
Delaygue–Rivoal–Roques (2017) : smallest N (here $N = 9^3$).

N-integrality and G-functions

The series $f(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{Q}[[x]]$ is globally bounded if its radius of convergence is finite and positive, and $(a_n)_{n\geq 0}$ is N-integral.

If f is moreover holonomic over $\mathbb{Q}$, then it is a G-function.

Conjecture by Christol (1987) : all such G-functions are diagonals of rational fractions. Bostan–Yurkevich (2022) : many recent examples.

When $v \leq w$ and one of the β_j is 1, the generalized hypergeometric series $\sum_{n\geq 0} Q_{\alpha,\beta}(n)x^n$ are holonomic. Therefore they belong to this particular subclass of G-functions if $v = w$ (radius of convergence 1) and the Christol criterion is satisfied.

Many of these are known to satisfy the Christol conjecture. But

$$ {}_3F_2\left(\begin{matrix}1/9, 4/9, 5/9\\ 1/3, 1\end{matrix}; x\right) := \sum_{n\geq 0} \frac{(1/9)_n(4/9)_n(5/9)_n}{(1/3)_n(1)_n^2}\, x^n $$

is an example of G-function for which we do not know if it is a diagonal of a rational fraction.

Our three goals

- Define appropriately q-analogs of the generalized hypergeometric terms $Q_{\alpha,\beta}(n)$: they have to be different from (though related to) the ones appearing in classical basic hypergeometric series, and will be defined via the usual q-Pochhammer symbols $(q^r; q^s)_n$.

- Find the ϕ_b-adic valuations of these $(q^r; q^s)_n$: we will need to extend Dwork maps to all positive integers b, and be able to find a uniform answer for any integers r, s, n.

- Prove an effective criterion of q-integrality for our q-generalized hypergeometric terms : a sequence $(R(q;n))_{n\geq 0}$ with values in $\mathbb{Q}(q)$ is said to be q-integral if there exists $N(q) \in \mathbb{Z}[q] \setminus \{0\}$ such that $N(q)^n R(q;n) \in \mathbb{Z}[q]$ for all $n \geq 1$.
  We will need generalizations of Christol step functions.

q-analogs of rising factorials for rational numbers

Recall for $n \in \mathbb{Z}_{\geq 0}$ the q-Pochhammer symbol $(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i)$.
Given $\alpha = r/s \in \mathbb{Q}$, note that

$$\lim_{q \to 1} \frac{(q^\alpha;q)_n}{(1-q)^n} = \lim_{q \to 1} \frac{(q^r;q^s)_n}{(1-q^s)^n} = (\alpha)_n$$

Which of these two choices is appropriate, the classical one or the second ?
Note that the second is obtained by setting $q \to q^s$ in the first...
For ϕ_b-valuations and q-integrality, enough to consider for $(r,s) \in \mathbb{Z} \times \mathbb{Z}^*$:

$$(q^r;q^s)_n = \prod_{i=0}^{n-1}(1-q^{r+si}) \in \mathbb{Z}[q^{-1},q]$$

This is $\neq 0$ iff $r/s \notin \mathbb{Z}_{\leq 0}$ or $n \leq -r/s$.

In combinatorics, for positive r, s, $(q^r;q^s)_n^{-1}$ is the generating series of integer partitions with parts congruent to $r \bmod s$ and largest part $\leq r+(n-1)s$.

ϕ_b-valuations of q-Pochhammer symbols

Proposition (ABDJ, 2022)

Set $b \in \mathbb{Z}_{>0}$ and the multiplicative set $S_b := \{k \in \mathbb{Z} : \gcd(k, b) = 1\}$. Let $\alpha \in S_b^{-1}\mathbb{Z}$, the localization of $\mathbb{Z}$ by S_b. Then there is a unique element $D_b(\alpha) \in S_b^{-1}\mathbb{Z}$ such that $bD_b(\alpha) - \alpha \in \{0, \ldots, b-1\}$.

Example. $D_4(1/3) = 1/3$

Theorem (ABDJ, 2022)

Let $(r, s) \in \mathbb{Z} \times \mathbb{Z}^*$, $\alpha := r/s$, $c := \gcd(r, s, b)$, $b' := b/c$, and $s' := s/c$. Let $n \in \mathbb{Z}_{\geq 0}$ be such that $(q^r; q^s)_n$ is non-zero. Then

$$v_{\phi_b}((q^r; q^s)_n) = \left\lfloor \frac{cn}{b} - \frac{\lfloor 1 - \alpha \rfloor}{b'} - D_{b'}(\alpha) + 1 \right\rfloor$$

if $\gcd(s', b') = 1$ and 0 otherwise.

Special cases

Our result holds for r, s non necessarily coprime (and any positive b), but when $\gcd(r,s)=1$ and $v_p((r/s)_n) \geq 0$, it extends Christol's result :

$$v_p((r/s)_n) = \sum_{\ell \geq 1} v_{\phi_{p^\ell}} \left(\frac{(q^r;q^s)_n}{(1-q^s)^n} \right)$$

For $(r,s,b) = (2,6,8)$, we get $c = \gcd(2,6,8) = 2$, $s' = 3$ and $b' = 4$, so

$$\begin{aligned} v_{\phi_8}((q^2;q^6)_n) &= \left\lfloor \frac{2n}{8} - \frac{\lfloor 1 - 1/3 \rfloor}{4} - D_4\left(1/3\right) + 1 \right\rfloor \\ &= \left\lfloor \frac{n}{4} + \frac{2}{3} \right\rfloor = \left\lfloor \frac{3n+8}{12} \right\rfloor \end{aligned}$$

while $\phi_8(q) = q^4 + 1$ and $(q^2;q^6)_n = (1-q^2)(1-q^8)\ldots(1-q^{2+6n-6})$.

q-hypergeometric sequences

Set $(r_i, s_i), (t_j, u_j)$ pairs of integers with $s_i, u_j \neq 0$ and

$$\mathbf{r} := ((r_1, s_1), \dots, (r_v, s_v)) \quad \text{and} \quad \mathbf{t} := ((t_1, u_1), \dots, (t_w, u_w))$$

Define the q-hypergeometric sequence :

$$\mathcal{Q}_{\mathbf{r},\mathbf{t}}(q;n) := \frac{(q^{r_1};q^{s_1})_n \cdots (q^{r_v};q^{s_v})_n}{(q^{t_1};q^{u_1})_n \cdots (q^{t_w};q^{u_w})_n} \qquad n \geq 0$$

To study q-integrality : well-defined $\forall n \geq 0$ when $\beta_j := t_j/u_j \notin \mathbb{Z}_{\leq 0}$, never vanish when $\alpha_i := r_i/s_i \notin \mathbb{Z}_{\leq 0}$.

Suitable q-analogs :

$$\lim_{q \to 1} \left(\frac{\prod_{j=1}^{w} (1 - q^{u_j})}{\prod_{i=1}^{v} (1 - q^{s_i})} \right)^n \mathcal{Q}_{\mathbf{r},\mathbf{t}}(q;n) = Q_{\alpha,\beta}(n)$$

A generalization of Christol step functions

Set $\alpha_i = r_i/s_i, \beta_j = t_j/u_j$, and $d_{\mathbf{r},\mathbf{t}}$ lcm of all s_i, u_j. Set $c_i := \gcd(r_i, s_i, b)$ and $d_j := \gcd(t_j, u_j, b)$. Consider

$$V_b := \{1 \leq i \leq v \,:\, \gcd(s_i, b) = c_i\} \quad W_b := \{1 \leq j \leq w \,:\, \gcd(u_j, b) = d_j\}$$

For such i, j, there exist positive integers e_i, f_j with $be_i \equiv c_i \mod s_i$ and $bf_j \equiv d_j \mod u_j$.

Let $\tilde{b}$ be the greatest divisor of b coprime to $d_{\mathbf{r},\mathbf{t}}$ and let a be the unique element of $\{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$ satisfying $a\tilde{b} \equiv 1 \mod d_{\mathbf{r},\mathbf{t}}$.
For $b \in \{1, \ldots, d_{\mathbf{r},\mathbf{t}}\}$, define the step function $\Xi_{\mathbf{r},\mathbf{t}}(b, x)$ as

$$\# \left\{ (i,k) \in V_b \times \{0, \ldots, c_i - 1\} \,:\, \frac{\langle e_i \alpha_i \rangle + k}{c_i} - \lfloor 1 - a\alpha_i \rfloor \;\preceq x \right\}$$
$$- \# \left\{ (j,\ell) \in W_b \times \{0, \ldots, d_j - 1\} \,:\, \frac{\langle f_j \beta_j \rangle + \ell}{d_j} - \lfloor 1 - a\beta_j \rfloor \preceq x \right\}.$$

If b is coprime to $d_{\mathbf{r},\mathbf{t}}$, they are equal to Christol step functions

$$\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a, x) := \#\{i \in \{1, \ldots, v\} \,:\, a\alpha_i \preceq x\} - \#\{j \in \{1, \ldots, w\} \,:\, a\beta_j \preceq x\}$$

An example : q-analog of Christol's one

Consider

$$\mathcal{Q}_{\mathbf{r},\mathbf{t}}(q;n) = \frac{(q;q^9)_n(q^4;q^9)_n(q^5;q^9)_n}{(q;q^3)_n(q;q)_n^2}$$

We saw $d_{\mathbf{r},\mathbf{t}} = 9$. So the only new step functions to consider are the one associated with $b \in \{3,6,9\}$.

For $b = 3$, we have $a = \tilde{b} = 1$ and $c_1 = c_2 = c_3 = 1$, $d_1 = d_2 = d_3 = 1$. Moreover $V_3 = \emptyset$ and $W_3 = \{2,3\}$ with $f_2 = f_3 = 1$. So we get

$$\begin{aligned} \Xi_{\mathbf{r},\mathbf{t}}(3,x) &= -\#\left\{(j,\ell) \in \{2,3\} \times \{0\} \,:\, \frac{\langle f_j\beta_j\rangle + \ell}{d_j} - \lfloor 1 - a\beta_j \rfloor \preceq x\right\} \\ &= -\#\left\{j \in \{2,3\} \,:\, \langle \beta_j \rangle - \lfloor 1 - \beta_j \rfloor \preceq x\right\} \end{aligned}$$

Note that $\Xi_{\mathbf{r},\mathbf{t}}(3,1) < 0$.

An effective q-integrality criterion

Theorem (ABDJ, 2022)

Assume that $Q_{\mathbf{r},\mathbf{t}}(q;n)$ is well-defined and non zero, and $s_1,\dots,s_v$ are positive. Then the two following assertions are equivalent.

(i) The sequence $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ is q-integral.

(ii) For every $b\in\{1,\dots,d_{\mathbf{r},\mathbf{t}}\}$ and all x in $\mathbb{R}$, we have $\Xi_{\mathbf{r},\mathbf{t}}(b,x)\geq 0$.

q-integrality implies N-integrality. Converse not always true : depends on the behaviour of $\Xi_{\mathbf{r},\mathbf{t}}(b,\cdot)$ for b not coprime to $d_{\mathbf{r},\mathbf{t}}$.

Example. $\mathbf{r}=((1,9),(4,9),(5,9))$ and $\mathbf{t}=((1,3),(1,1),(1,1))$. The classical Christol functions satisfy the criterion for b coprime to 9.
Not for all $b\in\{3,6,9\}$, as $\Xi_{\mathbf{r},\mathbf{t}}(3,1)<0$.

Thus $(Q_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ is not q-integral.

Back to Christol's example

We saw our q-analog of Christol's example was not q-integral. But this one is :

$$\frac{(q;q^9)_n(q^4;q^9)_n(q^5;q^9)_n(q^9;q^9)_n}{(q;q^3)_n(q;q)_n^3}$$

The $q^{1/9}$-integrality of $\widetilde{Q}_{\alpha,\beta}(q;n) := \frac{(q^{1/9};q)_n(q^{4/9};q)_n(q^{5/9};q)_n}{(q^{1/3};q)_n(q;q)_n^2}$ is equivalent to the q-integrality of

$$\widetilde{Q}_{\alpha,\beta}(q^9;n) = \frac{(q;q^9)_n(q^4;q^9)_n(q^5;q^9)_n}{(q^3;q^9)_n(q^9;q^9)_n^2} =: \mathcal{Q}_{\mathbf{r},\mathbf{t}}(q;n)$$

for a suitable choice of vectors $\mathbf{r},\mathbf{t}$.

We prove $(\mathcal{Q}_{\mathbf{r},\mathbf{t}}(q;n))_{n\geq 0}$ not q-integral. The above trick fails : multiplying $\mathcal{Q}_{\mathbf{r},\mathbf{t}}(q;n)$ by $(q^9;q^9)_n/(q;q)_n$ amounts to multiplying $\widetilde{Q}_{\alpha,\beta}(q;n)$ by $(q;q)_n/(q^{1/9};q^{1/9})_n$ which does not correspond to any choice of parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.