Some new identities for Schur functions

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Dedicated to Dominique Foata on the occasion of his 65th birthday

Abstract

Some new identities for Schur functions are proved. In particular, we settle in the affirmative a recent conjecture of Ishikawa-Wakayama [6] and solve a problem raised by Bressoud [2].

1 Introduction

We fix a positive integer n and let $X = (x_1, \dots, x_n)$ be a set of n independent variables. For each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ of length $\leq n$, the Schur function $s_\lambda(X)$ are usually defined as follows [7]:

$$s_\lambda(X) = \det\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n} / \det\left(x_i^{n-j}\right)_{1\leq i,j\leq n}.$$

In this paper we shall follow the standard definitions and notations of Macdonald's book [7]. Thus the *Ferrers diagram* of λ is the subset $\{(i,\, j)|j \geq 1, i \leq \lambda_j\}$ of $\mathbb{N}^2$. If the diagram of μ is included in that of λ we note $\mu \subseteq \lambda$ and the skew diagram λ/μ is called a *horizontal strip* (or h.s. for short) if there is at most one cell in each column of λ/μ. For any partition λ we note $c_j := c_j(\lambda)$ the number of columns of length j in λ, i.e. $c_j = \lambda_j - \lambda_{j+1}$ and define

$$f_\lambda(a,b) = \prod_{j \text{ odd}} \frac{a^{c_j+1} - b^{c_j+1}}{a-b} \prod_{j \text{ even}} \frac{1-(ab)^{c_j+1}}{1-ab}.$$

Since $f_\lambda(a,0) = a^{c(\lambda)}$, where $c(\lambda)$ is the number of columns of odd length of λ, a classical identity of Littlewood [7] reads then as follows :

$$\sum_\lambda f_\lambda(a,0)s_\lambda(X) = \prod_i (1-ax_i)^{-1} \prod_{j<k} (1-x_jx_k)^{-1}. \tag{1}$$

Set

$$\Phi(X;a,b) := \prod_i (1-ax_i)^{-1}(1-bx_i)^{-1}\prod_{j<k}(1-x_jx_k)^{-1}.$$

In a recent paper [6], Ishikawa and Wakayama gave the following extension of (1):

$$\sum_\lambda f_\lambda(a,b)\, s_\lambda(X) = \Phi(X;a,b). \tag{2}$$

As pointed by Bressoud [2], when $(a,b)=(1,0)$, $(1,-1)$ and $(0,0)$, identity (2) reduces to the following interesting known identities respectively:

$$\sum_\lambda s_\lambda(X) = \prod_{i=1}^n \frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}, \tag{3}$$

$$\sum_{\lambda\,\text{even}} s_\lambda(X) = \prod_{i=1}^n \frac{1}{1-x_i^2}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}, \tag{4}$$

$$\sum_{\lambda'\,\text{even}} s_\lambda(X) = \prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}, \tag{5}$$

where λ' is the conjugate of λ.

In this paper we shall give two generalizations of Ishikawa and Wakayama's formula (2). To state them we need some definitions.

For $r\ge 0$, let $h_r(X)$ (resp. $e_r(X)$) be the *homogeneous* (resp. *elementary*) symmetric function of X and set

$$P_r(a,b,c) = \sum_{k=0}^r \frac{a^{k+1}-b^{k+1}}{a-b}\,\frac{1-(ab)^{r-k+1}}{1-ab}c^k,$$

$$Q_r(a,b,c) = \sum_{k=0}^r h_{r-k}(a,b,c)(abc)^k.$$

For any positive integer sequence $\xi=(\xi_1,\ldots,\xi_r,\ldots)$, where $\xi_r\neq 0$ for only a finite number of integers r, set

$$F_\xi(a,b,c) = h_{\xi_1}(a,b,c)\prod_{k\ge 1}P_{\xi_{2k}}(a,b,c)Q_{\xi_{2k+1}}(a,b,c).$$

For any integer $i\ge 1$, let ε_i be the i^{th} vector of the canonical basis of $\mathbb{Z}^\infty$ and introduce the operator $\delta_i : \delta_i\xi = \xi-\varepsilon_i-\varepsilon_{i+1}$ for $\xi\in\mathbb{N}^\infty$. Set

$\delta_i F_\xi(a,b,c) = F_{\delta_i\xi}(a,b,c)$, where $P_k = Q_k = 0$ if $k < 0$ by convention. Hence, to any partition λ of length $\leq n$ we can associate the polynomial

$$f_\lambda(a,b,c) := \sum_{k=0}^{n}(-abc)^k \sum_{i_1<\cdots<i_k} \delta_{i_1}\cdots\delta_{i_k} F_{\Gamma(\lambda)}(a,b,c),$$

where $\Gamma(\lambda) = (c_1, c_2, \ldots)$ is the sequence of the multiplicities of parts in the dual of λ, or c_j is the number of columns of length j in λ.

Now we can state our first generalization of (2), which gives in fact a positive answer to a conjecture of Ishikawa and Wakayama [6].

Theorem 1 *We have*

$$\sum_\lambda f_\lambda(a,b,c)s_\lambda(X) = \Phi(X;a,b)\prod_i(1-cx_i)^{-1}.$$

On the other hand, Macdonald [7, p. 83-84], Désarménien-Stembridge [4, 9] and Okada [8] have given *bounded versions* of identities (3)-(5), respectively, as follows :

Theorem 2 (Macdonald) *For non negative integers m and n,*

$$\sum_{\lambda_1\leq m} s_\lambda(X) = \frac{\det\left(x_i^{j-1} - x_i^{m+2n-j}\right)}{\prod_{i=1}^{n}(1-x_i)\prod_{i<j}(x_i-x_j)(x_ix_j-1)}.$$

Theorem 3 (Désarménien-Stembridge) *For non negative integers m and n,*

$$\sum_{\substack{\lambda_1\leq 2m\\ \lambda\ even}} s_\lambda(X) = \frac{\det\left(x_i^{j-1} - x_i^{2m+2n+1-j}\right)}{\prod_{i=1}^{n}(1-x_i^2)\prod_{i<j}(x_ix_j-1)(x_i-x_j)}.$$

Remark. This result follows immediately from Macdonald's formula. Indeed Pieri's formula implies:

$$\sum_{k=0}^{n} e_k(X) \sum_{\substack{\lambda_1\leq 2m\\ \lambda \text{even}}} s_\lambda(X) = \sum_{\lambda_1\leq 2m+1} s_\lambda(X).$$

Since $\sum_{k=0}^{n} e_k(X) = \prod_{i=1}^{n}(1+x_i)$, we get immediately theorem 3 by applying Macdonald's formula.

Theorem 4 (Okada) *For non negative integers m and n which is even,*

$$\sum_{\substack{\lambda_1 \le m \\ \lambda' \ even}} s_\lambda(X) = \frac{1}{2} \frac{\det\left(x_i^{j-1} - x_i^{m+2n-1-j}\right) + \det\left(x_i^{j-1} + x_i^{m+2n-1-j}\right)}{\prod_{i<j}(x_i x_j - 1)(x_i - x_j)}.$$

After giving elementary proofs of (2) and of the last three identities [1, 2], Bressoud [2] raised the problem of finding an extension of (2) for bounded partitions. Our second generalization of (2) will give an answer to Bressoud's problem [2].

For any sequence $\xi \in \{\pm 1\}^n$, we denote by $|\xi|_{-1}$ the number of -1's in the sequence ξ, set $X^\xi = \{x_1^{\xi_1}, \ldots, x_n^{\xi_n}\}$ and

$$D(\xi, z) = 1 - z \prod_i x_i^{(\xi_i - 1)/2}.$$

Theorem 5 *For non negative integers m and n,*

$$\sum_{\lambda \subseteq (m^n)} f_\lambda(a,b) s_\lambda(X) = \sum_{\xi \in \{\pm 1\}^n} \beta(\xi, a, b) \Phi(X^\xi; a, b) \prod_i x_i^{m(1-\xi_i)/2}$$

where the coefficient $\beta(\xi, a, b)$ is equal to

$$\begin{cases} \left(\dfrac{a^{m+1}}{D(\xi, 1/a)} - \dfrac{b^{m+1}}{D(\xi, 1/b)}\right) \dfrac{D(\xi, a) D(\xi, b)}{a - b} & \text{if } |\xi|_{-1} \text{ odd,} \\[2ex] \left(\dfrac{1}{D(\xi, 1)} - \dfrac{(ab)^{m+1}}{D(\xi, 1/ab)}\right) \dfrac{D(\xi, 1) D(\xi, ab)}{1 - ab} & \text{if } |\xi|_{-1} \text{ even.} \end{cases}$$

Remark. Assume that $|a| < 1$, $|b| < 1$ and $|x_i| < 1$ $(1 \le i \le n)$ and let $m \to \infty$, then all the summands tend to 0 except the one corresponding to $|\xi|_{-1} = 0$, which tends to $\Phi(X; a, b)$. Therefore theorem 5 reduces to (2) when $m \to \infty$.

Note also that the special $b = 0$ case of theorem 5 was proved by Goulden [5].

We shall give the proof of theorem 1 in section 2 and that of theorem 5 in section 3 using Macdonald's approach [7]. Finally, in section 4, we will show that when $(a, b) = (1, 0)$, $(1, -1)$ and $(0, 0)$, theorem 5 reduces actually to the above results of Macdonald, Désarménien-Stembridge and Okada respectively.

2 Proof of theorem 1

Let $\mathcal{P}$ be the set of partitions of length $\leq n$. Given a partition $\lambda \in \mathcal{P}$, we denote by $H(\lambda)$ the set of partitions $\mu \in \mathcal{P}$ such that λ/μ is a horizontal strip. As noticed at the end of [6], identity (2) can be derived from Littlewood's formula (1) and the so-called Pieri formula (see [7]):

$$s_\mu(X)\, h_k(X) = \sum_{\substack{\lambda:\mu\in H(\lambda)\\ |\lambda/\mu|=k}} s_\lambda(X). \tag{6}$$

In the same vain, we shall derive theorem 1 from (2) and (6). We first review such a proof for (2). By virtue of (1) and (6), identity (2) is equivalent to the following:

$$f_\lambda(a,b) = \sum_{\mu\in H(\lambda)} b^{|\lambda/\mu|} a^{c(\mu)}. \tag{7}$$

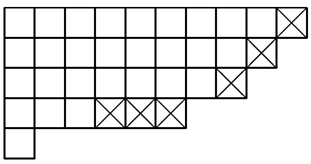
Let $A_j(\lambda)$ be the subdiagram of λ consisting of c_j columns of length j for $j \geq 1$. Thus choosing a partition μ in $H(\lambda)$ is equivalent to choose r left-most (resp. the remaining $c_j - r$) columns of length j (resp. $j-1$) for μ within each block $A_j(\mu)$. Clearly the corresponding weight is

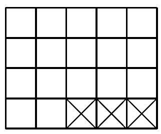
$$\begin{cases} \sum_{r=0}^{c_j} a^{c_j-r} b^r = \frac{a^{c_j+1}-b^{c_j+1}}{a-b} & \text{if } j \text{ is odd,} \\ \sum_{r=0}^{c_j} (ab)^r = \frac{1-(ab)^{c_j+1}}{1-ab} & \text{if } j \text{ is even.} \end{cases}$$

Multiplying the weights on all $j \geq 1$ yields (7).

Each pair (λ, μ) with $\mu \in H(\lambda)$ can be visualized by putting a cross $(\times)$ in each cell of λ/μ.

Example. For $\lambda = (10, 9, 8, 6, 1)$ and $\mu = (9, 8, 7, 3, 1) \in H(\lambda)$, their Ferrers diagrams and the block $A_4(\lambda)$ are represented as follows :

$\lambda =$ 

$A_4(\lambda) =$ 

Similarly, by (2) and (6), we see that theorem 5 is equivalent to the following:

$$f_\lambda(a,b,c) = \sum_{(\mu,\,\nu)\in C(\lambda)} a^{c(\nu)} b^{|\mu/\nu|} c^{|\lambda/\mu|}, \tag{8}$$

where $C(\lambda) = \{(\mu, \nu) \mid \mu \in H(\lambda), \nu \in H(\mu)\}$.

We shall compute the right-hand side of (8) using a sieve method. To this end we shall first enumerate a larger class of patterns whose generating function is equal to $F_{\Gamma(\lambda)}(a, b, c)$.

Recall that we identify a partition λ with its Ferrers diagram. We will say that a subset S of $\mathbb{N}^2$ is a *partition diagram* if $\{(x-k, y) | (x, y) \in S\}$ is a Ferrers diagram for some integer $k \geq 0$. Let $H'(\lambda)$ be the set of all subsets μ of λ such that $\mu \cap A_j(\lambda)$ is a partition diagram for all $j \geq 1$ and λ/μ is a horizontal strip. Define

$$B(\lambda) = \{(\mu, \nu) \mid \mu \in H(\lambda), \nu \in H'(\mu)\}.$$

Note that in the above definition, the subdiagram ν of λ is not necessarily a partition diagram. In this regard, the set $C(\lambda)$ can be described as follows:

$$C(\lambda) = \{(\mu, \nu) \in B(\lambda) \mid \nu \in H(\mu)\}.$$

Given $\nu \in H'(\mu)$, the jth row of ν is called *compatible* if

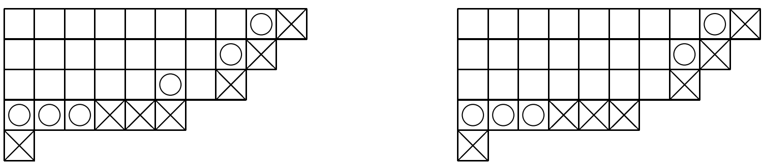
$$\forall x \geq 1, \quad (x+1; j) \in \nu \Longrightarrow (x; j) \in \nu.$$

For $p \geq 0$ let $B_p(\lambda)$ be the set of $(\mu, \nu) \in B(\lambda)$ such that ν has at least p non compatible rows. Clearly $B_0(\lambda) = B(\lambda)$ and $B(\lambda) \setminus C(\lambda) = B_1(\lambda)$, in other words, a pair $(\mu, \nu) \in B(\lambda)$ is an element of $C(\lambda)$ iff all the rows of ν are compatible. By the principle of inclusion-exclusion we obtain

$$\sum_{(\mu,\nu)\in C(\lambda)} a^{c(\nu)} b^{|\mu/\nu|} c^{|\lambda/\mu|} = \sum_{p=0}^{l(\lambda)} (-1)^p \sum_{(\mu,\nu)\in B_p(\lambda)} a^{c(\nu)} b^{|\mu/\nu|} c^{|\lambda/\mu|}. \qquad (9)$$

Each triple (λ, μ, ν) with $(\mu, \nu) \in B(\lambda)$ can be visualized by putting a circle $\circ$ (resp. cross $\times$) in each cell of μ/ν (resp. λ/μ).

Example. The following diagrams represent two triples (λ, μ, ν) :



(a) (b)

Clearly $\lambda = (10, 9, 8, 6, 1)$ and $\mu = (9, 8, 7, 3)$. In (a), the pair (μ, ν) is in $B_1(\lambda)$ because the third row of ν is not compatible, so $\nu \in H'(\mu) \setminus H(\mu)$ and ν is not a partition. In (b), the pair (μ, ν) is in $C(\lambda)$ because all the rows of ν are compatible, so $\nu = (8, 7, 7)$ is a partition in $H(\mu)$.

Lemma 1 *We have*

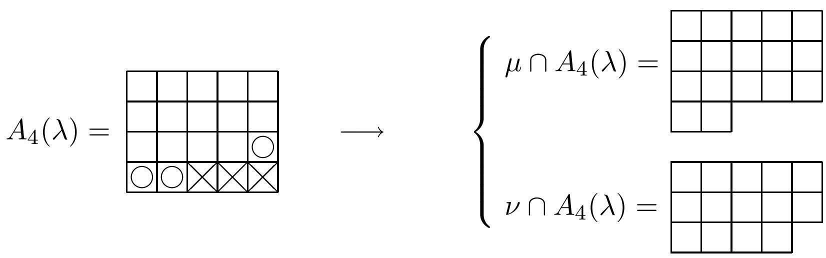
$$\sum_{(\mu,\nu)\in B(\lambda)} a^{c(\nu)} b^{|\mu/\nu|} c^{|\lambda/\mu|} = F_{\Gamma(\lambda)}(a,b,c).$$

Proof. As in the proof of (7), we divide the diagram λ into rectangular blocks $A_j(\lambda)$, $j \geq 1$, and compute the weight within each block $A_j(\lambda)$. Clearly choosing a pair (μ,ν) in $B(\lambda)$ is equivalent to, for each $j \geq 1$, first choose the p left-most (resp. the remaining $q = c_j - p$) columns of length j (resp. $j-1$) for μ in $A_j(\lambda)$, and then choose s (resp. $p-s$) left-most columns of length j (resp. $j-1$) for ν among the p columns of μ, also choose r (resp. the remaining $q-r$) left-most columns of length $j-1$ (resp. $j-2$) for ν. Thus the corresponding weight is $h_{c_1}(a,b,c)$ if $j=1$ and, for each $j \geq 2$,

$$\begin{cases} \displaystyle\sum_{p+q=c_j} c^q \left(\sum_{s=0}^{p} (ab)^s\right)\left(\sum_{r=0}^{q} a^r b^{q-r}\right) = P_{c_j}(a,b,c) & \text{if } j \text{ even;} \\ \displaystyle\sum_{p+q=c_j} c^q \left(\sum_{s=0}^{p} b^s\right)\left(\sum_{r=0}^{q} (ab)^r\right) = Q_{c_j}(a,b,c) & \text{if } j \text{ odd.} \end{cases}$$

Multiplying up over all $j \geq 1$ we get the desired formula. □

Example. Consider the (a) case of the previous example. The subdiagrams corresponding to the block $A_4(\lambda)$ are the following :



Note that $c_j = 5$, $p = 2$, $s = 0$ and $r = 2$.

For any set of integers $J = \{j_1, j_2, \ldots, j_p\}$ $(p \geq 1)$ let $B_J(\lambda)$ denote the set of all the pairs $(\mu,\nu) \in B(\lambda)$ such that the jth row of ν is not compatible for $j \in J$. Hence $B_J(\lambda) \in B_p(\lambda)$.

Lemma 2 *There holds*

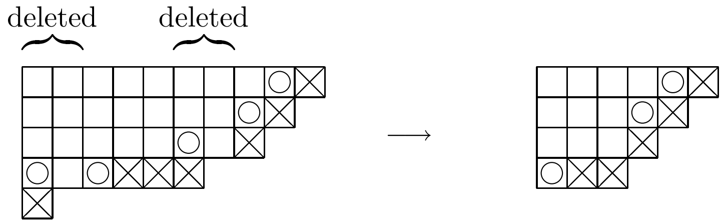
$$\sum_{(\mu,\nu)\in B_J(\lambda)} a^{c(\nu)} b^{|\mu/\nu|} c^{|\lambda/\mu|} = (abc)^p \delta_{j_1} \ldots \delta_{j_p} F_{\Gamma(\lambda)}(a,b,c).$$

Proof. Recall that $\lambda' = (1^{c_1}2^{c_2}\ldots)$. Suppose there exists a pair (μ,ν) in $B_J(\lambda)$, then there should be an integer $x_j \in \mathbb{N}$ such that $(x_j+1, j) \in \nu$ and $(x_j, j) \in \mu/\nu$ for any $j \in J$. In view of the definition of $B_J(\lambda)$ we must have $x_j = c_{l(\lambda)} + \cdots + c_{j+1}$ and $(x_j, j+1) \in \lambda/\mu$, for λ/μ is a horizontal strip. It follows that $c_j \geq 1$ and $c_{j+1} \geq 1$. Furthermore, if $j+1$ is also in J, we must have $c_{j+1} \geq 2$. Summarizing, we have the following equivalence:

$$B_J(\lambda) \neq \emptyset \Longleftrightarrow c_j c_{j+1} \neq 0 \ \forall j \in J \text{ and } \ c_{j+1} \geq 2 \text{ if } j, j+1 \in J.$$

It is easy to see that the last condition is equivalent to $\delta_{j_1}\cdots\delta_{j_p}\Gamma(\lambda) \in \mathbb{N}^\infty$ or $\delta_{j_1}\cdots\delta_{j_p}F_{\Gamma(\lambda)}(a,b,c) \neq 0$.

In what follows we shall assume that $B_J(\lambda) \neq \emptyset$. Thus we can define a unique partition $\delta_J(\lambda)$ such that $\Gamma(\delta_J(\lambda)) = \delta_{j_1}\ldots\delta_{j_p}\Gamma(\lambda)$. Graphically, the diagram $\delta_J(\lambda)$ can be obtained by deleting, successively for $j \in J$, the x_jth and (x_j+1)th columns and shift all the cells on the right of x_jth column of λ to left by two units. For $(\mu,\nu) \in B_J(\lambda)$, if we apply the same graphical operation to the μ and ν, we get a pair $(\delta_J(\mu), \delta_J(\nu)) \in B(\delta_J(\lambda))$.
For example, in the previous example, if $J = \{3,4\}$, then $\delta_J(\lambda) = (6,5,4,3)$. The corresponding triples (λ,μ,ν) with $(\mu,\nu) \in B(\lambda)$ and $(\delta_J(\lambda), \delta_J(\mu), \delta_J(\nu))$ with $(\delta_J(\mu), \delta_J(\nu)) \in B(\delta_J(\lambda))$ are illustrated as follows :



Since the weight corresponding to the deleted x_jth and x_{j+1}th columns of λ, μ and ν is abc for each $j \in J$, we have

$$a^{c(\nu)}b^{|\lambda/\nu|}c^{|\mu/\lambda|} = (abc)^p a^{c(\delta_J(\nu))}b^{|\delta_J(\lambda)/\delta_J(\nu)|}c^{|\delta_J(\mu)/\delta_J(\lambda)|}.$$

Therefore

$$\sum_{(\lambda,\nu)\in B_J(\lambda)} a^{c(\nu)}b^{|\lambda/\nu|}c^{|\mu/\lambda|} = (abc)^p \sum_{(\lambda,\nu)\in B(\delta_J(\lambda))} a^{c(\nu)}b^{|\lambda/\nu|}c^{|\mu/\lambda|}.$$

The lemma follows then immediately from lemma 1. □

It follows from lemma 2 that for $p \geq 1$

$$\sum_{(\mu,\nu)\in B_p(\lambda)} a^{c(\nu)}b^{|\mu/\nu|}c^{|\lambda/\mu|} = (abc)^p \sum_{1\leq j_1<\cdots<j_p\leq l(\lambda)} \delta_{j_1}\ldots\delta_{j_p}F_{\Gamma(\lambda)}(a,b,c).$$

Combining with (9) and lemma 1 we derive immediately theorem 1.

Remark. Similarly, using another identity of Littlewood [7]:

$$\sum_{\lambda} a^{r(\lambda)} s_\lambda(X) = \prod_i \frac{1+ax_i}{1-x_i^2} \prod_{j<k} (1-x_j x_k)^{-1}, \tag{10}$$

where $r(\lambda)$ is the number of rows of odd length of λ, we obtain:

$$\sum_{\lambda} f_{\lambda'}(a,b)\, s_\lambda(X) = \prod_i \frac{(1+ax_i)(1+bx_i)}{1-x_i^2} \prod_{j<k} (1-x_j x_k)^{-1}$$

and

$$\sum_{\lambda} f_{\lambda'}(a,b,c)\, s_\lambda(X) = \prod_i \frac{(1+ax_i)(1+bx_i)(1+cx_i)}{1-x_i^2} \prod_{j<k} (1-x_j x_k)^{-1}.$$

Note that $c_j(\lambda') = m_j(\lambda)$ is the multiplicity of j in λ.

3 Proof of theorem 5

Consider the generating function

$$S(u) = \sum_{\lambda_0, \lambda} f_\lambda(a,b)\, s_\lambda(X)\, u^{\lambda_0}$$

where the sum is over all $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$. Suppose λ is of form $\mu_1^{r_1}, \mu_2^{r_2}, \ldots \mu_k^{r_k}$, where $\mu_1 > \mu_2 > \cdots > \mu_k \geq 0$ and the r_i are positive integers whose sum is n. Let $S_n^\lambda = S_{r_1} \times \cdots S_{r_k}$ be the group of permutations leaving λ invariant. Then

$$\begin{aligned} s_\lambda(X) &= \sum_{w \in S_n} w\left(x_1^{\lambda_1} \ldots x_n^{\lambda_n} \prod_{i<j} \frac{x_i}{x_i - x_j} \right) \\ &= \sum_{w \in S_n / S_n^\lambda} w\left(x_1^{\lambda_1} \ldots x_n^{\lambda_n} \prod_{\lambda_i > \lambda_j} \frac{x_i}{x_i - x_j} \right), \end{aligned}$$

where the permutation w acts on the indices of the indeterminates. Each $w \in S_n/S_n^\lambda$ corresponds to a surjective mapping $f : X \longrightarrow \{1, 2, \ldots, k\}$ such that $|f^{-1}(i)| = r_i$. For any subset Y of X, let $p(Y)$ denote the product of the elements of Y. (In particular, $p(\emptyset) = 1$.) We can rewrite Schur functions as follows:

$$s_\lambda(X) = \sum_f p(f^{-1}(1))^{\mu_1} \cdots p(f^{-1}(k))^{\mu_k} \prod_{f(x_i)<f(x_j)} \frac{x_i}{x_i - x_j},$$

summed over all surjective mappings $f : X \longrightarrow \{1, 2, \ldots, k\}$ such that $|f^{-1}(i)| = r_i$. Furthermore, each such f determines a *filtration* of X:

$$\mathcal{F}: \quad \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = X,$$

according to the rule $x_i \in F_l \iff f(x_i) \leq l$ for $1 \leq l \leq k$. Conversely, such a filtration $\mathcal{F} = (F_0, F_1, \ldots, F_k)$ determines a surjection $f : X \longrightarrow \{1, 2, \ldots, k\}$ uniquely. Thus we can write:

$$s_\lambda(X) = \sum_{\mathcal{F}} \pi_{\mathcal{F}} \prod_{1 \leq i \leq k} p(F_i \setminus F_{i-1})^{\mu_i}, \tag{11}$$

summed over all the filtrations $\mathcal{F}$ such that $|F_i| = r_1 + r_2 + \cdots + r_i$ for $1 \leq i \leq k$, and

$$\pi_{\mathcal{F}} = \prod_{f(x_i)<f(x_j)} \frac{x_i}{x_i - x_j},$$

where f is the function defined by $\mathcal{F}$.

Now let $\nu_i = \mu_i - \mu_{i+1}$ if $1 \leq i \leq k-1$ and $\nu_k = \mu_k$, thus $\nu_i > 0$ if $i < k$ and $\nu_k \geq 0$. Since the lengths of columns of λ are $|F_j| = r_1 + \cdots + r_j$ with multiplicities ν_j for $1 \leq j \leq k$, we have

$$f_\lambda(a, b) = \prod_{|F_j|\,\text{odd}} \frac{a^{\nu_j+1} - b^{\nu_j+1}}{a - b} \prod_{|F_j|\,\text{even}} \frac{1 - (ab)^{\nu_j+1}}{1 - ab}. \tag{12}$$

Furthermore, let $\mu_0 = \lambda_0$ and $\nu_0 = \mu_0 - \mu_1$ in the definition of $S(u)$, so that $\nu_0 \geq 0$ and $\mu_0 = \nu_0 + \nu_1 + \cdots + \nu_k$. It follows from (11) and (12) that :

$$\begin{aligned} S(u) \quad = \quad & \sum_{\mathcal{F}} \pi_{\mathcal{F}} \sum_{\nu} u^{\nu_0} \prod_{|F_j|\,\text{odd}} \frac{a^{v_j+1} - b^{v_j+1}}{a - b} u^{v_j} p(F_j)^{v_j} \\ & \times \prod_{|F_j|\,\text{even}} \frac{1 - (ab)^{v_j+1}}{1 - ab} u^{v_j} p(F_j)^{v_j}, \end{aligned} \tag{13}$$

where the outer sum is over all filtrations $\mathcal{F}$ of X and the inner sum is over all integers $\nu_0, \nu_1, \ldots, \nu_k$ such that $\nu_0 \geq 0$, $\nu_k \geq 0$ and $\nu_i > 0$ for $1 \leq i \leq k-1$.

For any filtration $\mathcal{F}$ of X set

$$\begin{aligned}\mathcal{A}_{\mathcal{F}}(X,u) &= \prod_{|F_j|\,\mathrm{odd}} \left[\frac{a(a-b)^{-1}}{1-ap(F_j)u} - \frac{b(a-b)^{-1}}{1-bp(F_j)u} - \chi(F_j \neq X)\right] \\ &\quad \times \prod_{|F_j|\,\mathrm{even}\geq 1} \left[\frac{(1-ab)^{-1}}{1-p(F_j)u} - \frac{ab(1-ab)^{-1}}{1-abp(F_j)u} - \chi(F_j \neq X)\right],\end{aligned}$$

where $\chi(A)=1$ if A is true, and $\chi(A)=0$ if A is false. Then the inner sum of (13) is

$$(1-u)^{-1}\mathcal{A}_{\mathcal{F}}(X,u),$$

therefore

$$S(u) = (1-u)^{-1}\sum_{\mathcal{F}} \pi_{\mathcal{F}}\mathcal{A}_{\mathcal{F}}(X,u),$$

where the sum is over all the filtrations of X as before.

The above formula shows that $S(u)$ is a rational function of u whose denominator is the product of the form $1-p(Y)u$, $1-ap(Y)u$, $1-bp(Y)u$ or $1-abp(Y)u$, where $Y \subseteq X$. Therefore we have the following result.

Lemma 3 *The generating function $S(u)$ is of the form:*

$$\begin{aligned}S(u) &= \frac{c(\emptyset)}{1-u} + \sum_{\substack{Y\subseteq X\\ |Y|\ odd}} \left(\frac{a(Y)}{1-ap(Y)u} - \frac{b(Y)}{1-bp(Y)u}\right) \\ &\quad + \sum_{\substack{Y\subseteq X\\ |Y|\ even>0}} \left(\frac{c(Y)}{1-p(Y)u} - \frac{d(Y)}{1-abp(Y)u}\right).\end{aligned}$$

It remains to compute the residues. Let us start with $c(\emptyset)$. Writing $\lambda_0 = \lambda_1 + k$ with $k \geq 0$, we see that

$$\begin{aligned}S(u) &= \sum_{k\geq 0} u^k \sum_{\lambda} f_\lambda(a,b)\, s_\lambda(X) u^{\lambda_1} \\ &= (1-u)^{-1}\sum_{\lambda} f_\lambda(a,b)\, s_\lambda(X) u^{\lambda_1},\end{aligned}$$

it follows from (2) that

$$c(\emptyset) = (S(u)(1-u))\,|_{u=1} = \Phi(X;a,b).$$

For computations of the other residues, we introduce some more notations. For any $Y \subseteq X$, let $Y' = X \setminus Y$ and $-Y = \{x_i^{-1} : x_i \in Y\}$. For any subset Z of X or $-X$ let

$$\alpha(Z,u) = \begin{cases} (1-ap(Z)u)(1-bp(Z)u) & \text{if } |Z| \text{ odd;} \\ (1-p(Z)u)(1-abp(Z)u) & \text{if } |Z| \text{ even.} \end{cases}$$

As the computations of other residues are similar, we just give the details for $c(Y)$. Let $Y \subseteq X$ such that $|Y|$ is even. Then we have

$$c(Y) = \left[(1-u)^{-1} \sum_{\mathcal{F}} \pi_{\mathcal{F}} \mathcal{A}_{\mathcal{F}}(X;u)(1-p(Y)u) \right]_{u=p(-Y)}. \tag{14}$$

If $Y \notin \mathcal{F}$, the corresponding summand is equal to 0. Thus we need only to consider the following filtrations $\mathcal{F}$:

$$\emptyset = F_0 \subsetneq \cdots \subsetneq F_t = Y \subsetneq \cdots \subsetneq F_k = X \qquad 1 \leq t \leq k.$$

We may then split $\mathcal{F}$ into two filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$, of $-Y$ and $Y' = X \setminus Y$ respectively, as follows :

$$\begin{aligned} \mathcal{F}_1 &: \emptyset \subsetneq -(Y \setminus F_{t-1}) \subsetneq \cdots \subsetneq -(Y \setminus F_1) \subsetneq -Y, \\ \mathcal{F}_2 &: \emptyset \subsetneq F_{t+1} \setminus Y \subsetneq \cdots \subsetneq F_{k-1} \setminus Y \subsetneq Y'. \end{aligned}$$

Then, writing $v = p(Y)u$, we have

$$\begin{aligned} (1-u)^{-1}\mathcal{A}_{\mathcal{F}}(X;u)(1-p(Y)u) &= (1-p(-Y)v)^{-1}\mathcal{A}_{\mathcal{F}_1}(-Y;v)\mathcal{A}_{\mathcal{F}_2}(Y';v) \\ &\quad \times\alpha(-Y,v)\left[(1-ab)^{-1} - \beta(v)(1-v)\right], \end{aligned}$$

where $\beta(v) = ab/(1-abv)(1-ab) + \chi(Y \neq X)$, and

$$\pi_{\mathcal{F}}(X) = \pi_{\mathcal{F}_1}(-Y)\pi_{\mathcal{F}_2}(Y') \prod_{x_i \in Y, x_j \in Y'} (1 - x_i^{-1}x_j)^{-1},$$

As $u = p(-Y)$ is equivalent to $v = 1$, it follows from (14) that

$$\begin{aligned} c(Y) &= (1-ab)^{-1}(1-p(-Y))^{-1}\alpha(-Y,1) \prod_{x_i \in Y, x_j \in Y'} (1 - x_i^{-1}x_j)^{-1} \\ &\quad \times \left[\sum_{\mathcal{F}_1} \pi_{\mathcal{F}_1}(-Y)\mathcal{A}_{\mathcal{F}_1}(-Y;v)\right]_{v=1} \times \left[\sum_{\mathcal{F}_2} \pi_{\mathcal{F}_2}(Y')\mathcal{A}_{\mathcal{F}_2}(Y';v)\right]_{v=1}. \end{aligned}$$

Using the result of $c(\emptyset)$, which can be written:

$$\Phi(X;a,b) = \sum_{\mathcal{F}} \left(\pi_{\mathcal{F}}(X)\mathcal{A}_{\mathcal{F}}(X;u)\right)_{u=1},$$

we obtain:

$$c(Y) = \frac{\alpha(-Y,1)\Phi(-Y;a,b)\Phi(Y';a,b)}{(1-ab)(1-p(-Y))} \prod_{x_i\in Y, x_j\in Y'} (1-x_i^{-1}x_j)^{-1}.$$

Each subset Y of X can be encoded by a sequence $\xi \in \{\pm 1\}^n$ according to the rule : $\xi_i = 1$ if $x_i \notin Y$ and $\xi_i = -1$ if $x_i \in Y$. Hence

$$c(Y) = \frac{\Phi(x_1^{\xi_1},\ldots,x_n^{\xi_n};a,b)}{(1-ab)(1-p(-Y))}\alpha(-Y,1),$$

Note also that

$$p(Y) = \prod_i x_i^{(1-\xi_i)/2}, \qquad p(-Y) = \prod_i x_i^{(\xi_i-1)/2}.$$

In the same way, we find for any even size subset $Y \subseteq X$ that

$$d(Y) = \frac{ab\Phi(x_1^{\xi_1},\ldots,x_n^{\xi_n};a,b)}{(1-ab)(1-(ab)^{-1}p(-Y))}\alpha(-Y,1),$$

and for any odd size subset $Y \subseteq X$ that

$$\begin{aligned} a(Y) &= \frac{a\Phi(x_1^{\xi_1},\ldots,x_n^{\xi_n};a,b)}{(a-b)(1-a^{-1}p(-Y))}\alpha(-Y,1), \\ b(Y) &= \frac{b\Phi(x_1^{\xi_1},\ldots,x_n^{\xi_n};a,b)}{(a-b)(1-b^{-1}p(-Y))}\alpha(-Y,1). \end{aligned}$$

By virtue of lemma 1, extracting the coefficient of u^m in $S(u)$ yields

$$\begin{aligned} \sum_{\lambda\subseteq(m^n)} f_\lambda(a,b)s_\lambda(X) = \Phi(X;a,b) \; &+ \sum_{\substack{Y\subseteq X \\ |Y|\,\mathrm{odd}}} \left[a(Y)a^m - b(Y)b^m\right] p(Y)^m \\ &+ \sum_{\substack{Y\subseteq X \\ |Y|\,\mathrm{even}>0}} \left[c(Y) - d(Y)(ab)^m\right] p(Y)^m. \end{aligned}$$

Finally, substituting the values of $a(Y), b(Y), c(Y)$ and $d(Y)$ in the above formula we obtain theorem 5.

4 Three special cases

First we note that $f_\lambda(1,0)=1$,

$$f_\lambda(1,-1)=\begin{cases}0 & \text{if any } c_j \text{ is odd,}\\ 1 & \text{otherwise;}\end{cases}$$

and

$$f_\lambda(0,0)=\begin{cases}0 & \text{if any } c_j \text{ is positive for any odd } j,\\ 1 & \text{otherwise.}\end{cases}$$

On the other hand, we have

$$\beta(\xi,1,0)=1,$$

$$\beta(\xi,1,-1)=\begin{cases}1 & \text{if } m \text{ even,}\\ \prod_i x_i^{(\xi_i-1)/2} & \text{if } m \text{ odd;}\end{cases}$$

and

$$\beta(\xi,0,0)=\begin{cases}0 & \text{if } |\xi|_{-1} \text{ is odd,}\\ 1 & \text{otherwise.}\end{cases}$$

So we derive immediately from theorem 5 the following result.

Corollary 1 *The sums of Schur functions of shape in a given rectangle are:*

$$\sum_{\lambda\subseteq(m)^n} s_\lambda(X) = \sum_{\xi\in\{\pm1\}^n} \Phi(X^\xi;1,0)\prod_i x_i^{m(1-\xi_i)/2}, \tag{15}$$

$$\sum_{\substack{\lambda\subseteq(2m)^n\\ \lambda\ even}} s_\lambda(X) = \sum_{\xi\in\{\pm1\}^n} \Phi(X^\xi;1,-1)\prod_i x_i^{m(1-\xi_i)}, \tag{16}$$

$$\sum_{\substack{\lambda\subseteq(m)^n\\ \lambda'\ even}} s_\lambda(X) = \sum_{\substack{\xi\in\{\pm1\}^n\\ |\xi|_{-1}\ even}} \Phi(X^\xi;0,0)\prod_i x_i^{m(1-\xi_i)/2}, \tag{17}$$

where n is even in the last identity.

To see that the above corollary is equivalent to theorems 2, 3 and 4, we need only to appeal to Vandermonde's determinantal formula :

$$\sum_{\sigma\in S_n}\epsilon(\sigma)\prod_{i=1}^n x_{\sigma(i)}^{i-1}=\det(x_j^{i-1})=\prod_{1\le i<j\le n}(x_j-x_i). \tag{18}$$

Notice that for $\xi\in\{\pm1\}^n$ and $1\le i<j\le n$,

$$(x_i^{\xi_i}-x_j^{\xi_j})(x_i^{\xi_i}x_j^{\xi_j}-1)=(x_i-x_j)(x_ix_j-1)x_i^{\xi_i-1}x_j^{\xi_j-1},$$

therefore

$$\prod_{i<j}(x_i^{\xi_i}-x_j^{\xi_j})(x_i^{\xi_i}x_j^{\xi_j}-1)=\prod_{i<j}(x_i-x_j)(x_ix_j-1)\prod_i x_i^{(n-1)(\xi_i-1)}. \tag{19}$$

Note also that

$$\prod_i(1-x_i^{\xi_j})=(-1)^{|\xi|-1}\prod_{i<j}(1-x_i)\prod_i x_i^{(\xi_i-1)/2}. \tag{20}$$

The $(a,b)=(1,0)$ **case** : Set

$$\Delta_B=\frac{\prod_{i<j}(x_j-x_i)}{\Phi(X;1,0)}=\prod_i(1-x_i)\prod_{i<j}(x_i-x_j)(x_ix_j-1).$$

It follows from (18), (19) and (20) that

$$\Phi(X^\xi;1,0)=\frac{(-1)^{|\xi|-1}}{\Delta_B}\prod_i x_i^{(1-\xi_i)(n-1/2)}\sum_{\sigma\in S_n}\epsilon(\sigma)\prod_i x_{\sigma(i)}^{\xi_{\sigma(i)}(i-1)}.$$

So the right side of (15) is

$$\begin{aligned}
&\frac{1}{\Delta_B}\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{\xi\in\{\pm1\}^n}(-1)^{|\xi|-1}\prod_i x_{\sigma(i)}^{(m+2n-1)(1-\xi_{\sigma(i)})/2+\xi_{\sigma(i)}(i-1)}\\
&=\frac{1}{\Delta_B}\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{\xi\in\{\pm1\}^n}\prod_{\xi_{\sigma(i)}=1}x_{\sigma(i)}^{i-1}\prod_{\xi_{\sigma(i)}=-1}\left(-x_{\sigma(i)}^{m+2n-i}\right)\\
&=\frac{1}{\Delta_B}\det\left(x_i^{j-1}-x_i^{m+2n-j}\right).
\end{aligned}$$

Hence theorem 2 is equivalent to (15).

The $(a,b)=(1,-1)$ **case** : Set

$$\Delta_C=\frac{\prod_{i<j}(x_j-x_i)}{\Phi(X;1,-1)}=\prod_i(1-x_i^2)\prod_{i<j}(x_i-x_j)(x_ix_j-1).$$

It follows from (18), (19) and (20) that

$$\Phi(X^\xi;1,-1)=\frac{(-1)^{|\xi|-1}}{\Delta_C}\prod_i x_i^{n(1-\xi_i)}\sum_{\sigma\in S_n}\epsilon(\sigma)\prod_i x_{\sigma(i)}^{\xi_{\sigma(i)}(i-1)},$$

and the right hand side of (16) is

$$\begin{aligned}
&\frac{1}{\Delta_C}\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{\xi\in\{\pm1\}^n}(-1)^{|\xi|_{-1}}\prod_i x_{\sigma(i)}^{(n+m)(1-\xi_{\sigma(i)})+(i-1)\xi_{\sigma(i)}}\\
&=\frac{1}{\Delta_C}\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{\xi\in\{\pm1\}^n}\prod_{\xi_{\sigma(i)}=1}x_{\sigma(i)}^{i-1}\prod_{\xi_{\sigma(i)}=-1}\left(-x_{\sigma(i)}^{2n+2m-i+1}\right)\\
&=\frac{1}{\Delta_C}\det\left(x_i^{j-1}-x_i^{2m+2n+1-j}\right).
\end{aligned}$$

So Theorem 3 is equivalent to (16).

The $(a,b)=(0,0)$ **case** : Set

$$\Delta_D=\frac{\prod_{i<j}(x_j-x_i)}{\Phi(X;0,0)}=\prod_{i<j}(x_i-x_j)(x_ix_j-1).$$

It follows from (18), (19) and (20) that

$$\Phi(X^\xi;0,0)=\frac{1}{\Delta_D}\prod_i x_i^{(n-1)(1-\xi_i)}\sum_{\sigma\in S_n}\epsilon(\sigma)\prod_i x_{\sigma(i)}^{\xi_{\sigma(i)}(i-1)}$$

and the right side of (17) is

$$\begin{aligned}
&\frac{1}{\Delta_D}\sum_{\sigma\in S_n}\epsilon(\sigma)\sum_{\substack{\xi\in\{\pm1\}^n\\ |\xi|_{-1}\text{even}}}\prod_{\xi_{\sigma(i)}=1}x_{\sigma(i)}^{i-1}\prod_{\xi_{\sigma(i)}=-1}x_{\sigma(i)}^{2n+m-i-1}\\
&=\frac{1}{2\Delta_D}\left[\det\left(x_i^{j-1}-x_i^{m+2n-1-j}\right)+\det\left(x_i^{j-1}+x_i^{m+2n-1-j}\right)\right].
\end{aligned}$$

So theorem 4 is equivalent to (17).

When $m=0$, as the left sides of (15), (16) and (17) are equal to 1, we obtain the following result.

Corollary 2 *For any non negative integer n, we have*

$$\begin{aligned}
\det\left(x_i^{j-1}-x_i^{2n-j}\right) &= \prod_i(1-x_i)\prod_{i<j}(x_i-x_j)(x_ix_j-1),\\
\det\left(x_i^{j-1}-x_i^{2n-j+1}\right) &= \prod_i(1-x_i^2)\prod_{i<j}(x_i-x_j)(x_ix_j-1),\\
\det\left(x_i^{j-1}+x_i^{2n-1-j}\right) &= 2\prod_{i<j}(x_i-x_j)(x_ix_j-1).
\end{aligned}$$

These are actually Weyl's denominator formulas for root systems of type B_n, C_n and D_n ([3], p. 68-69) respectively.

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