New Identities of Hall-Littlewood Polynomials and Applications

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Abstract

Starting from Macdonald's summation formula of Hall-Littlewood polynomials over bounded partitions and its even partition analogue, Stembridge (1990, Trans. Amer. Math. Soc., **319**, no.2, 469-498) derived sixteen multiple q-identities of Rogers-Ramanujan type. Inspired by our recent results on Schur functions (2001, Adv. Appl. Math., **27**, 493-509) and based on computer experiments we obtain two further such summation formulae of Hall-Littlewood polynomials over bounded partitions and derive six new multiple q-identities of Rogers-Ramanujan type.

1 Introduction

The Rogers-Ramanujan identities (see [1, 3]) :

$$\sum_{n=0}^{\infty} \frac{q^{n^2+an}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{n=1 \\ n\equiv\pm(a+1) \pmod 5}}^{\infty} (1-q^n)^{-1},$$

where $a = 0$ or 1, are among the most famous q-series identities in partitions and combinatorics. Since their discovery the Rogers-Ramanujan identities have been proved and generalized in various ways (see [1, 3, 4, 12] and the references cited there). In [12], by adapting a method of Macdonald for calculating partial fraction expansions of symmetric formal power series, Stembridge gave an unusual proof of Rogers-Ramanujan identities as well as fourteen other non trivial q-series identities of Rogers-Ramanujan type and their multiple analogs. Although it is possible to describe his proof within the setting of q-series, two summation formulas of Hall-Littlewood polynomials were a crucial source of inspiration for such kind of identities. One of our original motivations was to look for new multiple q-identities of Rogers-Ramanujan type through this approach, but we think that the new summation formulae of Hall-Littlewood polynomials are interesting for their own.

Throughout this paper we will use the standard notations of q-series (see, for example, [5]). Set $(x)_0 := (x;q)_0 = 1$ and for $n \geq 1$

$$\begin{aligned} (x)_n &:= (x;q)_n = \prod_{k=1}^{n}(1-xq^{k-1}), \\ (x)_\infty &:= (x;q)_\infty = \prod_{k=1}^{\infty}(1-xq^{k-1}). \end{aligned}$$

For $n \geq 0$ and $r \geq 1$, set

$$(a_1, \cdots, a_r;\, q)_n = \prod_{i=1}^{r}(a_i)_n\ , \qquad (a_1, \cdots, a_r; q)_\infty = \prod_{i=1}^{r}(a_i)_\infty.$$

Let $n \geq 1$ be a fixed integer and S_n the group of permutations of the set $\{1, 2, \ldots, n\}$. Let $X = \{x_1, \ldots, x_n\}$ be a set of indeterminates and q a parameter. For each *partition* $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leq n$, if $m_i := m_i(\lambda)$ is the multiplicity of i in λ, then we also note λ by $(1^{m_1}\, 2^{m_2}\, \ldots)$. Recall that the Hall-Littlewood polynomials $P_\lambda(X, q)$ are defined by [9, p.208] :

$$P_\lambda(X,q) = \prod_{i\geq 1}\frac{(1-q)^{m_i}}{(q)_{m_i}}\sum_{w\in S_n} w\left(x_1^{\lambda_1}\ldots x_n^{\lambda_n}\prod_{i<j}\frac{x_i - qx_j}{x_i - x_j}\right),$$

where the factor is added to ensure the coefficient of $x_1^{\lambda_1}\ldots x_n^{\lambda_n}$ in P_λ is 1.

For a parameter α define the auxiliary function

$$\Psi_q(X;\alpha) := \prod_i (1-x_i)^{-1}(1-\alpha x_i)^{-1}\prod_{j<k}\frac{1-qx_jx_k}{1-x_jx_k}.$$

Then it is well-known [9, p. 230] that the sums of $P_\lambda(X, q)$ over all partitions and even partitions are given by the following formulae :

$$\sum_\lambda P_\lambda(X,q) = \Psi_q(X;0), \qquad (1)$$

$$\sum_\lambda P_{2\lambda}(X,q) = \Psi_q(X;-1). \qquad (2)$$

For any sequence $\xi \in \{\pm 1\}^n$ set $X^\xi = \{x_1^{\xi_1}, \cdots, x_n^{\xi_n}\}$ and denote by $|\xi|_{-1}$ the number of -1's in ξ. Then, by summing P_λ over partitions with bounded parts, Macdonald [9, p. 232] and Stembridge [12] have respectively generalized (1) and (2) as follows :

$$\sum_{\lambda_1 \le k} P_\lambda(X, q) = \sum_{\xi \in \{\pm 1\}^n} \Psi_q(X^\xi; 0) \prod_i x_i^{k(1-\xi_i)/2}, \tag{3}$$

$$\sum_{\substack{\lambda_1 \le 2k \\ \lambda \ even}} P_\lambda(X, q) = \sum_{\xi \in \{\pm 1\}^n} \Psi_q(X^\xi; -1) \prod_i x_i^{k(1-\xi_i)}. \tag{4}$$

Now, for parameters α, β define another auxiliary function

$$\Phi_q(X; \alpha, \beta) := \prod_i \frac{1 - \alpha x_i}{1 - \beta x_i} \prod_{j<k} \frac{1 - q x_j x_k}{1 - x_j x_k}.$$

Then the following summation formulae similar to (1) and (2) for Hall-Littlewood polynomials hold true [9, p.232] :

$$\sum_{\lambda' \text{ even}} c_\lambda(q)\, P_\lambda(X, q) = \Phi_q(X; 0, 0), \tag{5}$$

$$\sum_{\lambda} d_\lambda(q)\, P_\lambda(X, q) = \Phi_q(X; q, 1), \tag{6}$$

where λ' is the conjugate of λ and

$$c_\lambda(q) = \prod_{i \ge 1} (q; q^2)_{m_i(\lambda)/2}, \qquad d_\lambda(q) = \prod_{i \ge 1} \frac{(q)_{m_i(\lambda)}}{(q^2; q^2)_{[m_i(\lambda)/2]}}.$$

In view of the numerous applications of (3) and (4) it is natural to seek such extensions for (5) and (6). However, as remarked by Stembridge [12, p. 475], in these other cases there arise complications which render *doubtful* the existence of expansions as explicit as those of (3) and (4). We noticed that these complications arise if one wants to keep exactly the same coefficients $c_\lambda(q)$ and $d_\lambda(q)$ as in (5) and (6) for the sums over bounded partitions. Actually we have the following

Theorem 1 *For $k \ge 1$,*

$$\sum_{\substack{\lambda_1 \le k \\ \lambda' \ even}} c_{\lambda,k}(q) P_\lambda(X, q) = \sum_{\substack{\xi \in \{\pm 1\}^n \\ |\xi|_{-1} \ even}} \Phi_q(X^\xi; 0, 0) \prod_i x_i^{k(1-\xi_i)/2}, \tag{7}$$

$$\sum_{\lambda_1 \le k} d_{\lambda,k}(q) P_\lambda(X, q) = \sum_{\xi \in \{\pm 1\}^n} \Phi_q(X^\xi; q, 1) \prod_i x_i^{k(1-\xi_i)/2}, \tag{8}$$

where

$$c_{\lambda,k}(q)=\prod_{i=1}^{k-1}(q;q^2)_{m_i(\lambda)/2},\quad d_{\lambda,k}(q)=\prod_{i=1}^{k-1}\frac{(q)_{m_i(\lambda)}}{(q^2;q^2)_{[m_i(\lambda)/2]}}. \tag{9}$$

Remark. We were led to such extensions by starting from the right-hand side instead of the left-hand side and inspired by the similar formulae corresponding to the case $q=0$ of Hall-Littlewood polynomials [7], i.e., Schur functions. In the initial stage we made also the Maple tests using the package ACE [13]. In the case $q=0$, the right-hand sides of (3), (4), (7) and (8) can be written as quotients of determinants and the formulae reduce to the known identities of Schur functions [7].

For any partition λ it will be convenient to adopt the following notation :

$$(x)_\lambda := (x;\,q)_\lambda = (x)_{\lambda_1-\lambda_2}(x)_{\lambda_2-\lambda_3}\cdots,$$

and to introduce the general q-binomial coefficients

$$\begin{bmatrix} n\\ \lambda\end{bmatrix} := \frac{(q)_n}{(q)_{n-\lambda_1}(q)_\lambda},$$

with the convention that $\begin{bmatrix} n\\ \lambda\end{bmatrix}=0$ if $\lambda_1>n$. If $\lambda=(\lambda_1)$ we recover the classical q-binomial coefficient. Finally, for any partition λ we denote by $l(\lambda)$ the length of λ, i.e., the number of its positive parts, and $n(\lambda):=\sum_i\binom{\lambda_i}{2}$.

The following is the key q-identity which allows to produce identities of Rogers-Ramanujan type.

Theorem 2 *For $k\geq 1$,*

$$\sum_{l(\lambda)\leq k} z^{|\lambda|}q^{n(2\lambda)}\frac{(a,\,b;q^{-2})_{\lambda_1}}{(q^2;q^2)_\lambda(q;q^2)_{\lambda_k}} = \frac{(z;q^2)_\infty}{(abzq;q^2)_\infty} \tag{10}$$

$$\times\sum_{r\geq 0} z^{kr}q^{(k+1)\binom{2r}{2}}\frac{(a,b;q^{-2})_r(aq^{2r+1}z,bq^{2r+1}z;q^2)_\infty}{(q)_{2r}(zq^{2r-1})_\infty}(1-zq^{4r-1}).$$

Here is an outline of this paper. in section 2 we first derive from Theorem 2 six multiple analogs of Rogers-Ramanujan type identities. In section 3 we give the proof of Theorem 1 and some consequences, and defer the elementary proof, i.e., without using the Hall-Littlewood polynomials, of Theorem 2 and other multiple q-series identities to section 4. To prove theorems 1, 2 and 4 (see section 3.3) we apply the generating function technique and the computation of residues, but theorem 4 can also be derived from theorem 1. In section 5 we will show how to derive some of our q-identities, which imply the six multianalogs of Rogers-Ramanujan type identities, from Andrews formula [3, Thm. 3.4], which was proved using Bailey's method.

2 Multiple identities of Rogers-Ramanujan type

We need the *Jacobi triple product* identity [1, p.21] :

$$J(x,\,q) := 1+\sum_{r=1}^{\infty}(-1)^r x^r q^{\binom{r}{2}}(1+q^r/x^{2r}) = (q,\,x,\,q/x)_\infty. \qquad (11)$$

For any partition λ set $n_2(\lambda)=\sum_i \lambda_i^2$. We derive then from Theorem 2 the following identities of Rogers-Ramanujan type.

Theorem 3 *For $k\ge 1$,*

$$\sum_{l(\lambda)\le k}\frac{q^{2n_2(\lambda)}}{(q;q^2)_{\lambda_k}(q^2;q^2)_\lambda} = \prod_n (1-q^n)^{-1} \qquad (12)$$

where $n \equiv \pm(2k+1),\ \pm(2k+3), \pm 2, \pm 4,\ \dots, \pm 4k \pmod{8k+8}$*;*

$$\sum_{l(\lambda)\le k}\frac{q^{2n_2(\lambda)-2\lambda_1}}{(q;q^2)_{\lambda_k}(q^2;q^2)_\lambda}(1-q^{2\lambda_1}) = \frac{(q^{2k-1},\ q^{6k+9};\ q^{8k+8})_\infty}{\prod_n (1-q^n)} \qquad (13)$$

where $n \equiv \pm(2k+5), \pm 2,\ \dots, \pm 4k, \pm(4k+2) \pmod{8k+8}$*;*

$$\sum_{l(\lambda)\le k}\frac{q^{2n_2(\lambda)-\lambda_1^2}}{(q;q^2)_{\lambda_k}(q^2;q^2)_\lambda}(-q;q^2)_{\lambda_1}$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{4k+2},\ -q^{2k}, -q^{2k+2};\ q^{4k+2})_\infty; \qquad (14)$$

$$\sum_{l(\lambda)\le k}\frac{q^{2n_2(\lambda)-\lambda_1^2-\lambda_1}}{(q;q^2)_{\lambda_k}(q^2;q^2)_\lambda}(-1;q^2)_{\lambda_1}(1-q^{2\lambda_1})$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}(q^{4k+2},\ -q^{2k-1},\ -q^{2k+3};\ q^{4k+2})_\infty; \qquad (15)$$

$$\sum_{l(\lambda)\le k}\frac{q^{2n_2(\lambda)-2\lambda_1^2+\lambda_1}}{(q;q^2)_{\lambda_k}(q^2;q^2)_\lambda}(-1;q^2)_{\lambda_1}(-q;q^2)_{\lambda_1}$$
$$= \frac{(-q)_\infty}{(q)_\infty}(q^{4k}, -q^{2k}, -q^{2k}; q^{4k})_\infty; \qquad (16)$$

$$\sum_{l(\lambda)\le k}\frac{q^{2n_2(\lambda)-\lambda_1^2+\lambda_1}}{(q;q^2)_{\lambda_k}(q^2;q^2)_\lambda}(-1;q^2)_{\lambda_1}$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}(q^{4k+2},\ -q^{2k+1},\ -q^{2k+1};\ q^{4k+2})_\infty. \qquad (17)$$

Proof. When $z = q$, we can rewrite (10) as follows :

$$\sum_{l(\lambda)\le k} q^{2n_2(\lambda)-2\lambda_1^2+2\lambda_1}\frac{(a^{-1},\, b^{-1};\, q^2)_{\lambda_1}}{(q^2;\, q^2)_\lambda (q;\, q^2)_{\lambda_k}}(ab)^{\lambda_1} \tag{18}$$

$$= \frac{(aq^2, bq^2;\, q^2)_\infty}{(abq^2;\, q^2)_\infty (q^2;\, q^2)_\infty}\left(1+\sum_{r\ge 1} q^{2kr^2+r}\frac{(a^{-1}, b^{-1};\, q^2)_r}{(aq^2, bq^2;\, q^2)_r}(ab)^r(1+q^{2r})\right).$$

For (12), letting a and b tend to 0 in (18) we obtain

$$\sum_{l(\lambda)\le k}\frac{q^{2n_2(\lambda)}}{(q;q^2)_{\lambda_k}(q^2;q^2)_\lambda} = (q^2;q^2)_\infty^{-1} J(-q^{2k+1},\, q^{4k+4}).$$

The right side of (12) follows then from (11) after simple manipulations.

For (13), let $a \to 0$ in (18) and multiply both sides by $1-q^{-2}$. Identifying the coefficients of b we obtain :

$$\sum_{l(\lambda)\le k}\frac{q^{2n_2(\lambda)-2\lambda_1}}{(q;q^2)_{\lambda_k}(q^2;q^2)_\lambda}(1-q^{2\lambda_1}) = (q^2;q^2)_\infty^{-1}\, J(-q^{2k-1},\, q^{4k+4}).$$

The result follows from (11) after simple manipulations.

Identity (14) follows from (18) with $a = -q^{-1}$ and $b \to 0$ and then by applying (11) with q replaced by q^{4k+2} and $x = -q^{2k}$.

For (15), we choose $a = -1$ in (18) and multiply both sides by $1 - q^{-2}$, then identify the coefficient of b. The identity follows then by applying (11) with q replaced by q^{4k+2} and $x = -q^{2k-1}$.

Identity (16) follows from (18) by taking $a = -q^{-1}$ and $b = -1$ and then applying (11) with q replaced by q^{4k} and $x = -q^{2k}$. For (17), we choose $a = -1$ and $b \to 0$ in (18). The identity follows then by applying (11) with q replaced by q^{4k+2} and $x = -q^{2k+1}$. □

When $k = 1$ the above six identities reduce respectively to the following Rogers-Ramanujan type identities :

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q)_{2n}} = \prod_{\substack{n=1\\ n\equiv \pm2,\pm3,\pm4,\pm5 \pmod{16}}}^{\infty}\frac{1}{1-q^n}, \tag{19}$$

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q)_{2n+1}} = \prod_{\substack{n=1\\ n\equiv \pm1,\pm4,\pm6,\pm7 \pmod{16}}}^{\infty}\frac{1}{1-q^n}, \tag{20}$$

$$\sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(q)_{2n}} = \frac{(q^6,q^6,q^{12};\,q^{12})_\infty}{(q)_\infty}, \tag{21}$$

$$\sum_{n=0}^{\infty} q^{n^2+n} \frac{(-q^2;q^2)_n}{(q)_{2n+1}} = \frac{(q^3,q^9,q^{12};\,q^{12})_\infty}{(q)_\infty}, \tag{22}$$

$$1+2\sum_{n\geq 1} q^{n} \frac{(-q)_{2n-1}}{(q)_{2n}} = \frac{(q^4,\,-q^2,\,-q^2;\,q^4)_\infty}{(q)_\infty (q;q^2)_\infty}, \tag{23}$$

$$1+2\sum_{n\geq 1} q^{n(n+1)} \frac{(-q^2;q^2)_{n-1}}{(q)_{2n}} = \frac{(q^6,-q^3,-q^3;\,q^6)_\infty}{(q)_\infty\,(-q;q^2)_\infty}. \tag{24}$$

Note that (19), (20), (21) and (22) are already known, they correspond to Eqs. (39), (38), (29) and (28) in Slater's list [11], respectively. Identity (23) can be derived from the q-Kummer identity [5, p. 236] by the substitution $q \leftarrow q^2$, $a=-1$ and $b=-q$, but (24) seems to be new.

3 Proof of Theorem 1 and consequences

3.1 Proof of identity (7)

For any statement A it will be convenient to use the true or false function $\chi(A)$, which is 1 if A is true and 0 if A is false. Consider the generating function

$$S(u) = \sum_{\lambda_0,\lambda} \chi(\lambda' \text{ even})\, c_{\lambda,\lambda_0}(q) P_\lambda(X,q)\, u^{\lambda_0}$$

where the sum is over all partitions $\lambda=(\lambda_1,\ldots,\lambda_n)$ and the integers $\lambda_0 \geq \lambda_1$. Suppose $\lambda = (\mu_1^{r_1}\,\mu_2^{r_2}\,\ldots\mu_k^{r_k})$, where $\mu_1>\mu_2>\cdots>\mu_k\geq 0$ and $(r_1,\ldots,r_k)$ is a composition of n.

Let S_n^λ be the set of permutations of S_n which fix λ. Each $w\in S_n/S_n^\lambda$ corresponds to a surjective mapping $f: X \longrightarrow \{1,2,\ldots,k\}$ such that $|f^{-1}(i)|=r_i$. For any subset Y of X, let $p(Y)$ denote the product of the elements of Y (in particular, $p(\emptyset)=1$). We can rewrite Hall-Littlewood functions as follows :

$$P_\lambda(X,q) = \sum_f p(f^{-1}(1))^{\mu_1}\cdots p(f^{-1}(k))^{\mu_k} \prod_{f(x_i)<f(x_j)} \frac{x_i-qx_j}{x_i-x_j},$$

summed over all surjective mappings $f: X \longrightarrow \{1,2,\ldots,k\}$ such that $|f^{-1}(i)|=r_i$. Furthermore, each such f determines a *filtration* of X :

$$\mathcal{F}: \quad \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = X, \tag{25}$$

according to the rule $x_i \in F_l \iff f(x_i) \le l$ for $1 \le l \le k$. Conversely, such a filtration $\mathcal{F} = (F_0, F_1, \dots, F_k)$ determines a surjection $f : X \longrightarrow \{1, 2, \dots, k\}$ uniquely. Thus we can write :

$$P_\lambda(X, q) = \sum_{\mathcal{F}} \pi_{\mathcal{F}} \prod_{1 \le i \le k} p(F_i \setminus F_{i-1})^{\mu_i}, \tag{26}$$

summed over all the filtrations $\mathcal{F}$ such that $|F_i| = r_1 + r_2 + \cdots + r_i$ for $1 \le i \le k$, and

$$\pi_{\mathcal{F}} = \prod_{f(x_i) < f(x_j)} \frac{x_i - q x_j}{x_i - x_j},$$

where f is the function defined by $\mathcal{F}$.

Now let $\nu_i = \mu_i - \mu_{i+1}$ if $1 \le i \le k-1$ and $\nu_k = \mu_k$, thus $\nu_i > 0$ if $i < k$ and $\nu_k \ge 0$. Since the lengths of columns of λ are $|F_j| = r_1 + \cdots + r_j$ with multiplicities ν_j for $1 \le j \le k$, we have

$$\chi(\lambda' \,\text{even}) = \prod_{j=1}^{k} \chi(|F_j|_{\text{even}}). \tag{27}$$

A filtration $\mathcal{F}$ is called *even* if $|F_j|$ is even for $j \ge 1$. Furthermore, let $\mu_0 = \lambda_0$ and $\nu_0 = \mu_0 - \mu_1$ in the definition of $S(u)$, so that $\nu_0 \ge 0$ and $\mu_0 = \nu_0 + \nu_1 + \cdots + \nu_k$. Define $\varphi_{2n}(q) = (1-q)(1-q^3)\cdots(1-q^{2n-1})$ and $c_{\mathcal{F}}(q) = \prod_{i=1}^{k} \varphi_{|F_i \setminus F_{i-1}|}(q)$ for even filtrations $\mathcal{F}$. Thus, since $r_j = m_{\mu_j}(\lambda)$ for $j \ge 1$, we have

$$\begin{aligned} c_{\lambda,\,\lambda_0}(q) \;&=\; c_{\mathcal{F}}(q) \left(\chi(\nu_k = 0)\varphi_{|F_k \setminus F_{k-1}|}(q) + \chi(\nu_k \ne 0)\right)^{-1} \\ &\qquad \times \left(\chi(\nu_0 = 0)\varphi_{|F_1|}(q) + \chi(\nu_0 \ne 0)\right)^{-1}. \end{aligned}$$

Let $F(X)$ be the set of filtrations of X. Summarizing we obtain

$$\begin{aligned} S(u) \;&=\; \sum_{\mathcal{F} \in F(X)} c_{\mathcal{F}}\, \pi_{\mathcal{F}}\, \chi(\mathcal{F}_{\text{even}}) \sum_{\nu_1 > 0} (u\, p(F_1))^{\nu_1} \cdots \sum_{\nu_{k-1} > 0} (u\, p(F_{k-1}))^{\nu_{k-1}} \\ &\qquad \times \sum_{\nu_0 \ge 0} \frac{u^{\nu_0}}{\chi(\nu_0 = 0)\, \varphi_{|F_1|}(q) + \chi(\nu_0 \ne 0)} \\ &\qquad \times \sum_{\nu_k \ge 0} \frac{u^{\nu_k}\, p(F_k)^{\nu_k}}{\chi(\nu_k = 0)\, \varphi_{|F_k \setminus F_{k-1}|}(q) + \chi(\nu_k \ne 0)}. \end{aligned} \tag{28}$$

For any filtration $\mathcal{F}$ of X set

$$\mathcal{A}_{\mathcal{F}}(X, u) = c_{\mathcal{F}}(q) \prod_{|F_j|\,\text{even}} \left[\frac{p(F_j)u}{1 - p(F_j)u} + \frac{\chi(F_j = X)}{\varphi_{|F_j \setminus F_{j-1}|}(q)} + \frac{\chi(F_j = \emptyset)}{\varphi_{|F_1|}(q)} \right]$$

if $\mathcal{F}$ is even, and 0 otherwise. It follows from (28) that

$$S(u) = \sum_{\mathcal{F}\in F(X)} \pi_{\mathcal{F}} \mathcal{A}_{\mathcal{F}}(X,u).$$

Hence $S(u)$ is a rational function of u with simple poles at $1/p(Y)$, where Y is a subset of X such that $|Y|$ is even. We are now proceeding to compute the corresponding residue $c(Y)$ at each pole $u = 1/p(Y)$.

Let us start with $c(\emptyset)$. Writing $\lambda_0 = \lambda_1 + k$ with $k \geq 0$, we see that

$$\begin{aligned} S(u) &= \sum_{\lambda} \chi(\lambda' \,\text{even})\, c_\lambda(q) P_\lambda(X,q) u^{\lambda_1} \sum_{k\geq 0} \frac{u^k}{\chi(k=0)\varphi_{m_{\lambda_1}}(q) + \chi(k\neq 0)} \\ &= \sum_{\lambda} \chi(\lambda' \,\text{even})\, c_\lambda(q) P_\lambda(X,q) u^{\lambda_1} \left(\frac{u}{1-u} + \frac{1}{\varphi_{m_{\lambda_1}}(q)} \right). \end{aligned}$$

It follows from (5) that

$$c(\emptyset) = [S(u)(1-u)]_{u=1} = \Phi_q(X;0,0).$$

For the computations of other residues, we need some more notations. For any $Y \subseteq X$, let $Y' = X \setminus Y$ and $-Y = \{x_i^{-1} : x_i \in Y\}$. Let $Y \subseteq X$ such that $|Y|$ is even. Then

$$c(Y) = \left[\sum_{\mathcal{F}} \pi_{\mathcal{F}} \mathcal{A}_{\mathcal{F}}(X,\, u)(1 - p(Y)u) \right]_{u=p(-Y)}. \tag{29}$$

If $Y \notin \mathcal{F}$, the corresponding summand is equal to 0. Thus we need only to consider the following filtrations $\mathcal{F}$:

$$\emptyset = F_0 \subsetneq \cdots \subsetneq F_t = Y \subsetneq \cdots \subsetneq F_k = X \qquad 1 \leq t \leq k.$$

We may then split $\mathcal{F}$ into two filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$:

$$\begin{aligned} \mathcal{F}_1 &: \emptyset \subsetneq -(Y \setminus F_{t-1}) \subsetneq \cdots \subsetneq -(Y\setminus F_1) \subsetneq -Y, \\ \mathcal{F}_2 &: \emptyset \subsetneq F_{t+1}\setminus Y \subsetneq \cdots \subsetneq F_{k-1}\setminus Y \subsetneq Y'. \end{aligned}$$

Then, writing $v = p(Y)u$ and $c_{\mathcal{F}} = c_{\mathcal{F}_1} \times c_{\mathcal{F}_2}$, we have

$$\pi_{\mathcal{F}}(X) = \pi_{\mathcal{F}_1}(-Y)\pi_{\mathcal{F}_2}(Y') \prod_{x_i\in Y, x_j\in Y'} \frac{1-qx_i^{-1}x_j}{1-x_i^{-1}x_j},$$

and $\mathcal{A}_{\mathcal{F}}(X,\, u)(1-p(Y)u)$ is equal to

$$\mathcal{A}_{\mathcal{F}_1}(-Y,\, v)\mathcal{A}_{\mathcal{F}_2}(Y',\, v)(1-v)\left(\frac{v}{1-v}+\frac{\chi(Y=X)}{\varphi_{|Y\setminus F_{t-1}|}(q)}\right)$$
$$\times\left(\frac{v}{1-v}+\frac{1}{\varphi_{|Y\setminus F_{t-1}|}(q)}\right)^{-1}\left(\frac{v}{1-v}+\frac{1}{\varphi_{|F_{t+1}\setminus Y|}(q)}\right)^{-1}.$$

Thus when $u=p(-Y)$, i.e., $v=1$,

$$\begin{aligned}&[\pi_{\mathcal{F}}(X)\mathcal{A}_{\mathcal{F}}(X,\, u)(1-p(Y)u)]_{u=p(-Y)} =\\ &\quad\left[\pi_{\mathcal{F}_1}(-Y)\mathcal{A}_{\mathcal{F}_1}(-Y,\, v)(1-v)\pi_{\mathcal{F}_2}(Y')\mathcal{A}_{\mathcal{F}_2}(Y',\, v)(1-v)\right]_{v=1}\\ &\qquad\times\prod_{x_i\in Y, x_j\in Y'}\frac{1-qx_i^{-1}x_j}{1-x_i^{-1}x_j}.\end{aligned}$$

Using (29) and the result of $c(\emptyset)$, which can be written

$$\left[\sum_{\mathcal{F}}\pi_{\mathcal{F}}\mathcal{A}_{\mathcal{F}}(X,u)(1-u)\right]_{u=1}=\Phi_q(X;0,0),$$

we get

$$c(Y)=\Phi_q(-Y;0,0)\Phi_q(Y';0,0)\prod_{x_i\in Y, x_j\in Y'}\frac{1-qx_i^{-1}x_j}{1-x_i^{-1}x_j}.$$

Each subset Y of X can be encoded by a sequence $\xi\in\{\pm1\}^n$ according to the rule : $\xi_i=1$ if $x_i\notin Y$ and $\xi_i=-1$ if $x_i\in Y$. Hence

$$c(Y)=\Phi_q(X^{\xi};0,0).$$

Note also that

$$p(Y)=\prod_i x_i^{(1-\xi_i)/2},\qquad p(-Y)=\prod_i x_i^{(\xi_i-1)/2}.$$

Now, extracting the coefficients of u^k in the equation :

$$S(u)=\sum_{\substack{Y\subseteq X\\ |Y|\text{ even}>0}}\frac{c(Y)}{1-p(Y)u},$$

yields

$$\sum_{\substack{\lambda_1\le k\\ \lambda'\ even}}c_{\lambda,k}(q)P_\lambda(X,q)=\sum_{\substack{Y\subseteq X\\ |Y|\text{ even}}}c(Y)p(Y)^k.$$

Finally, substituting the value of $c(Y)$ in the above formula we obtain (7).
Remark. Stembridge's formula (4) can be derived from Macdonald's (3) and Pieri's formula for Hall-Littlewood polynomials. Indeed, one of Pieri's formulas states that [9, p. 215] :

$$P_\mu(X,q)e_m(X) = \sum_\lambda \prod_{i\geq 1} \begin{bmatrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix} P_\lambda(X,q), \tag{30}$$

where the sum is over all partitions λ such that $\mu \subseteq \lambda$ with $|\lambda/\mu| = m$ and there is at most one cell in each row of the Ferrers diagram of λ/μ. It follows from (30) that

$$\sum_{\substack{\mu_1\leq 2k \\ \mu\ even}} P_\mu(X,q) \sum_{m\geq 0} e_m(X) = \sum_{\lambda_1\leq 2k+1} P_\lambda(X,q),$$

noticing that λ determines in a unique way μ even by deleting a cell in each odd part of λ, and thus $\begin{bmatrix} \lambda'_i-\lambda'_{i+1} \\ \lambda'_i-\mu'_i \end{bmatrix} = 1$. Finally we obtain the result, using the fact that $\prod_i (1+x_i^{\xi_i})^{-1} = \prod_i (1+x_i)^{-1} \times \prod_i x_i^{(1-\xi_i)/2}$. It would be interesting to give a similar proof of (7) using (3) and another Pieri's formula [9, p. 218].

3.2 Proof of identity (8)

As in the proof of (7), we compute the generating function

$$F(u) = \sum_{\lambda_0,\lambda} d_{\lambda,\lambda_0}(q) P_\lambda(X;q)\, u^{\lambda_0}$$

where the sum is over all partitions $\lambda = (\lambda_1,\ldots,\lambda_n)$ and integers $\lambda_0 \geq \lambda_1$. For any filtration $\mathcal{F}$ of X (cf. (25)) set

$$d_{\mathcal{F}}(q) = \prod_{i=1}^{k} \psi_{|F_i\setminus F_{i-1}|}(q), \quad \text{where} \quad \psi_n(q) = (q)_n \prod_{j=1}^{[n/2]} (1-q^{2j})^{-1}.$$

Thus, as $r_j = m_{\mu_j}(\lambda)$, $j\geq 1$, we have

$$\begin{aligned} d_{\lambda,\,\lambda_0}(q) \;&=\; d_{\mathcal{F}}(q)\left(\chi(\nu_k=0)\psi_{|F_k\setminus F_{k-1}|}(q) + \chi(\nu_k\neq 0)\right)^{-1} \\ &\qquad \times \left(\chi(\nu_0=0)\psi_{|F_1|}(q) + \chi(\nu_0\neq 0)\right)^{-1}. \end{aligned}$$

In view of (26) we have

$$F(u)=\sum_{\mathcal{F}\in F(X)}\pi_{\mathcal{F}}\mathcal{B}_{\mathcal{F}}(X,u),$$

where

$$\mathcal{B}_{\mathcal{F}}(X,u)=d_{\mathcal{F}}\prod_{j}\left[\frac{p(F_j)u}{1-p(F_j)u}+\frac{\chi(F_j=X)}{\psi_{|F_j\setminus F_{j-1}|}(q)}+\frac{\chi(F_j=\emptyset)}{\psi_{|F_1|}(q)}\right].$$

It follows that $F(u)$ is a rational function of u and can be written as :

$$F(u)=\frac{c(\emptyset)}{1-u}+\sum_{\substack{Y\subseteq X\\ |Y|>0}}\frac{c(Y)}{1-p(Y)u}.$$

Extracting the coefficient of u^k in the above identity yields

$$\sum_{\lambda_1\leq k}d_{\lambda,k}(q)P_\lambda(X,q)=\sum_{Y\subseteq X}c(Y)p(Y)^k. \tag{31}$$

It remains to compute the residues. Writing $\lambda_0=\lambda_1+r$ with $r\geq 0$, then

$$\begin{aligned}F(u) &= \sum_{\lambda}d_\lambda(q)P_\lambda(X,q)u^{\lambda_1}\sum_{r\geq 0}\frac{u^r}{\chi(r=0)\psi_{m_{\lambda_1}}(q)+\chi(r\neq 0)}\\ &= \sum_{\lambda}d_\lambda(q)P_\lambda(X,q)u^{\lambda_1}\left(\frac{u}{1-u}+\frac{1}{\psi_{m_{\lambda_1}}(q)}\right),\end{aligned}$$

it follows from (6) that

$$c(\emptyset)=(F(u)(1-u))\,|_{u=1}=\Phi_q(X;q,1). \tag{32}$$

For computations of the other residues, set $Y'=X\setminus Y$ and define, for $Y=F_t$, the two filtrations :

$$\begin{aligned}\mathcal{F}_1 &: \emptyset\subsetneq -(Y\setminus F_{t-1})\subsetneq\cdots\subsetneq -(Y\setminus F_1)\subsetneq -Y,\\ \mathcal{F}_2 &: \emptyset\subsetneq F_{t+1}\setminus Y\subsetneq\cdots\subsetneq F_{k-1}\setminus Y\subsetneq Y'.\end{aligned}$$

Then, writing $v=p(Y)u$ and $d_{\mathcal{F}}=d_{\mathcal{F}_1}\times d_{\mathcal{F}_2}$, we have

$$\pi_{\mathcal{F}}(X)=\pi_{\mathcal{F}_1}(-Y)\pi_{\mathcal{F}_2}(Y')\prod_{x_i\in Y,x_j\in Y'}\frac{1-qx_i^{-1}x_j}{1-x_i^{-1}x_j},$$

and $\mathcal{B}_{\mathcal{F}}(X, u)(1-p(Y)u)$ can be written as

$$\mathcal{B}_{\mathcal{F}_1}(-Y, v)\mathcal{B}_{\mathcal{F}_2}(Y', v)(1-v)\left(\frac{v}{1-v}+\frac{\chi(Y=X)}{\psi_{|Y\setminus F_{t-1}|}}\right)$$
$$\times\left(\frac{v}{1-v}+\frac{1}{\psi_{|Y\setminus F_{t-1}|}(q)}\right)^{-1}\left(\frac{v}{1-v}+\frac{1}{\psi_{|F_{t+1}\setminus Y|}(q)}\right)^{-1}.$$

Rewriting (32) as

$$\left[\sum_{\mathcal{F}}\pi_{\mathcal{F}}\mathcal{B}_{\mathcal{F}}(X, u)(1-u)\right]_{u=1}=\Phi_q(X;q,1),$$

we get

$$\begin{aligned} c(Y) &= \left[\sum_{\mathcal{F}}\pi_{\mathcal{F}}\mathcal{B}_{\mathcal{F}}(X;u)(1-p(Y)u)\right]_{u=p(-Y)} \\ &= \Phi_q(-Y;q,1)\Phi_q(Y';q,1)\prod_{x_i\in Y, x_j\in Y'}\frac{1-qx_i^{-1}x_j}{1-x_i^{-1}x_j}. \end{aligned}$$

Finally, the proof is completed by substituting the values of $c(Y)$ in (31).

3.3 Some direct consequences on q-series

The following corollary of Theorem 1 will be useful in the proof of identities of Rogers-Ramanujan type.

Theorem 4 *For $k\geq 1$,*

$$\sum_{l(\lambda)\leq k}\frac{(q;q^2)_\lambda}{(q;q^2)_{\lambda_k}}z^{|\lambda|}q^{n(2\lambda)}\begin{bmatrix}n\\2\lambda\end{bmatrix}=(z;q^2)_n\sum_{r\geq 0}z^{kr}q^{(k+1)\binom{2r}{2}}$$
$$\times\begin{bmatrix}n\\2r\end{bmatrix}\frac{1-zq^{4r-1}}{(zq^{2r-1})_{n+1}}. \tag{33}$$

$$\sum_{l(\lambda)\leq k}\prod_{i=1}^{k-1}\frac{(q)_{\lambda_i-\lambda_{i+1}}}{(q^2;q^2)_{[(\lambda_i-\lambda_{i+1})/2]}}z^{|\lambda|}q^{n(\lambda)}\begin{bmatrix}n\\\lambda\end{bmatrix}=(z^2;q^2)_n\sum_{r\geq 0}z^{kr}q^{r+(k+1)\binom{r}{2}}$$
$$\times\begin{bmatrix}n\\r\end{bmatrix}\frac{(1-zq^{-1})(1-z^2q^{2r-1})(1-zq^n)}{(1-zq^{r-1})(1-zq^r)(z^2q^{r-1})_{n+1}}. \tag{34}$$

Proof. We know [9, p. 213] that if $x_i = z^{1/2}q^{i-1}$ $(1 \le i \le n)$ then :

$$P_{\lambda'}(X, q) = z^{|\lambda|/2} q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix}. \tag{35}$$

In view of (9) we have

$$c_{(2\lambda)',k}(q) = \frac{(q;q^2)_\lambda}{(q;q^2)_{\lambda_k}}.$$

Replacing λ by 2λ and taking the conjugation in the left-hand side of (7) we obtain the left-hand side of (33). On the other hand, for any $\xi \in \{\pm 1\}^n$ such that the number of $\xi_i = -1$ is r, $0 \le r \le n$, we have

$$\Phi_q(X^\xi; 0, 0) = \Psi_q(X^\xi; -1) \prod_i (1 - x_i^{2\xi_i}), \tag{36}$$

which is readily seen to equal 0 unless $\xi \in \{-1\}^r \times \{1\}^{n-r}$. Now, in the latter case, we have $\prod_i x_i^{k(1-\xi_i)/2} = z^{kr/2} q^{k\binom{r}{2}}$,

$$\prod_{i=1}^{n} (1 - x_i^{2\xi_i}) = (-1)^r z^{-r} q^{-2\binom{r}{2}} (z;q^2)_n, \tag{37}$$

and [12, p. 476] :

$$\Psi_q(X^\xi; -1) = (-1)^r z^r q^{3\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix} \frac{1 - zq^{2r-1}}{(zq^{r-1})_{n+1}}. \tag{38}$$

Substituting these into the right side of (7) with r replaced by $2r$ we obtain the right side of (33).

Next, by (9) we have

$$d_{\lambda',k}(q) = \prod_{i=1}^{k-1} \frac{(q)_{\lambda_i - \lambda_{i+1}}}{(q^2;q^2)_{[(\lambda_i - \lambda_{i+1})/2]}}.$$

Similarly, in (8), replacing x_i by zq^{i-1} $(1 \le i \le n)$ and invoking (35) we see that the left side of (8) reduces to that of (34). On the other hand, since

$$\Phi_q(X^\xi; q, 1) = \Phi_q(X^\xi; 0, 0) \prod_{i=1}^{n} \frac{1 - qx_i^{\xi_i}}{1 - x_i^{\xi_i}},$$

by (36), this is equal to zero unless $\xi \in \{-1\}^r \times \{1\}^{n-r}$ for some r, $0 \le r \le n$. In the latter case, we have

$$\prod_{i=1}^{n} \frac{1 - qx_i^{\xi_i}}{1 - x_i^{\xi_i}} = q^r \frac{1 - zq^{-1}}{1 - zq^{r-1}} \frac{1 - zq^n}{1 - zq^r}, \tag{39}$$

and invoking (36), (37) and (38) with z replaced by z^2,

$$\Phi_q(X^\xi;0,0) = q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix} (1-z^2q^{2r-1}) \frac{(z^2;q^2)_n}{(z^2q^{r-1})_{n+1}} \tag{40}$$

Plunging these into the right side of (8) yields that of (34). □

When $n \to +\infty$, since $\begin{bmatrix} n \\ \lambda \end{bmatrix} \to \frac{1}{(q)_\lambda}$, equations (33) and (34) reduce respectively to :

$$\sum_{l(\lambda)\le k} \frac{z^{|\lambda|}q^{n(2\lambda)}}{(q^2;q^2)_\lambda (q;q^2)_{\lambda_k}} = (z;q^2)_\infty \sum_{r\ge 0} \frac{z^{kr}q^{(k+1)\binom{2r}{2}}}{(q)_{2r}(zq^{2r-1})_\infty}(1-zq^{4r-1}), \tag{41}$$

$$\sum_{l(\lambda)\le k} \frac{z^{|\lambda|}q^{n(\lambda)}}{(q)_{\lambda_k} \prod_{i=1}^{k-1}(q^2;q^2)_{[(\lambda_i-\lambda_{i+1})/2]}} \tag{42}$$

$$= (z^2;q^2)_\infty \sum_{r\ge 0} z^{kr} q^{r+(k+1)\binom{r}{2}} \frac{1-zq^{-1}}{(q)_r(1-zq^{r-1})} \frac{1-z^2q^{2r-1}}{(1-zq^r)(z^2q^{r-1})_\infty}.$$

Furthermore, setting $z=q$ in (41) and (42) we obtain respectively (11) and

$$\sum_{l(\lambda)\le k} \frac{q^{|\lambda|+n(\lambda)}}{(q)_{\lambda_k} \prod_{i=1}^{k-1}(q^2;q^2)_{[(\lambda_i-\lambda_{i+1})/2]}} = \frac{1}{(q;q^2)_\infty}. \tag{43}$$

4 Elementary approach to multiple q-identities

4.1 Preliminaries

Recall [1, pp. 36-37] that the binomial formula has the following q-analog :

$$(z)_n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^m z^m q^{m(m-1)/2}. \tag{44}$$

Since the *elementary symmetric functions* $e_r(X)$ $(0 \le r \le n)$ satisfy

$$(1+x_1z)(1+x_2z)\cdots(1+x_nz) = \sum_{r=0}^{n} e_r(X)z^r,$$

it follows from (44) that for integers $i \ge 0$ and $j \ge 1$:

$$e_r(q^i,\, q^{i+1},\, \ldots,\, q^{i+j-1}) = q^{ir}e_r(1,\, q,\, \ldots,\, q^{j-1}) = q^{ir+\binom{r}{2}} \begin{bmatrix} j \\ r \end{bmatrix}. \tag{45}$$

The following result can be derived from the Pieri's rule for Hall-Littlewood polynomials [9, p. 215], but our proof is elementary.

Lemma 1 *For any partition μ such that $\mu_1 \le n$ there holds*

$$q^{\binom{m}{2}+n(\mu)} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} n \\ \mu \end{bmatrix} = \sum_{\lambda} q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix} \prod_{i \ge 1} \begin{bmatrix} \lambda_i - \lambda_{i+1} \\ \lambda_i - \mu_i \end{bmatrix}, \tag{46}$$

where the sum is over all partitions λ such that λ/μ is an m-horizontal strip, i.e., $\mu \subseteq \lambda$, $|\lambda/\mu| = m$ and there is at most one cell in each column of the Ferrers diagram of λ/μ.

Proof. Let $l := l(\mu)$ and $\mu_0 = n$. Partition the set $\{1, 2, \ldots, n\}$ into $l+1$ subsets :

$$X_i = \{j \mid 1 \le j \le n \text{ and } \mu'_j = i\} = \{j \mid \mu_{i+1} + 1 \le j \le \mu_i\}, \qquad 0 \le i \le l.$$

Using (45) to extract the coefficients of z^m in the following identity :

$$(1+z)(1+zq)\cdots(1+zq^{n-1}) = \prod_{i=0}^{l} \prod_{j \in X_i} (1+zq^{j-1}),$$

we obtain

$$q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix} = \sum_{\mathbf{r}} \prod_{i=0}^{l} q^{r_i \mu_{i+1} + \binom{r_i}{2}} \begin{bmatrix} \mu_i - \mu_{i+1} \\ r_i \end{bmatrix}, \tag{47}$$

where $\mathbf{r} = (r_0, r_1, \ldots, r_l)$ is a composition of m. For any such composition $\mathbf{r}$ we define a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ by

$$\lambda_i = \mu_i + r_{i-1}, \qquad 1 \le i \le l+1.$$

Then λ/μ is a m-horizontal strip. So (47) can be written as

$$q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix} = \sum_{\lambda} \prod_{i=0}^{l} q^{(\lambda_{i+1}-\mu_{i+1})\mu_{i+1} + \binom{\lambda_{i+1}-\mu_{i+1}}{2}} \begin{bmatrix} \mu_i - \mu_{i+1} \\ \mu_i - \lambda_{i+1} \end{bmatrix}, \tag{48}$$

where the sum is over all partitions λ such that λ/μ is an m-horizontal strip. Now, since

$$(\lambda_{i+1} - \mu_{i+1})\mu_{i+1} + \binom{\lambda_{i+1} - \mu_{i+1}}{2} + \binom{\mu_{i+1}}{2} = \binom{\lambda_{i+1}}{2}, \qquad 0 \le i \le l,$$

and $\left[\begin{smallmatrix} n \\ \mu \end{smallmatrix}\right] \prod_{i=0}^{l} \left[\begin{smallmatrix} \mu_i - \mu_{i+1} \\ \mu_i - \lambda_{i+1} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right] \prod_{i \ge 1} \left[\begin{smallmatrix} \lambda_i - \lambda_{i+1} \\ \lambda_i - \mu_i \end{smallmatrix}\right]$ are equal because they are both equal to

$$\frac{(q)_n}{(q)_{n-\lambda_1}(q)_{\lambda_1-\mu_1}(q)_{\mu_1-\lambda_2}\cdots(q)_{\mu_l}},$$

multiplying (48) by $q^{n(\mu)}\left[\begin{smallmatrix} n \\ \mu \end{smallmatrix}\right]$ yields (46). □

Lemma 2 *There hold the following identities :*

$$\sum_{\lambda} z^{|\lambda|} q^{2n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix} = \frac{1}{(z)_n}, \tag{49}$$

$$\sum_{\lambda} z^{|\lambda|} q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix} = \frac{(-z)_n}{(z^2)_n}, \tag{50}$$

$$\sum_{\lambda} (q, q^2)_\lambda \; z^{|\lambda|} q^{n(2\lambda)} \begin{bmatrix} n \\ 2\lambda \end{bmatrix} = \frac{(z; q^2)_n}{(z)_n}. \tag{51}$$

Proof. Identity (49) is due to Hall [6] and can be proved by using the q-binomial identity [8]. Stembridge [12] proved (50) using the q-binomial identity. Now, writing

$$\frac{(z^2;\, q^2)_n}{(z^2)_n} = (z)_n \, \frac{(-z)_n}{(z^2)_n}$$

and applying successively (44), (50) and (46) we obtain

$$\begin{aligned} \frac{(z^2;\, q^2)_n}{(z^2)_n} &= \sum_{\mu,m} (-1)^m z^{m+|\mu|} q^{\binom{m}{2}+n(\mu)} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} n \\ \mu \end{bmatrix} \\ &= \sum_{\mu,m} (-1)^m z^{m+|\mu|} \sum_{\lambda:\, \lambda/\mu = m-hs} \prod_{i\geq 1} \begin{bmatrix} \lambda_i - \lambda_{i+1} \\ \lambda_i - \mu_i \end{bmatrix} q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix} \\ &= \sum_{\lambda} z^{|\lambda|} q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix} \prod_{i\geq 1} \sum_{r_i \geq 0} (-1)^{r_i} \begin{bmatrix} \lambda_i - \lambda_{i+1} \\ r_i \end{bmatrix}. \end{aligned}$$

The identity (51) follows then from

$$\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} = \begin{cases} (q;\, q^2)_n & \text{if} \quad m = 2n, \\ 0 & \text{if } m \text{ is odd}, \end{cases}$$

which can be proved using the q-binomial formula [1, p. 36]. □

Remark. When $n \to \infty$ the above identities reduce respectively to the following :

$$\sum_{\lambda} \frac{z^{|\lambda|} q^{2n(\lambda)}}{(q)_\lambda} = \frac{1}{(z)_\infty}, \tag{52}$$

$$\sum_{\lambda} \frac{z^{|\lambda|} q^{n(\lambda)}}{(q)_\lambda} = \frac{(-z)_\infty}{(z^2)_\infty}, \tag{53}$$

$$\sum_{\lambda} \frac{z^{|\lambda|} q^{n(2\lambda)}}{(q^2; q^2)_\lambda} = \frac{1}{(zq; q^2)_\infty}. \tag{54}$$

Also (52) and (54) are actually equivalent since the later can be derived from (52) by substituting q by q^2 and z by zq.

The following is the q-Gauss sum [5, p.10] due to Heine :

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ x \end{matrix}; q; \frac{x}{ab}\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(q)_n (x)_n} \left(\frac{x}{ab}\right)^n = \frac{(x/a,\ x/b)_\infty}{(x, x/ab)_\infty}. \quad (55) $$

Lemma 3 *We have*

$$ \sum_{\lambda} z^{|\lambda|} q^{n(2\lambda)} \frac{(a,\ b; q^{-2})_{\lambda_1}}{(q^2; q^2)_\lambda} = \frac{(azq,\ bzq; q^2)_\infty}{(zq,\ abzq; q^2)_\infty}. \quad (56) $$

Proof. Substituting q^2 by q and z by zq, the identity is equivalent to

$$ \sum_{\lambda} z^{|\lambda|} q^{2n(\lambda)} \frac{(a,\ b; q^{-1})_{\lambda_1}}{(q)_\lambda} = \frac{(az,\ bz)_\infty}{(z,\ abz)_\infty}. \quad (57) $$

Now, writing $k = \lambda_1$ and $\mu = (\lambda_2, \lambda_3, \cdots)$, and using (49) we get

$$ \begin{aligned} \sum_{\lambda} z^{|\lambda|} q^{2n(\lambda)} \frac{(a,\ b; q^{-1})_{\lambda_1}}{(q)_\lambda} &= \sum_{k\geq 0} z^k q^{k(k-1)} \frac{(a, b; q^{-1})_k}{(q)_k} \sum_{\mu} z^{|\mu|} q^{2n(\mu)} \begin{bmatrix} k \\ \mu \end{bmatrix} \\ &= \sum_{k\geq 0} (abz)^k \frac{(a^{-1}, b^{-1})_k}{(q)_k (z)_k}. \end{aligned} $$

Identity (57) follows then from (55). □

Remark. Formula (57) was derived in [12] from a more general formula of Hall-Littlewood polynomials.

4.2 Elementary proof of Theorem 4

We shall only prove (33) when n is even and leave the case when n is odd and (34) to the interested reader because their proofs are very similar. Consider the generating function of the left-hand side of (33) with $n = 2r$:

$$ \begin{aligned} \varphi(u) &= \sum_{k\geq 0} u^k \sum_{l(\lambda)\leq k} \frac{(q; q^2)_\lambda}{(q; q^2)_{\lambda_k}} z^{|\lambda|} q^{n(2\lambda)} \begin{bmatrix} 2r \\ 2\lambda \end{bmatrix} \\ &= \sum_{\lambda} u^{l(\lambda)} z^{|\lambda|} q^{n(2\lambda)} (q; q^2)_\lambda \begin{bmatrix} 2r \\ 2\lambda \end{bmatrix} \sum_{k\geq 0} \frac{u^k}{(q; q^2)_{\lambda_{k+l(\lambda)}}} \\ &= \sum_{\lambda} u^{l(\lambda)} z^{|\lambda|} q^{n(2\lambda)} (q; q^2)_\lambda \begin{bmatrix} 2r \\ 2\lambda \end{bmatrix} \left(\frac{u}{1-u} + \frac{1}{(q; q^2)_{\lambda_{l(\lambda)}}} \right). \quad (58) \end{aligned} $$

Now, each partition λ with parts bounded by r can be encoded by a pair of sequences $\nu = (\nu_0, \nu_1, \cdots, \nu_l)$ and $\mathbf{m} = (m_0, \cdots, m_l)$ such that $\lambda = (\nu_0^{m_0}, \ldots, \nu_l^{m_l})$, where $r = \nu_0 > \nu_1 > \cdots > \nu_l > 0$ and ν_i has multiplicity $m_i \geq 1$ for $1 \leq i \leq l$ and $\nu_0 = r$ has multiplicity $m_0 \geq 0$. Using the notation :

$$< \alpha >= \frac{\alpha}{1-\alpha}, \quad u_i = z^i q^{i(2i-1)} \quad \text{for} \quad i \geq 0,$$

we can then rewrite (58) as follows :

$$\begin{aligned}
\varphi(u) &= \sum_{\nu} (q;q^2)_\nu \begin{bmatrix} 2r \\ 2\nu \end{bmatrix} \left(< u > + \frac{1}{(q;q^2)_{\nu_l}} \right) \\
&\quad \times \sum_{\mathbf{m}} \left((u_r u)^{m_0} + \frac{\chi(m_0 = 0)}{(q;q^2)_{r-\nu_1}} \right) \prod_{i=1}^{l} (u_{\nu_i} u)^{m_i} \\
&= \sum_{\nu} \frac{(q)_{2r}}{(q^2;q^2)_\nu} B_\nu,
\end{aligned} \tag{59}$$

where the sum is over all strict partitions $\nu = (\nu_0, \nu_1, \ldots, \nu_l)$ and

$$B_\nu = \left(< u > + \frac{1}{(q;q^2)_{\nu_l}} \right) \left(< u_r u > + \frac{1}{(q;q^2)_{r-\nu_1}} \right) \prod_{i=1}^{l} < u_{\nu_i} u > .$$

So $\varphi(u)$ is a rational fraction with simple poles at u_p^{-1} for $0 \leq p \leq r$. Let $b_p(z,r)$ be the corresponding residue of $\varphi(u)$ at u_p^{-1} for $0 \leq p \leq r$. Then, it follows from (59) that

$$b_p(z,r) = \sum_{\nu} \frac{(q)_{2r}}{(q^2;q^2)_\nu} \left[B_\nu (1 - u_p u) \right]_{u=u_p^{-1}} . \tag{60}$$

We shall first consider the cases where $p = 0$ or r. Using (58) and (51) we have

$$b_0(z,r) = [\varphi(u)(1-u)]_{u=1} = \frac{(z;q^2)_{2r}}{(z)_{2r}}. \tag{61}$$

Now, by (59) and(60) we have

$$b_0(z,r) = \sum_{\nu} \frac{(q)_{2r}}{(q^2;q^2)_\nu} \left(< u_r > + \frac{1}{(q;q^2)_{r-\nu_1}} \right) \prod_{i=1}^{l} < u_{\nu_i} >, \tag{62}$$

and

$$b_r(z,r) = \sum_{\nu} \frac{(q)_{2r}}{(q^2;q^2)_\nu} \left(< 1/u_r > + \frac{1}{(q;q^2)_{\nu_l}} \right) \prod_{i=1}^{l} < u_{\nu_i}/u_r >, \tag{63}$$

which, by setting $\mu_i = r - \nu_{l+1-i}$ for $1 \le i \le l$ and $\mu_0 = r$, can be written as

$$b_r(z,r) = \sum_\mu \frac{(q)_{2r}}{(q^2;q^2)_\mu} \left(< 1/u_r > + \frac{1}{(q;q^2)_{r-\mu_1}} \right) \prod_{i=1}^{l} < u_{r-\mu_i}/u_r > . \quad (64)$$

Comparing (64) with (62) we see that $b_r(z,r)$ is equal to $b_0(z,r)$ with z replaced by $z^{-1}q^{-2(2r-1)}$. Il follows from (61) that

$$b_r(z,r) = b_0(z^{-1}q^{-2(2r-1)}, r) = (z;q^2)_{2r} q^{r(2r-1)} \frac{1 - zq^{4r-1}}{(zq^{2r-1})_{2r+1}}. \quad (65)$$

Consider now the case where $0 < p < r$. Clearly, for each partition ν, the corresponding summand in (60) is not zero only if $\nu_j = p$ for some j, $0 \le j \le r$. Furthermore, each such partition ν can be splitted into two strict partitions $\rho = (\rho_0, \rho_1, \ldots, \rho_{j-1})$ and $\sigma = (\sigma_0, \ldots, \sigma_{l-j})$ such that $\rho_i = \nu_i - p$ for $0 \le i \le j-1$ and $\sigma_s = \nu_{j+s}$ for $0 \le s \le l-j$. So we can write (60) as follows :

$$b_p(z,r) \quad = \quad \begin{bmatrix} 2r \\ 2p \end{bmatrix} \sum_\rho \frac{(q)_{2r-2p}}{(q^2;q^2)_\rho} F_\rho(p) \times \sum_\sigma \frac{(q)_{2p}}{(q^2;q^2)_\sigma} G_\sigma(p)$$

where for $\rho = (\rho_0, \rho_1, \ldots, \rho_l)$ with $\rho_0 = r - p$,

$$F_\rho(p) = \left(< u_r/u_p > + \frac{1}{(q;q^2)_{r-p-\rho_1}} \right) \prod_{i=1}^{l(\rho)} < u_{\rho_i+p}/u_p >,$$

and for $\sigma = (\sigma_0, \ldots, \sigma_l)$ with $\sigma_0 = p$,

$$G_\sigma(p) = \left(< 1/u_p > + \frac{1}{(q;q^2)_{\sigma_l}} \right) \prod_{i=1}^{l(\sigma)} < u_{\sigma_i}/u_p > .$$

Comparing with (62) and (64) and using (61) and (65) we obtain

$$\begin{aligned} b_p(z,r) \quad &= \quad \begin{bmatrix} 2r \\ 2p \end{bmatrix} b_0(zq^{4p}, r-p)\, b_p(z,p) \\ &= \quad \begin{bmatrix} 2r \\ 2p \end{bmatrix} (z;q^2)_{2r} q^{\binom{2r}{2p}} \frac{1 - zq^{4p-1}}{(zq^{2p-1})_{2r+1}}. \end{aligned}$$

Finally, extracting the coefficients of u^k in the equation

$$\varphi(u) = \sum_{p=0}^{r} \frac{b_p(z,r)}{1 - u_p u},$$

and using the values for $b_p(z,r)$ we obtain(33).

4.3 Proof of Theorem 2

Consider the generating function of the left-hand side of (10) :

$$\begin{aligned}
\varphi_{ab}(u) &:= \sum_{k\geq 0} u^k \sum_{l(\lambda)\leq k} z^{|\lambda|} q^{n(2\lambda)} \frac{(a,\, b; q^{-2})_{\lambda_1}}{(q^2;q^2)_\lambda (q;q^2)_{\lambda_k}} \\
&= \sum_{\lambda}\sum_{k\geq 0} u^{k+l(\lambda)} z^{|\lambda|} q^{n(2\lambda)} \frac{(a,\, b; q^{-2})_{\lambda_1}}{(q^2;q^2)_\lambda (q;q^2)_{\lambda_{l(\lambda)+k}}} \\
&= \sum_{\lambda} u^{l(\lambda)} z^{|\lambda|} q^{n(2\lambda)} \frac{(a,\, b; q^{-2})_{\lambda_1}}{(q^2;q^2)_\lambda} \left(\frac{u}{1-u} + \frac{1}{(q;q^2)_{\lambda_{l(\lambda)}}}\right), \qquad (66)
\end{aligned}$$

where the sum is over all the partitions λ. As in the elementary proof of Theorem 4, we can replace any partition λ by a pair $(\nu, \mathbf{m})$, where ν is a strict partition consisting of distinct parts $\nu_1, \cdots, \nu_l$ of λ, so that $\nu_1 > \cdots > \nu_l > 0$, and $\mathbf{m} = (m_1, \ldots, m_l)$ is the sequence of multiplicities of ν_i for $1 \leq i \leq l$. Therefore

$$\begin{aligned}
\varphi_{ab}(u) &= \sum_{\nu,\,\mathbf{m}} \frac{(a,\, b; q^{-2})_{\nu_1}}{(q^2;q^2)_\nu} \left(\frac{u}{1-u} + \frac{1}{(q;q^2)_{\nu_l}}\right) \prod_{i=1}^{l} (u_{\nu_i} u)^{m_i} \\
&= \sum_{\nu} \frac{(a,\, b; q^{-2})_{\nu_1}}{(q^2;q^2)_\nu} \left(< u > + \frac{1}{(q;q^2)_{\nu_l}}\right) \prod_{i=1}^{l} < u_{\nu_i} u >, \qquad (67)
\end{aligned}$$

where the sum is over all the strict partitions ν. Each of the terms in this sum, as a rational function of u, has a finite set of simple poles, which may occur at the points u_r^{-1} for $r \geq 0$. Therefore, each term is a linear combination of partial fractions. Moreover, the sum of their expansions converges coefficientwise. So φ_{ab} has an expansion

$$\varphi_{ab}(u) = \sum_{r\geq 0} \frac{c_r}{1 - u z^r q^{r(2r-1)}},$$

where c_r denotes the formal sum of partial fraction coefficients contributed by the terms of (67). It remains to compute these residues c_r $(r \geq 0)$. By using (56) and (66), we get immediately

$$c_0 = [\varphi_{ab}(u)(1-u)]_{u=1} = \frac{(azq,\, bzq; q^2)_\infty}{(zq,\, abzq; q^2)_\infty}.$$

In view of (67), this yields the identity

$$\sum_{\nu} \frac{(a,\, b; q^{-2})_{\nu_1}}{(q^2;q^2)_\nu} \prod_{i=1}^{l} < u_{\nu_i} > = \frac{(azq,\, bzq; q^2)_\infty}{(zq,\, abzq; q^2)_\infty}. \qquad (68)$$

Clearly, a summand in (67) has a non zero contribution to c_r ($r > 0$) only if the corresponding partition ν has a part equal to r. For any partition ν such that $\exists j \,|\, \nu_j = r$, set $\rho_i := \nu_i - r$ for $1 \le i < j$ and $\sigma_i := \nu_{i+j}$ for $0 \le i \le l-j$, we then get two partitions ρ and σ, with σ_i bounded by r. Multiplying (67) by $(1 - u_r u)$ and setting $u = 1/u_r$ we obtain

$$\begin{aligned} c_r &= \sum_{\rho} \frac{(a, b; q^{-2})_{\rho_1 + r}}{(q^2; q^2)_\rho} \prod_{i=1}^{j-1} < u_{r+\rho_i}/u_r > \\ &\quad \times \sum_{\sigma} \frac{1}{(q^2; q^2)_\sigma} \left(< 1/u_r > + \frac{1}{(q; q^2)_{\sigma_{l-j}}} \right) \prod_{i=1}^{l-j} < u_{\sigma_i}/u_r > . \end{aligned}$$

In view of (63) the inner sum over σ is equal to $b_r(z, r)/(q)_{2r}$, applying (65), we get

$$\begin{aligned} c_r &= (z; q^2)_{2r} q^{\binom{2r}{2}} \frac{1 - zq^{4r-1}}{(zq^{2r-1})_{2r+1}} \frac{(a, b; q^{-2})_r}{(q)_{2r}} \\ &\qquad \times \sum_{\rho} \frac{(aq^{-2r}, bq^{-2r}; q^{-2})_{\rho_1}}{(q^2; q^2)_\rho} \prod_{i=1}^{j-1} < u_{r+\rho_i}/u_r > . \end{aligned}$$

Now, the sum over ρ can be computed using (68) with a, b and z replaced by aq^{-2r}, bq^{-2r} and zq^{4r}, respectively. After simplification, we obtain

$$c_r = q^{\binom{2r}{2}} \frac{(z; q^2)_\infty}{(zq^{2r-1})_\infty} \frac{(a, b; q^{-2})_r (azq^{2r+1}, bzq^{2r+1}; q^2)_\infty}{(q)_{2r} (abzq; q^2)_\infty} (1 - zq^{4r-1}),$$

which completes the proof.

5 Proofs through Bailey's method

A classical approach to identities of Rogers-Ramanujan type is based on Bailey's method (see [3, 14]). Recall that a pair of sequences (α_n, β_n) is a *Bailey pair* if there are two parameters x and q such that (see for example [3, p. 25-26]) :

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r} (xq)_{n+r}} \qquad \forall n \ge 0. \tag{69}$$

If (α_n, β_n) is a Bailey pair then Bailey's lemma [3, p. 25-26] states that (α'_n, β'_n) is also a Bailey pair, where

$$\alpha'_n = \frac{(\rho_1)_n (\rho_2)_n (xq/\rho_1\rho_2)^n}{(xq/\rho_1)_n (xq/\rho_2)_n} \alpha_n$$

and

$$\beta'_n = \sum_{j\geq 0} \frac{(\rho_1)_j(\rho_2)_j(xq/\rho_1\rho_2)^j}{(q)_{n-j}(xq/\rho_1)_n(xq/\rho_2)_n}\beta_j.$$

In [2, 3] Andrews noticed that applying Bailey's lemma to the same Bailey pair iteratively leads to a Bailey chain, which yields almost straightforwardly multiple identities of Rogers-Ramanujan type.

In what follows we shall briefly indicate how to derive our identity (18), from which we derived our six multisum identities (12)-(17), through this method.

Our starting point is Theorem 3.4 of Andrews [3]. Indeed, letting $N \to \infty$ and for $i = 1, \cdots, k-1$, letting $b_i \to \infty$, $c_i \to \infty$ and setting $b_k = a^{-1}$ and $c_k = b^{-1}$ in [3, Theorem 3.4], we obtain

$$\frac{(xq,\, abxq)_\infty}{(axq,\, bxq)_\infty} \sum_{l(\lambda)\leq k} q^{n_2(\lambda)-\lambda_1^2+\lambda_1} x^{|\lambda|} (a^{-1}, b^{-1})_{\lambda_1} (ab)^{\lambda_1} \frac{(q)_{\lambda_k}}{(q)_\lambda} \beta_{\lambda_k} \tag{70}$$

$$= \sum_{n\geq 0} q^{(k-1)n^2+n} x^{kn} \frac{(a^{-1}, b^{-1})_n (ab)^n}{(axq,\, bxq)_n} \alpha_n,$$

where $(\alpha_n,\, \beta_n)$ is a Bailey pair.

Now, invoking the following Bailey pair $(\alpha_n,\, \beta_n)$ [10, F(1)] : $\alpha_0 = \beta_0 = 1$ and for $n \geq 1$

$$\alpha_n = q^{n^2}(q^{n/2} + q^{-n/2}), \qquad \beta_n = \frac{1}{(q^{1/2}, q)_n}, \tag{71}$$

and plugging it in (70) with $x = 1$ yields (18) after replacing q by q^2.

It is interesting to note that (23) and (24) are consequences of Bailey's lemma with Slater's pair (71), but they did not appear in [10, 11].

We note that Stembridge [12] derived his sixteen multianalogs of Rogers-Ramanujan type from the following specializations of his Theorem 3.4 :

$$\frac{(q, abq)_\infty}{(aq, bq)_\infty} \sum_{l(\lambda)\leq k} q^{n_2(\lambda)-\lambda_1^2+\lambda_1} (ab)^{\lambda_1} \frac{(a^{-1}, b^{-1})_{\lambda_1}}{(q)_\lambda} \tag{72}$$

$$= \sum_{n\geq 0} q^{(k+\frac{1}{2})n^2+\frac{1}{2}n} (-ab)^n \frac{(a^{-1}, b^{-1})_n}{(aq, bq)_n} (1+q^n),$$

$$\frac{(q, abq^2)_\infty}{(aq^2, bq^2)_\infty} \sum_{l(\lambda)\leq k} q^{n_2(\lambda)+|\lambda|-\lambda_1^2+\lambda_1} (ab)^{\lambda_1} \frac{(a^{-1}, b^{-1})_{\lambda_1}}{(q)_\lambda} \tag{73}$$

$$= \sum_{n\ge 0} q^{(k+\frac12)n^2+(k+\frac32)n}(-ab)^n \frac{(a^{-1},b^{-1})_n}{(aq^2,bq^2)_n}(1-q^{2n+1}),$$

$$\frac{(-aq,q)_\infty}{(-q,aq^2)_\infty}\sum_{l(\lambda)\le k} q^{\frac12(n_2(\lambda)+|\lambda|-\lambda_1^2+\lambda_1)}(-a)^{\lambda_1}\frac{(a^{-1})_{\lambda_1}}{(q)_\lambda} \tag{74}$$

$$= \sum_{n\ge 0} q^{\frac{k+1}{2}(n^2+n)} a^n \frac{(a^{-1})_n}{(aq^2)_n}(1-q^{2n+1}),$$

$$\frac{(-aq^{1/2},q)_\infty}{(-q^{1/2},aq)_\infty}\sum_{l(\lambda)\le k} q^{\frac12(n_2(\lambda)-\lambda_1^2+\lambda_1)}(-a)^{\lambda_1}\frac{(a^{-1})_{\lambda_1}}{(q)_\lambda} \tag{75}$$

$$= \sum_{n\ge 0} q^{\frac{k+1}{2}n^2} a^n \frac{(a^{-1})_n}{(aq)_n}(1+q^n).$$

In the same vein we can derive the above four identities from [3, Theorem 3.4]. For example, for (72) take $x=1$ in (70) and use the Bailey pair B(1) of [10], and for (73) take $x=q$ in (70) and use the Bailey pair B(3) of [10]. For (74) and (75), we need another specialization of [3, Theorem 3.4]. Letting $N\to\infty$, $b_i\to\infty$ for $i=1,\cdots,k-1$ and setting $b_k=a^{-1}$ and $c_i=-\sqrt{xq}$ for $i=1,\cdots,k$ in [3, Theorem 3.4] we obtain

$$\frac{(xq,\,-a\sqrt{xq})_\infty}{(axq,\,-\sqrt{xq})_\infty}\sum_{l(\lambda)\le k} q^{\frac12(n_2(\lambda)-\lambda_1^2+\lambda_1)}x^{\frac12|\lambda|}(a^{-1})_{\lambda_1}(-a)^{\lambda_1} \tag{76}$$

$$\times\frac{(-\sqrt{xq},q)_{\lambda_k}}{(q)_\lambda}\beta_{\lambda_k} = \sum_{n\ge 0} q^{\frac12((k-1)n^2+n)}x^{\frac12 kn}\frac{(a^{-1})_n(-a)^n}{(axq)_n}\alpha_n,$$

where $(\alpha_n,\,\beta_n)$ is a Bailey pair.

Taking $x=q$ in (76) and using the Bailey pair E(3) of Slater [10] yields (74). For (75), take $x=1$ in (76) and use the following Bailey pair [10, p. 468] : $\alpha_0=\beta_0=1$ and for $n\ge 1$

$$\alpha_n=(-1)^n q^{n^2}(q^{n/2}+q^{-n/2}),\qquad \beta_n=\frac{1}{(-q^{1/2},q)_n}. \tag{77}$$

Recently, Bressoud, Ismail and Stanton [4] have pointed out that the sixteen multisum identities, but not the above four more general identities, in Stembridge [12] can be proved by means of *change of base in Bailey pairs.*

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