BILATERAL BAILEY LATTICES AND ANDREWS–GORDON TYPE IDENTITIES

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Abstract. We show that the Bailey lattice can be extended to a bilateral version in just a few lines from the bilateral Bailey lemma, using a very simple lemma transforming bilateral Bailey pairs related to a into bilateral Bailey pairs related to a/q. Using this lemma and similar ones, we give bilateral versions and simple proofs of other (new and known) Bailey lattices, among which a Bailey lattice of Warnaar and the inverses of Bailey lattices of Lovejoy. As consequences of our bilateral point of view, we derive new m-versions of the Andrews–Gordon identities, Bressoud's identities, a new companion to Bressoud's identities, and the Bressoud–Göllnitz–Gordon identities. Finally, we give a new elementary proof of another very general identity of Bressoud using one of our Bailey lattices.

1. Introduction and statement of results

A classical approach to obtain and prove q-series identities is the Bailey lemma, originally found by Bailey [Bai49], and whose iterative strength was later highlighted by Andrews [And84, And86, AAR99] through the so-called Bailey chain. Fix a complex number a. Recall [AAR99] that a Bailey pair $(\alpha_n, \beta_n)_{n\geq 0}$ related to a is a pair of sequences satisfying:

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}} \quad \forall\, n \in \mathbb{N}. \tag{1.1}$$

Here and throughout the paper, we use standard q-series notations which can be found in [GR04]:

$$(a)_\infty = (a;q)_\infty := \prod_{j\geq 0}(1-aq^j) \quad \text{and} \quad (a)_k = (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty},$$

where k is any integer, and

$$(a_1,\dots,a_m)_k := (a_1)_k \cdots (a_m)_k,$$

where k is an integer or infinity.

The Bailey lemma describes how, from a Bailey pair, one can produce infinitely many of them.

Theorem 1.1 (Bailey lemma). *If (α_n, β_n) is a Bailey pair related to a, then so is (α'_n, β'_n), where*

$$\alpha'_n = \frac{(\rho,\sigma)_n (aq/\rho\sigma)^n}{(aq/\rho, aq/\sigma)_n}\,\alpha_n$$

and

$$\beta'_n = \sum_{j=0}^{n} \frac{(\rho,\sigma)_j (aq/\rho\sigma)_{n-j}(aq/\rho\sigma)^j}{(q)_{n-j}(aq/\rho, aq/\sigma)_n}\,\beta_j.$$

Despite its quite elementary proof, as it only requires the q-analog of the Pfaff–Saalschütz formula (see [GR04, Appendix (II.12)]), which is itself consequence of the q-binomial theorem, it yields many formulas in q-series, some of which are highly non trivial. For instance, in [AAR99], the following unit Bailey pair (related to a) is considered (proving that it is indeed a Bailey pair is elementary, it can be done either directly or by inverting the relation (1.1)):

$$\alpha_n = (-1)^n q^{\binom{n}{2}} \frac{1-aq^{2n}}{1-a}\frac{(a)_n}{(q)_n}, \qquad \beta_n = \delta_{n,0}. \tag{1.2}$$

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Applying twice Theorem 1.1 to the unit Bailey pair (1.2) yields a simple proof of the famous Rogers–Ramanujan identities [RR19].

Theorem 1.2 (Rogers–Ramanujan identities)**.** *Let $i = 0$ or 1. Then*

$$\sum_{n\geq 0} \frac{q^{n^2+(1-i)n}}{(q)_n} = \frac{1}{(q^{2-i}, q^{3+i}; q^5)_\infty}.$$

Iterating $r \geq 2$ times this process yields the $i = 1$ and $i = r$ special instances of the Andrews–Gordon identities [And74, Gor61].

Theorem 1.3 (Andrews–Gordon identities)**.** *Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. We have*

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0} \frac{q^{s_1^2+\cdots+s_{r-1}^2+s_i+\cdots+s_{r-1}}}{(q)_{s_1-s_2}\dots(q)_{s_{r-2}-s_{r-1}}(q)_{s_{r-1}}} = \frac{(q^{2r+1}, q^i, q^{2r-i+1}; q^{2r+1})_\infty}{(q)_\infty}. \tag{1.3}$$

However it is not possible to prove the cases $1 < i < r$ of the Andrews–Gordon identities with only the Bailey chain. Thus the Bailey lattice was developed in [AAB87] as a more general tool which enabled the authors to give a proof of the full Andrews–Gordon identities. The key point is to change the parameter a to a/q at some point before iterating the Bailey lemma, therefore providing a concept of Bailey lattice instead of the classical Bailey chain described above.

Here is the classical Bailey lattice proved in [AAB87]. Its proof is more involved than for Theorem 1.1 as it relies on identities for terminating q-series generalizing the above-mentionned q-Pfaff–Saalschütz formula.

Theorem 1.4 (Bailey lattice)**.** *If (α_n, β_n) is a Bailey pair related to a, then (α'_n, β'_n) is a Bailey pair related to a/q, where*

$$\alpha'_0 = \alpha_0, \quad \alpha'_n = \frac{(\rho, \sigma)_n (a/\rho\sigma)^n}{(a/\rho, a/\sigma)_n}(1-a)\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right),$$

and

$$\beta'_n = \sum_{j=0}^{n} \frac{(\rho, \sigma)_j (a/\rho\sigma)_{n-j} (a/\rho\sigma)^j}{(q)_{n-j}(a/\rho, a/\sigma)_n} \beta_j.$$

Alternatively, Andrews, Schilling and Warnaar showed in [ASW99, Section 3] that it is possible to prove (1.3) by combining the Bailey lemma with tricky calculations, therefore bypassing the Bailey lattice. In [BIS00], it is also explained how a change of base allows one to avoid using the Bailey lattice. Recenty, McLaughlin [McL18] showed that (1.3) can be proved much more easily by combining the classical Bailey Lemma with a simple lemma (see also the result of Lovejoy [Lov22, Lemma 2.2] which corresponds to the case $a = q$ of McLaughlin's result).

Lemma 1.5 (McLaughlin)**.** *If (α_n, β_n) is a Bailey pair related to a, then (α'_n, β'_n) is a Bailey pair related to a/q, where*

$$\alpha'_0 = \alpha_0, \quad \alpha'_n = (1-a)\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right), \quad \beta'_n = \beta_n.$$

In this paper, we will show, among other things, that this lemma and the Bailey lattice can be extended to bilateral versions.

As noted in [BMS96] and [Jou10], it is possible to define for all $n \in \mathbb{Z}$ a *bilateral Bailey pair* (α_n, β_n) related to a by the relation:

$$\beta_n = \sum_{j\leq n} \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}} \quad \forall\, n \in \mathbb{Z}. \tag{1.4}$$

Remark 1.6. The relation (1.1) defining classical (unilateral) Bailey pairs is a special instance of the above relation defining bilateral ones, as choosing $\alpha_n = 0$ for negative integers n in (1.4) implies $\beta_n = 0$ for n negative. Actually, the converse is also true, as the classical Bailey inversion holds for bilateral Bailey pairs: (α_n, β_n) is a bilateral Bailey pair related to a if and only if

$$\alpha_n = \frac{1-aq^{2n}}{1-a} \sum_{j\leq n} \frac{(a)_{n+j}}{(q)_{n-j}} (-1)^{n-j} q^{\binom{n-j}{2}} \beta_j \quad \forall\, n \in \mathbb{Z}. \tag{1.5}$$

Thus, from all our results in this paper, one can deduce the corresponding unilateral results by setting $\alpha_n = 0$ (or equivalently $\beta_n = 0$) for all $n < 0$.

In [BMS96], the Bailey lemma is extended in the following way.

Theorem 1.7 (Bilateral Bailey lemma). *If (α_n, β_n) is a bilateral Bailey pair related to a, then so is (α'_n, β'_n), where*

$$\alpha'_n = \frac{(\rho, \sigma)_n (aq/\rho\sigma)^n}{(aq/\rho, aq/\sigma)_n} \alpha_n,$$

and

$$\beta'_n = \sum_{j \leq n} \frac{(\rho, \sigma)_j (aq/\rho\sigma)_{n-j} (aq/\rho\sigma)^j}{(q)_{n-j} (aq/\rho, aq/\sigma)_n} \beta_j,$$

subject to convergence conditions on the sequences α_n and β_n, which make the relevant infinite series absolutely convergent.

Our first result is an extension of the Bailey lattice to the bilateral case.

Theorem 1.8 (Bilateral Bailey lattice). *If (α_n, β_n) is a bilateral Bailey pair related to a, then (α'_n, β'_n) is a bilateral Bailey pair related to a/q, where*

$$\alpha'_n = \frac{(\rho, \sigma)_n (a/\rho\sigma)^n}{(a/\rho, a/\sigma)_n} (1-a) \left(\frac{\alpha_n}{1 - aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1 - aq^{2n-2}} \right),$$

and

$$\beta'_n = \sum_{j \leq n} \frac{(\rho, \sigma)_j (a/\rho\sigma)_{n-j} (a/\rho\sigma)^j}{(q)_{n-j} (a/\rho, a/\sigma)_n} \beta_j,$$

subject to convergence conditions on the sequences α_n and β_n, which make the relevant infinite series absolutely convergent.

While, as mentioned above, several proofs have been given for the (unilateral) Bailey lattice, none of them can be considered very simple. On the other hand, simpler proofs were given to prove the Andrews–Gordon identities without using the Bailey lattice. Here we give a very simple proof of our bilateral Bailey lattice, which when considering Bailey pairs such that $\alpha_n = 0$ for $n < 0$ reduces to the unilateral Bailey lattice. Hence we provide in particular the first very simple proof of the classical Bailey lattice.

The key in our proof is the following simple lemma, which generalises McLaughlin's unilateral Lemma 1.5 and transforms bilateral Bailey pairs related to a into bilateral Bailey pairs related to a/q.

Lemma 1.9 (Key lemma 1). *If (α_n, β_n) is a bilateral Bailey pair related to a, then (α'_n, β'_n) is a bilateral Bailey pair related to a/q, where*

$$\alpha'_n = (1-a) \left(\frac{\alpha_n}{1 - aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1 - aq^{2n-2}} \right), \quad \beta'_n = \beta_n, \tag{1.6}$$

subject to convergence conditions on the sequences α_n and β_n, which make the relevant infinite series absolutely convergent.

While it is not customary to do so, we give the proof in this introduction to show that it is just a few lines long and only requires the definition of bilateral Bailey pairs and elementary sum manipulations which are similar to the ones for the unilateral version.

Proof of Lemma 1.9. For all $n \in \mathbb{Z}$, we have

$$\begin{aligned}
\sum_{j \leq n} \frac{\alpha'_j}{(q)_{n-j}(a)_{n+j}} &= \sum_{j \leq n} \frac{(1-a)}{(q)_{n-j}(a)_{n+j}} \left(\frac{\alpha_j}{1 - aq^{2j}} - \frac{aq^{2j-2}\alpha_{j-1}}{1 - aq^{2j-2}} \right) \\
&= \sum_{j \leq n} \frac{(1-a)\alpha_j}{(q)_{n-j}(a)_{n+j}(1 - aq^{2j})} - \sum_{j \leq n} \frac{(1-a)(1-q^{n-j})aq^{2j}\alpha_j}{(q)_{n-j}(a)_{n+j+1}(1 - aq^{2j})} \\
&= \sum_{j \leq n} \frac{(1-a)\alpha_j}{(q)_{n-j}(a)_{n+j+1}(1 - aq^{2j})} \left((1 - aq^{n+j}) - aq^{2j}(1 - q^{n-j}) \right)
\end{aligned}$$

$$= \sum_{j \le n} \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}} = \beta_n = \beta'_n,$$

which is the desired result by (1.4). □

With this lemma, we can give an extremely simple proof of the bilateral Bailey lattice.

Proof of the bilateral Bailey lattice. Start from a bilateral Bailey pair (α_n, β_n) related to a. Then apply Lemma 1.9 to obtain a bilateral Bailey pair $(\tilde{\alpha}_n, \tilde{\beta}_n)$ related to a/q and satisfying (1.6). Applying the bilateral Bailey lemma (Theorem 1.7) to $(\tilde{\alpha}_n, \tilde{\beta}_n)$ with a replaced by a/q gives the desired bilateral Bailey pair (α'_n, β'_n) related to a/q. □

In addition to Lemma 1.9, let us give a similar simple lemma whose unilateral version is also due to McLaughlin in [McL18, Lemma 13.1 (2)] (see also Lovejoy [Lov22, Lemma 3.1]), and whose proof is very similar to the one of Lemma 1.9.

Lemma 1.10 (Key lemma 2)**.** *If (α_n, β_n) is a bilateral Bailey pair related to a, then (α'_n, β'_n) is a bilateral Bailey pair related to a/q, where*

$$\alpha'_n = (1-a)\left(\frac{q^n \alpha_n}{1-aq^{2n}} - \frac{q^{n-1}\alpha_{n-1}}{1-aq^{2n-2}}\right), \quad \beta'_n = q^n \beta_n, \tag{1.7}$$

subject to convergence conditions on the sequences α_n and β_n, which make the relevant infinite series absolutely convergent.

Using, as in the short proof of the bilateral Bailey lattice above, Lemma 1.10 followed by Theorem 1.7 with a replaced by a/q, we obtain the following new bilateral Bailey lattice, similar to Theorem 1.8. As far as we know, its unilateral version was also unknown until now.

Theorem 1.11 (New bilateral Bailey lattice)**.** *If (α_n, β_n) is a bilateral Bailey pair related to a, then (α'_n, β'_n) is a bilateral Bailey pair related to a/q, where*

$$\alpha'_n = \frac{(\rho,\sigma)_n (a/\rho\sigma)^n}{(a/\rho, a/\sigma)_n}(1-a)\left(\frac{q^n \alpha_n}{1-aq^{2n}} - \frac{q^{n-1}\alpha_{n-1}}{1-aq^{2n-2}}\right),$$

and

$$\beta'_n = \sum_{j \le n} \frac{(\rho,\sigma)_j (a/\rho\sigma)_{n-j} (a/\rho\sigma)^j}{(q)_{n-j}(a/\rho, a/\sigma)_n}\, q^j \beta_j,$$

subject to convergence conditions on the sequences α_n and β_n, which make the relevant infinite series absolutely convergent.

Lemmas 1.9 and 1.10 can be generalised by adding an extra parameter b.

Lemma 1.12 (General lemma)**.** *If (α_n, β_n) is a bilateral Bailey pair related to a, then (α'_n, β'_n) is a bilateral Bailey pair related to a/q, where*

$$\alpha'_n = (1-a)\left(\frac{1-bq^n}{1-b}\frac{\alpha_n}{1-aq^{2n}} - \frac{q^{n-1}(aq^{n-1}-b)}{1-b}\frac{\alpha_{n-1}}{1-aq^{2n-2}}\right) \tag{1.8}$$

and

$$\beta'_n = \frac{(bq)_n}{(b)_n}\beta_n. \tag{1.9}$$

Remark 1.13. Lemma 1.9 is the case $b = 0$ and Lemma 1.10 is the case $b \to \infty$ of Lemma 1.12.

Remark 1.14. While at first glance Lemma 1.12 seems more general than Lemmas 1.9 and 1.10, it is actually equivalent to these two lemmas taken together. Indeed, the bilateral Bailey pair in Lemma 1.12 is equal to $1/(1-b)$ times the bilateral Bailey pair of Lemma 1.9 minus $b/(1-b)$ times the bilateral Bailey pair of Lemma 1.10. Using the fact that being a bilateral Bailey pair is stable under linear combination, Lemmas 1.9 and 1.10 imply Lemma 1.12. Note that it is also possible to prove Lemma 1.12 directly with a similar method to the proof of Lemma 1.9.

Despite following from Lemmas 1.9 and 1.10, this general Lemma 1.12 is still interesting as it provides in the unilateral case an "inverse" to Lovejoy's Lemma 2.3 of [Lov22], which he first stated in [Lov04, (2.4) and (2.5)].

Lemma 1.15 (Lovejoy)**.** *If (α_n, β_n) is a Bailey pair related to a, then (α'_n, β'_n) is a Bailey pair related to aq, where*

$$\alpha'_n = \frac{(1-aq^{2n+1})(aq/b)_n(-b)^n q^{n(n-1)/2}}{(1-aq)(bq)_n} \sum_{r=0}^{n} \frac{(b)_r}{(aq/b)_r}(-b)^{-r}q^{-r(r-1)/2}\alpha_r,$$

and

$$\beta'_n = \frac{(b)_n}{(bq)_n}\beta_n.$$

Moreover, from Lemma 1.12, we deduce a very general theorem transforming bilateral Bailey pairs related to a into bilateral Bailey pairs related to aq^{-N}. Recall the M-th elementary symmetric polynomial in N variables defined for $0 \leq M \leq N$ as

$$e_M(X_1, \ldots, X_N) = \sum_{\substack{I \subseteq \{1,\ldots,N\} \\ s.t. |I| = M}} \prod_{i \in I} X_i,$$

and $e_M(X_1, \ldots, X_N) = 0$ if $M < 0$ or $M > N$. Recall also that for all $0 \leq j \leq N$, the q-binomial coefficient is defined by

$$\begin{bmatrix} N \\ j \end{bmatrix}_q = \begin{bmatrix} N \\ j \end{bmatrix} := \frac{(q)_N}{(q)_j(q)_{N-j}}.$$

We extend this definition to $j < 0$ and $j > N$ by setting $\begin{bmatrix} N \\ j \end{bmatrix} = 0$, which is consistent with the definition of q-Pochhammer symbols with negative indices given above. The general theorem can be stated as follows.

Theorem 1.16 (New bilateral Bailey lattice in higher dimension)**.** *Let (α_n, β_n) be a bilateral Bailey pair related to a. For all $N \geq 1$, define the pair $(\alpha_n^{(N)}, \beta_n^{(N)})$ by*

$$\alpha_n^{(N)} = \frac{(1-aq^{2n-N})(aq^{1-N})_N}{(1-b_1)\cdots(1-b_N)} \sum_{j\in\mathbb{Z}} (-1)^j \frac{q^{jn-j(j+1)/2} f_{N,j,n}(b_1,\ldots,b_N)}{(aq^{2n-N-j})_{N+1}} \alpha_{n-j}, \tag{1.10}$$

where

$$f_{N,j,n}(b_1,\ldots,b_N) := \sum_{M\in\mathbb{Z}} \left((-1)^M \sum_{u\in\mathbb{Z}} a^{j-u} q^{(M-u)(n-u)+(j-u)(n-N)} \begin{bmatrix} M \\ u \end{bmatrix} \begin{bmatrix} N-M \\ j-u \end{bmatrix} \right) e_M(b_1,\ldots,b_N), \tag{1.11}$$

and

$$\beta_n^{(N)} = \frac{(b_1q,\ldots,b_Nq)_n}{(b_1,\ldots,b_N)_n}\beta_n. \tag{1.12}$$

Then, $(\alpha_n^{(N)}, \beta_n^{(N)})$ is a bilateral Bailey pair related to aq^{-N}.

Remark 1.17. When $j < 0$ or $j > N$, we have $f_{N,j,n}(b_1,\ldots,b_N) = 0$ because of the q-binomial coefficients in its definition. Therefore the sum in (1.10) is actually finite.

Again, Theorem 1.16 can be seen in the unilateral case as the inverse of a theorem of Lovejoy [Lov04, Theorem 2.3].

Theorem 1.18 (Lovejoy)**.** *Let (α_n, β_n) be a Bailey pair related to a. For all $N \geq 1$, define the pair $(\alpha_n^{(N)}, \beta_n^{(N)})$ by*

$$\alpha_n^{(N)} = \frac{(1-aq^{2n+N})(aq^N/b_N)_n(-b_N)^n q^{n(n-1)/2}}{(1-aq^N)(b_Nq)_n} \times \sum_{n\geq n_N\geq\cdots\geq n_1\geq 0} \frac{(1-aq^{2n_2+1})\cdots(1-aq^{2n_N+N-1})(aq/b_1)_{n_2}\cdots(aq^{N-1}/b_{N-1})_{n_N}}{(1-aq)\cdots(1-aq^{N-1})(aq/b_1)_{n_1}\cdots(aq^N/b_N)_{n_N}}$$

$$\times \frac{(b_1)_{n_1}\cdots(b_N)_{n_N}}{(b_1q)_{n_2}\cdots(b_{N-1}q)_{n_N}} b_1^{n_2-n_1}\cdots b_{N-1}^{n_N-n_{N-1}} b_N^{-n_N}(-1)^{n_1}q^{-n_1(n_1-1)/2}\alpha_{n_1},$$

and

$$\beta_n^{(N)} = \frac{(b_1,\dots,b_N)_n}{(b_1q,\dots,b_Nq)_n}\beta_n.$$

Then, $(\alpha_n^{(N)}, \beta_n^{(N)})$ *is a Bailey pair related to* aq^N.

As particular cases of Theorem 1.16, we recover and generalise to the bilateral case some Bailey lattices due to Warnaar, as well as discover new simple ones (see Section 3).

Moreover, we take advantage of the bilateral aspect of our results by using a bilateral Bailey pair (see (2.6)) instead of the classical unit Bailey pair, and obtain new generalisations, which we call m-versions, of the Andrews–Gordon identities, the Bressoud identities, and new companions to Bressoud's identities which we very recently discovered combinatorially [DJK23+] (see (2.14)-(2.15)). The m-version of the Andrews–Gordon identities is as follows.

Theorem 1.19 (m-version of the Andrews–Gordon identities)**.** *Let* $m \geq 0$*,* $r \geq 2$*, and* $0 \leq i \leq r$ *be three integers. We have*

$$\sum_{s_1\geq\cdots\geq s_r\geq-\lfloor m/2\rfloor} \frac{q^{s_1^2+\cdots+s_r^2+m(s_1+\cdots+s_r)-s_1-\cdots-s_i}}{(q)_{s_1-s_2}\cdots(q)_{s_{r-1}-s_r}}(-1)^{s_r}q^{\binom{s_r}{2}}\begin{bmatrix} m+s_r \\ m+2s_r\end{bmatrix}$$
$$= \sum_{k=0}^{i} q^{mk}\frac{(q^{2r+1},q^{(m+1)r-i+2k},q^{(1-m)r+i-2k+1};q^{2r+1})_\infty}{(q)_\infty}. \quad (1.13)$$

The m-version of our new companions to Bressoud's identities is the following.

Theorem 1.20 (m-version of our identities)**.** *Let* $m \geq 0$*,* $r \geq 2$*, and* $0 \leq i \leq r$ *be three integers. We have*

$$\sum_{s_1\geq\cdots\geq s_r\geq-\lfloor m/2\rfloor} \frac{q^{s_1^2+\cdots+s_r^2+m(s_1+\cdots+s_{r-1})-s_1-\cdots-s_i+s_{r-1}-2s_r}(-q)_{m+2s_r}}{(q)_{s_1-s_2}\cdots(q)_{s_{r-2}-s_{r-1}}(q^2;q^2)_{s_{r-1}-s_r}}(-1)^{s_r}\begin{bmatrix} m+s_r \\ m+2s_r\end{bmatrix}_{q^2}$$
$$= \sum_{k=0}^{i} q^{mk}\frac{(q^{2r},q^{(m+1)(r-1)-i+2k},q^{(1-m)r+m+i-2k+1};q^{2r})_\infty}{(q)_\infty}. \quad (1.14)$$

The m-version of the classical Bressoud identities (namely the even moduli counterpart of the Andrews–Gordon identities) is a bit less elegant (see Theorem 2.11), which seems to indicate that our new companions are actually more natural with regard to the Bailey lattice approach.

There are other famous identities of the Rogers–Ramanujan type, which where found by Göllnitz [Go67] and Gordon [Gor65] independently, and can be stated as follows.

Theorem 1.21 (Göllnitz–Gordon identities)**.** *Let* $i = 0$ *or* 1*. Then*

$$\sum_{n\geq 0}\frac{q^{n^2+2(1-i)n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^{3-2i},q^4,q^{5+2i};q^8)_\infty}.$$

As for the Rogers–Ramanujan identities, there are combinatorial interpretations and multisum generalizations of the Göllnitz–Gordon identities in the spirit of the Andrews–Gordon identities (1.3) (see for instance the recent paper [HZ23]). Actually, Bressoud proved in [Bre80] three different such generalizations, which are listed as (3.6)–(3.8) in his paper (he also proved another formula of the same kind, namely [Bre80, (3.9)], which is so similar to [Bre80, (3.8)] that it is considered in [HZ23] as a generalization of the Göllnitz–Gordon identities, although it is not *stricto sensu* the case).

While looking for m-versions of all these Bressoud–Göllnitz–Gordon identities, we discovered the following result, which surprisingly interpolates between the classical Bressoud identities and [Bre80, (3.6)].

Theorem 1.22 (m-version of the Bressoud and Bressoud–Göllnitz–Gordon identities)**.** *Let $m \geq 0$, $r \geq 2$, and $0 \leq i \leq r$ be three integers. We have*

$$\sum_{s_1\geq\cdots\geq s_r\geq -\lfloor m/2\rfloor} \frac{q^{s_1^2+\cdots+s_r^2+m(s_1+\cdots+s_r)-s_1-\cdots-s_i-(m+1)s_r/2}(-q^{(m+1)/2})_{s_r}}{(q)_{s_1-s_2}\dots(q)_{s_{r-1}-s_r}(-q^{(m+1)/2})_{s_{r-1}}}(-1)^{s_r}\begin{bmatrix} m+s_r \\ m+2s_r \end{bmatrix}$$
$$= \sum_{k=0}^{i} q^{mk}\frac{(q^{2r}, q^{(m+1)r-i+2k-(m+1)/2}, q^{(1-m)r+i-2k+(m+1)/2}; q^{2r})_\infty}{(q)_\infty}. \quad (1.15)$$

Actually, in [Bre80], Bressoud proved a very general multi-parameter identity (see Theorem 4.1 below), of which the cases $m = 0$ and $m = 1$ of Theorems 1.19, 2.11, and 1.22 are particular cases. This led us to believe that Theorem 4.1 could be proved using the classical Bailey lattice (Theorem 1.4), but we did not succeed. However we managed to prove it in a simple way by using the unilateral version of our new Bailey lattice (Theorem 1.11), see Section 4. Moreover, the cases $m = 0$ and $m = 1$ of Theorem 1.20 do not seem to follow from Theorem 4.1. So the Bailey lattice approach appears to be more general.

The paper is organised as follows. In Section 2, we use our bilateral Bailey lattices to prove general results and deduce m-versions of many classical identities, among which Theorems 1.19–1.22. In Section 3, we give N-iterations of our Bailey lattices, generalise some N-Bailey lattices of Warnaar to the bilateral case, and prove Theorem 1.16. In Section 4, we show how to derive a new proof of Bressoud's theorem with our new Bailey lattice of Theorem 1.11, and why we fail when trying to do the same using the classical Bailey lattice.

2. New m-versions of the Andrews–Gordon identities and others

2.1. **Combining bilateral Bailey lemmas and lattices.** In [AAB87], many applications of the Bailey lattice (Theorem 1.4) are provided, among which a general result, obtained in [AAB87, Theorem 3.1] by iterating $r-i$ times Theorem 1.1, then using Theorem 1.4, and finally $i-1$ times Theorem 1.1 with a replaced by a/q. Using the same process in our bilateral point of view, replacing Theorem 1.1 (resp. Theorem 1.4) by Theorem 1.7 (resp. Theorem 1.8), we derive the following generalisation of [AAB87, Theorem 3.1].

Theorem 2.1. *If (α_n, β_n) is a bilateral Bailey pair related to a, then for all integers $0 \leq i \leq r$ and $n \in \mathbb{Z}$, we have:*

$$\sum_{n\geq s_1\geq\cdots\geq s_r} \frac{a^{s_1+\cdots+s_r}q^{s_{i+1}+\cdots+s_r}\beta_{s_r}}{(\rho_1\sigma_1)^{s_1}\dots(\rho_r\sigma_r)^{s_r}}\frac{(\rho_1,\sigma_1)_{s_1}\dots(\rho_r,\sigma_r)_{s_r}}{(q)_{n-s_1}(q)_{s_1-s_2}\dots(q)_{s_{r-1}-s_r}}$$
$$\times \frac{(a/\rho_1\sigma_1)_{n-s_1}(a/\rho_2\sigma_2)_{s_1-s_2}\dots(a/\rho_i\sigma_i)_{s_{i-1}-s_i}}{(a/\rho_1,a/\sigma_1)_n(a/\rho_2,a/\sigma_2)_{s_1}\dots(a/\rho_i,a/\sigma_i)_{s_{i-1}}}\frac{(aq/\rho_{i+1}\sigma_{i+1})_{s_i-s_{i+1}}\dots(aq/\rho_r\sigma_r)_{s_{r-1}-s_r}}{(aq/\rho_{i+1},aq/\sigma_{i+1})_{s_i}\dots(aq/\rho_r,aq/\sigma_r)_{s_{r-1}}}$$
$$= \sum_{j\leq n}\frac{(\rho_1,\sigma_1,\dots,\rho_i,\sigma_i)_j(\rho_1\sigma_1\dots\rho_i\sigma_i)^{-j}a^{ij}(1-a)}{(q)_{n-j}(a)_{n+j}(a/\rho_1,a/\sigma_1,\dots,a/\rho_i,a/\sigma_i)_j}$$
$$\times\left(\frac{(\rho_{i+1},\sigma_{i+1},\dots,\rho_r,\sigma_r)_j(\rho_{i+1}\sigma_{i+1}\dots\rho_r\sigma_r)^{-j}(aq)^{(r-i)j}\alpha_j}{(aq/\rho_{i+1},aq/\sigma_{i+1},\dots,aq/\rho_r,aq/\sigma_r)_j(1-aq^{2j})}\right.$$
$$\left.-\frac{(\rho_{i+1},\sigma_{i+1},\dots,\rho_r,\sigma_r)_{j-1}(\rho_{i+1}\sigma_{i+1}\dots\rho_r\sigma_r)^{-j+1}(aq)^{(r-i)(j-1)}aq^{2j-2}\alpha_{j-1}}{(aq/\rho_{i+1},aq/\sigma_{i+1},\dots,aq/\rho_r,aq/\sigma_r)_{j-1}(1-aq^{2j-2})}\right), \quad (2.1)$$

subject to convergence conditions on the sequences α_n and β_n, which make the relevant infinite series absolutely convergent.

In this section we will consider the special case below where all parameters $\rho_j, \sigma_j \to \infty$ and at the end $n \to +\infty$, which is a bilateral generalisation of [AAB87, Corollary 4.2]. (We also shifted the index j to $j+1$ in the terms involving α_{j-1}.)

Corollary 2.2. *If (α_n, β_n) is a bilateral Bailey pair related to a, then for all integers $0 \leq i \leq r$, we have:*

$$\sum_{s_1\geq\cdots\geq s_r}\frac{a^{s_1+\cdots+s_r}q^{s_1^2+\cdots+s_r^2-s_1-\cdots-s_i}}{(q)_{s_1-s_2}\dots(q)_{s_{r-1}-s_r}}\beta_{s_r} = \frac{1}{(aq)_\infty}\sum_{j\in\mathbb{Z}} a^{rj}q^{rj^2-ij}\frac{1-a^{i+1}q^{2j(i+1)}}{1-aq^{2j}}\alpha_j, \quad (2.2)$$

subject to convergence conditions on the sequences α_n and β_n, which make the relevant infinite series absolutely convergent.

In [AAB87], Agarwal, Andrews and Bressoud prove the Andrews–Gordon identities (1.3) in the following way. They apply Corollary 2.2 to the unit Bailey pair (1.2) (which we recall is unilateral) with $a = q$, factorise the right-hand side using the Jacobi triple product identity [GR04, Appendix, (II.28)]

$$\sum_{j\in\mathbb{Z}}(-1)^j z^j q^{j(j-1)/2} = (q, z, q/z; q)_\infty, \tag{2.3}$$

and replace i by $i-1$.

Regarding m-versions of Bressoud and Bressoud–Göllnitz–Gordon type identities, we will also need the more general case below where all parameters except ρ_1, ρ_r tend to ∞.

Corollary 2.3. *If (α_n, β_n) is a bilateral Bailey pair related to a, then for all integers $0 \le i \le r$, we have:*

$$\sum_{s_1\geq\cdots\geq s_r} \frac{a^{s_1+\cdots+s_r} q^{s_1^2/2+s_2^2+\cdots+s_{r-1}^2+s_r^2/2-s_1/2-s_2-\cdots-s_i+s_r/2}}{(q)_{s_1-s_2}\ldots(q)_{s_{r-1}-s_r}}(-1)^{s_1+s_r}\frac{(\rho_1)_{s_1}(\rho_r)_{s_r}}{\rho_1^{s_1}\rho_r^{s_r}(aq/\rho_r)_{s_{r-1}}}\beta_{s_r}$$
$$= \frac{(a/\rho_1)_\infty}{(aq)_\infty}\sum_{j\in\mathbb{Z}}\frac{a^{rj}q^{(r-1)j^2-ij+j}}{1-aq^{2j}}\frac{(\rho_1,\rho_r)_j}{\rho_1^j\rho_r^j(a/\rho_1, aq/\rho_r)_j}\left(1+a^{i+1}q^{j(2i+1)}\frac{1-\rho_1 q^j}{\rho_1-aq^j}\right)\alpha_j, \tag{2.4}$$

subject to convergence conditions on the sequences α_n and β_n, which make the relevant infinite series absolutely convergent.

Of course when $\rho_1, \rho_r \to \infty$ in (2.4), one gets (2.2)

Remark 2.4. One can wonder whether applying the same process as above to the other bilateral Bailey lattice of Theorem 1.11 could bring anything interesting. It is of course possible de prove a result similar to Theorem 2.1 by using Theorem 1.11 instead of Theorem 1.8. However it is useless for our purpose, as this would yield a consequence equivalent to the one above (namely Corollary 2.3 with i replaced by $i-1$).

2.2. **Bilateral Bailey pairs.** In [Jou10], the bilateral Bailey lemma given in Theorem 1.7 is studied in particular by considering the case where $a = q^m$ for a nonnegative integer m (this instance is called shifted Bailey lemma in [Jou10]). The following bilateral Bailey pair, which was already mentioned in another form in [ASW99], is considered:

$$\alpha_n = (-1)^n q^{\binom{n}{2}} \quad \text{and} \quad \beta_n = (q)_m(-1)^n q^{\binom{n}{2}}\begin{bmatrix} m+n \\ m+2n\end{bmatrix}. \tag{2.5}$$

Taking $m = 0$ and $m = 1$ in (2.5) yields Bailey pairs equivalent to the cases $a = 1$ and $a = q$ of the unit Bailey pair (1.2). Note that choosing $\beta_n = \delta_{n,0}$ and computing α_n by the inversion (1.5) would not provide a new bilateral Bailey pair, as can be seen by Remark 1.6: it would give back the usual unit Bailey pair (1.2). However, to use in full generality the bilateral point of view while keeping a general, it would be natural to consider

$$\alpha_n = (-1)^{n+m} q^{\binom{n+m}{2}}\frac{1-aq^{2n}}{1-a}\frac{(a)_{n-m}}{(q)_{n+m}} \quad \text{and} \quad \beta_n = \delta_{n,-m}, \tag{2.6}$$

where we made use of the inversion (1.5). However, applying Corollary 2.2 to the bilateral Bailey pair (2.6) does not provide any interesting generalisation (like the m-versions of the next section in the case of (2.5)) of (1.3), but a formula which is equivalent to (1.3) for all m. Indeed, by applying (2.2) to (2.6) and replacing the index j by $j-m$, we obtain:

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq -m}\frac{a^{s_1+\cdots+s_{r-1}-m}q^{s_1^2+\cdots+s_{r-1}^2+m^2-s_1-\cdots-s_i}}{(q)_{s_1-s_2}\ldots(q)_{s_{r-1}+m}} = \frac{a^{-rm}q^{rm^2+im}}{(aq)_\infty}$$
$$\times\sum_{j\in\mathbb{Z}}a^{rj}q^{rj^2-2rmj-ij}\frac{1-a^{i+1}q^{2(j-m)(i+1)}}{1-a}(-1)^j q^{\binom{j}{2}}\frac{(a)_{j-2m}}{(q)_j}.$$

Shifting s_k to s_k+m for all k, and replacing a by q^{2m+1} then yields

$$q^{-rm^2+im-rm}\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0}\frac{q^{s_1^2+\cdots+s_{r-1}^2+s_{i+1}+\cdots+s_{r-1}}}{(q)_{s_1-s_2}\dots(q)_{s_{r-1}}}=\frac{q^{-rm^2+im-rm}}{(q^{2m+2})_\infty}$$
$$\times\sum_{j\geq 0}q^{rj^2+rj-ij}\frac{1-q^{(2j+1)(i+1)}}{1-q^{2m+1}}(-1)^jq^{\binom{j}{2}}\frac{(q^{2m+1})_{j-2m}}{(q)_j}.$$

Using $(q)_{2m}(q^{2m+1})_{j-2m}=(q)_j$, we see that the dependence in m disappears and we get (1.3) by replacing i by $i-1$ and using (2.3).

2.3. **m-versions of the Andrews–Gordon identities.** Recall that the Andrews–Gordon identities (1.3) arise in [Bre80] in pair with a similar formula [Bre80, (3.3)], valid for all integers $r\geq 2$ and $0\leq i\leq r-1$:

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0}\frac{q^{s_1^2+\cdots+s_{r-1}^2-s_1-\cdots-s_i}}{(q)_{s_1-s_2}\dots(q)_{s_{r-1}}}=\sum_{k=0}^{i}\frac{(q^{2r+1},q^{r-i+k},q^{r+i-k+1};q^{2r+1})_\infty}{(q)_\infty}. \tag{2.7}$$

Note that there is a small mistake in Bressoud's paper: in his formula [Bre80, (3.3)], $\pm(k-r+i)$ (in his notation) has to be changed to $\pm(k-r+i+1)$. Identity (2.7) is explained combinatorially in [DJK23+], while it is used in [ADJM23] to solve a combinatorial conjecture of Afsharijoo arising from commutative algebra.

We show that (1.3) and (2.7) can be embedded in a single formula involving the integer m from the previous subsection: this is Theorem 1.19, the m-version of the Andrews–Gordon identities. Our proof relies on Corollary 2.2, which itself is a consequence of our bilateral Bailey lattice.

Proof of Theorem 1.19. Apply Corollary 2.2 to the bilateral Bailey pair (2.5) with $a=q^m$ and divide both sides by $(q)_m$, this yields the desired left-hand side of (1.13). Regarding the right-hand side, one gets

$$\frac{1}{(q)_\infty}\sum_{j\in\mathbb{Z}}q^{rj^2-ij+mrj}\frac{1-q^{(m+2j)(i+1)}}{1-q^{m+2j}}(-1)^jq^{\binom{j}{2}},$$

which by expanding the quotient yields

$$\frac{1}{(q)_\infty}\sum_{k=0}^{i}q^{mk}\sum_{j\in\mathbb{Z}}q^{rj^2-ij+mrj+2kj}(-1)^jq^{\binom{j}{2}}=\frac{1}{(q)_\infty}\sum_{k=0}^{i}q^{mk}\sum_{j\in\mathbb{Z}}(-1)^jq^{(2r+1)\binom{j}{2}}q^{j((m+1)r-i+2k)}.$$

This gives the result by using the Jacobi triple product identity (2.3). □

The case $i=0$ of Theorem 1.19 is Theorem 2.3 (2.3) in [Jou10], where specialisations of this formula are also studied further. Taking $m=0$ in (1.13) forces the index s_r to be 0, therefore the left-hand side is the one of (2.7). The right-hand sides actually also coincide: it is obvious for the even indices $2k$ on the right-hand side of (2.7) (for $0\leq 2k\leq i$), while the odd indices $2k+1$ correspond to indices $i-k$ on the right-hand side of (1.13). Taking $m=1$ in (1.13) also yields s_r to be 0, therefore the left-hand side is the one of (1.3) (in which i is replaced by $i+1$). Regarding the right-hand sides, the one of (1.3) is given by the first term $k=0$ in (1.13) (with i replaced by $i-1$), while the sum from 1 to i actually cancels, even though it is not immediate at first sight.

2.4. **m-versions of Bressoud's even moduli counterparts.** In [Bre79], Bressoud found the counterpart for even moduli to the Andrews–Gordon identities (1.3):

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0}\frac{q^{s_1^2+\cdots+s_{r-1}^2+s_i+\cdots+s_{r-1}}}{(q)_{s_1-s_2}\dots(q)_{s_{r-2}-s_{r-1}}(q^2;q^2)_{s_{r-1}}}=\frac{(q^{2r},q^i,q^{2r-i};q^{2r})_\infty}{(q)_\infty}, \tag{2.8}$$

where $r\geq 2$ and $1\leq i\leq r$ are fixed integers. As for the Andrews–Gordon identities, there is a counterpart for (2.8) similar to (2.7) which is proved in [Bre80, (3.5)] and explained combinatorially in [DJK23+]:

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0}\frac{q^{s_1^2+\cdots+s_{r-1}^2-s_1-\cdots-s_i}}{(q)_{s_1-s_2}\dots(q)_{s_{r-2}-s_{r-1}}(q^2;q^2)_{s_{r-1}}}=\sum_{k=0}^{i}\frac{(q^{2r},q^{r-i+2k},q^{r+i-2k};q^{2r})_\infty}{(q)_\infty}, \tag{2.9}$$

for all integers $r\geq 2$ and $0\leq i\leq r-1$.

In this subsection, we aim to find a generalisation of both formulas above, in the spirit of Theorem 1.19. To do so, we will need the following bilateral version of [BIS00, Theorem 2.5], which changes the basis q to q^2.

Theorem 2.5. *If $(\alpha_n,\,\beta_n)$ is a bilateral Bailey pair related to a, then so is $(\alpha'_n,\,\beta'_n)$, where*

$$\alpha'_n = \frac{(-b)_n}{(-aq/b)_n}\frac{1+a}{1+aq^{2n}}b^{-n}q^{n-\binom{n}{2}}\alpha_n(a^2,q^2)$$

and

$$\beta'_n = \sum_{j\le n}\frac{(-a)_{2j}(b^2;q^2)_j(q^{-j+1}/b,bq^j)_{n-j}}{(b,-aq/b)_n(q^2;q^2)_{n-j}}b^{-j}q^{j-\binom{j}{2}}\beta_j(a^2,q^2),$$

provided the relevant series are absolutely convergent. Here $\alpha_n(a^2,q^2)$ and $\beta_n(a^2,q^2)$ means that a and q are replaced by a^2 and q^2 in the bilateral Bailey pair.

Proof. As in [BIS00], we only need to use the definition (1.4) of a bilateral Bailey pair, interchange summations and apply Formula (2.2) in [BIS00]. □

As a consequence, letting $b\to+\infty$, we derive the following bilateral Bailey pair, therefore generalizing (D4) in [BIS00]:

$$\alpha'_n = \frac{1+a}{1+aq^{2n}}q^n\alpha_n(a^2,q^2) \quad\text{and}\quad \beta'_n = \sum_{j\le n}\frac{(-a)_{2j}}{(q^2;q^2)_{n-j}}q^j\beta_j(a^2,q^2). \tag{2.10}$$

Now we are ready to give our result.

Theorem 2.6 (m-version of the Bressoud identities)**.** *Let $m\ge 0$, $r\ge 2$, and $0\le i\le r$ be three integers. We have*

$$\sum_{s_1\ge\cdots\ge s_r\ge -\lfloor m/2\rfloor}\frac{q^{s_1^2+\cdots+s_r^2+m(s_1+\cdots+s_{r-1})-s_1-\cdots-s_i}(-q)_{m+2s_r-1}}{(q)_{s_1-s_2}\cdots(q)_{s_{r-2}-s_{r-1}}(q^2;q^2)_{s_{r-1}-s_r}}(-1)^{s_r}\begin{bmatrix}m+s_r\\ m+2s_r\end{bmatrix}_{q^2} = a_m, \tag{2.11}$$

where

$$a_{2m} = \sum_{k=0}^{i}\sum_{\ell=0}^{2m}(-1)^\ell q^{2mk+2m\ell}\frac{(q^{2r},q^{2m(r-1)+r-i+2k+2\ell},q^{r+i-2m(r-1)-2k-2\ell};q^{2r})_\infty}{2(q)_\infty},$$

and

$$a_{2m+1} = (-1)^m q^{(2-r)m^2+(1+i-r)m}\sum_{\ell=0}^{m}q^{2\ell}\frac{(q^{2r},q^{2r-2m-1-i+4\ell},q^{i+2m+1-4\ell};q^{2r})_\infty}{(q)_\infty}.$$

Proof. We start from the bilateral Bailey pair (2.5) with $a=q^m$, to which we apply (2.10). This results in the bilateral Bailey pair:

$$\alpha_n = (-1)^n q^{n^2}\frac{1+q^m}{1+q^{m+2n}} \quad\text{and}\quad \beta_n = (q^2;q^2)_m\sum_{j\le n}(-1)^j q^{j^2}\frac{(-q^m)_{2j}}{(q^2;q^2)_{n-j}}\begin{bmatrix}m+j\\ m+2j\end{bmatrix}_{q^2}.$$

Then apply Corollary 2.2 with $a=q^m$ and r replaced by $r-1$ to the above bilateral Bailey pair and divide both sides by $(1+q^m)(q)_m$: the left-hand side is the desired one (β_n above is $\beta_{s_{r-1}}$ while $j=s_r$). The right-hand side is equal to

$$a_m = \frac{1}{(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+m(r-1)j}\frac{1-q^{(m+2j)(i+1)}}{1-q^{m+2j}}\frac{1}{1+q^{m+2j}}. \tag{2.12}$$

Shifting the index of summation j to $-j-m$ yields after rearranging:

$$a_m = \frac{1}{(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+m(r-1)j}\frac{1-q^{(m+2j)(i+1)}}{1-q^{m+2j}}\frac{(-1)^m q^{(m+2j)(m+1)}}{1+q^{m+2j}}. \tag{2.13}$$

Therefore we get by adding (2.12) and (2.13):

$$a_{2m} = \frac{1}{2(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+2m(r-1)j}\frac{1-q^{(2m+2j)(i+1)}}{1-q^{2m+2j}}\frac{1+q^{(2m+2j)(2m+1)}}{1+q^{2m+2j}},$$

in which we can expand both quotients and obtain the desired result by using (2.3). Summing (2.12) and (2.13) gives in the odd case:

$$a_{2m+1} = \frac{1}{2(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{rj^2-ij+(2m+1)(r-1)j} \left(1-q^{(2m+1+2j)(i+1)}\right) \frac{1-q^{(2m+1+2j)(2m+2)}}{1-q^{2(2m+1+2j)}}.$$

Expanding the quotient and using (2.3) yields

$$a_{2m+1} = \frac{1}{2(q)_\infty} \left(\sum_{\ell=0}^{m} q^{(4m+2)\ell} (q^{2r}, q^{(2m+1)(r-1)+r-i+4\ell}, q^{i-(2m+1)(r-1)-r-4\ell}; q^{2r})_\infty \right.$$
$$\left. - \sum_{\ell=0}^{m} q^{(2m+1)(i+1)+(4m+2)\ell} (q^{2r}, q^{(2m+1)(r-1)+r+i+2+4\ell}, q^{-i-2-(2m+1)(r-1)-r-4\ell}; q^{2r})_\infty \right).$$

Then observe that in the first sum we can use

$$q^{(4m+2)\ell}(q^{(2m+1)(r-1)+r-i+4\ell}, q^{i-(2m+1)(r-1)-r-4\ell}; q^{2r})_\infty$$
$$= (-1)^m q^{(2-r)m^2+(1+i-r)m+2\ell} (q^{2r-2m-1-i+4\ell}, q^{i+2m+1-4\ell}; q^{2r})_\infty,$$

while in the second we have

$$q^{(2m+1)(i+1)+((4m+2)\ell}(q^{(2m+1)(r-1)+r+i+2\ell}, q^{-i-2-(2m+1)(r-1)-r-4\ell}; q^{2r})_\infty$$
$$= (-1)^{m+1} q^{(2-r)m^2+(1+i-r)m-2\ell+2m} (q^{-2m+1+i+4\ell}, q^{2r+2m-i-1-4\ell}; q^{2r})_\infty.$$

Therefore replacing ℓ by $m-\ell$ in the second sum in a_{2m+1} above yields the result. □

Taking $m = 0$ in (2.11) forces the index s_r to be 0, therefore we obtain the identity (2.9) multiplied by 1/2. Taking $m = 1$ in (2.11) also yields s_r to be 0, therefore we get (2.8) in which i is replaced by $i + 1$.

Remark 2.7. Contrary to Theorem 1.19, we had to consider the parity of m to use the Jacobi triple product (2.3) in (2.12). However, we managed to find a general expression for a_m, but only when i is odd, writing

$$a_m = \frac{1}{(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{rj^2-ij+m(r-1)j} \frac{1-q^{(2m+4j)(i+1)/2}}{1-q^{2m+4j}}.$$

This gives for odd i and any nonnegative inter m:

$$a_m = \sum_{k=0}^{(i-1)/2} q^{2mk} \frac{(q^{2r}, q^{m(r-1)+r-i+4k}, q^{r+i-m(r-1)-4k}; q^{2r})_\infty}{(q)_\infty}.$$

2.5. **m-versions of new even moduli counterparts.** In [DJK23+], while studying combinatorial interpretations of the Andrews–Gordon and Bressoud identities ((1.3), (2.7) and (2.8)–(2.9)), the authors discovered in a purely combinatorial way the following pair of formulas, to be compared with (2.8) and (2.9):

$$(1+q) \sum_{s_1\geq\cdots\geq s_{r-1}\geq 0} \frac{q^{s_1^2+\cdots+s_{r-1}^2+s_i+\cdots+s_{r-2}+2s_{r-1}}}{(q)_{s_1-s_2}\cdots(q)_{s_{r-2}-s_{r-1}}(q^2;q^2)_{s_{r-1}}}$$
$$= \frac{1}{(q)_\infty}\left((q^{2r}, q^{2r-i-1}, q^{i+1}; q^{2r})_\infty + q(q^{2r}, q^{2r-i+1}, q^{i-1}; q^{2r})_\infty\right), \quad (2.14)$$

where $r \geq 2$ and $1 \leq i \leq r$, and

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0} \frac{q^{s_1^2+\cdots+s_{r-1}^2-s_1-\cdots-s_i+s_{r-1}}}{(q)_{s_1-s_2}\cdots(q)_{s_{r-2}-s_{r-1}}(q^2;q^2)_{s_{r-1}}} = \sum_{k=0}^{i} \frac{(q^{2r}, q^{r-i+2k-1}, q^{r+i-2k+1}; q^{2r})_\infty}{(q)_\infty}, \quad (2.15)$$

where $r \geq 2$ and $0 \leq i \leq r-1$.

Again, we are able to embed (2.14) and (2.15) into a general m-version, namely Theorem 1.20. To do this, instead of (2.10), we use the following bilateral Bailey pair generalizing (D1) in [BIS00], and which is considered in [Jou10]:

$$\alpha'_n = \alpha_n(a^2,q^2) \quad \text{and} \quad \beta'_n = \sum_{j\leq n} \frac{(-aq)_{2j}}{(q^2;q^2)_{n-j}} q^{n-j}\beta_j(a^2,q^2). \tag{2.16}$$

Proof of Theorem 1.20. We start from the bilateral Bailey pair (2.5) with $a = q^m$, to which we apply (2.16). This results in the bilateral Bailey pair:

$$\alpha_n = (-1)^n q^{n^2-n} \quad \text{and} \quad \beta_n = (q^2;q^2)_m \sum_{j\leq n} (-1)^j q^{j^2+n-2j} \frac{(-q^{1+m})_{2j}}{(q^2;q^2)_{n-j}} \begin{bmatrix} m+j \\ m+2j \end{bmatrix}_{q^2}.$$

Then apply Corollary 2.2 with $a = q^m$ and r replaced by $r-1$ to the above bilateral Bailey pair and divide both sides by $(q)_m$, the left-hand side is the desired one (β_n above is $\beta_{s_{r-1}}$ while $j = s_r$). The right-hand side is equal to

$$\frac{1}{(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{rj^2-(i+1)j+m(r-1)j} \frac{1-q^{(m+2j)(i+1)}}{1-q^{m+2j}},$$

which by expanding the quotient yields

$$\frac{1}{(q)_\infty} \sum_{k=0}^{i} q^{mk} \sum_{j\in\mathbb{Z}} (-1)^j q^{rj^2-(i+1)j+m(r-1)j+2kj} = \frac{1}{(q)_\infty} \sum_{k=0}^{i} q^{mk} \sum_{j\in\mathbb{Z}} (-1)^j q^{2r\binom{j}{2}} q^{j((m+1)(r-1)-i+2k)}.$$

This gives the result by using the Jacobi triple product identity (2.3). □

The case $i = 0$ is Theorem 3.2 in [Jou10]. Taking $m = 0$ in (1.14) forces the index s_r to be 0, therefore we get (2.15). Taking $m = 1$ in (1.14) also yields s_r to be 0, therefore the left-hand side is the one of (2.14) (in which i is replaced by $i+1$). Regarding the right-hand sides, the one of (2.14) is given by the two first terms $k = 0$ and $k = 1$ in (1.14) (with i replaced by $i-1$), while the sum from 2 to i actually cancels, even though it is not immediate at first sight.

2.6. **m-versions of the Bressoud and Bressoud–Göllnitz–Gordon identities.** In [Bre80], in addition to (1.3) and (2.7)–(2.9), Bressoud proved four identities of the same kind, denoted (3.6)–(3.9) in his paper, among which (3.6)–(3.8) generalize the Göllnitz–Gordon identities of Theorem 1.21. In this section, we will give m-versions for all ot these. More precisely, as our m-versions yield nice simplifications in the cases $m = 0$ and $m = 1$, all formulas come in pairs, as in the previous subsections. Formulas (3.6) and (3.7) of [Bre80] will surprisingly arise in pairs with (2.8) and (2.9) respectively, while each of (3.8) and (3.9) of [Bre80] will be associated with formulas which seem to be new.

2.6.1. *m-version of* [Bre80, (3.6)]. First recall (3.6) in [Bre80]:

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0} \frac{q^{2(s_1^2+\cdots+s_{r-1}^2-s_1-\cdots-s_i)}(-q^{1+2s_{r-1}};q^2)_\infty}{(q^2;q^2)_{s_1-s_2}\dots(q^2;q^2)_{s_{r-2}-s_{r-1}}(q^2;q^2)_{s_{r-1}}}$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{k=0}^{i} (q^{4r}, q^{2r-2i+2k-1}, q^{2r+2i-2k+1}; q^{4r})_\infty, \tag{2.17}$$

where $r \geq 2$ and $0 \leq i \leq r-1$ are fixed integers. Note that the parameters in Bressoud's work are renamed $(k,r,i) \to (r,i+1,k)$ to fit with our notation. The appropriate m-version of this formula is given by (1.15) that we prove below. Surprisingly it also gives a m-version of (2.8).

Proof of Theorem 1.22. We start from the bilateral Bailey pair (2.5) with $a = q^m$, to which we apply Corollary 2.3 with $a = q^m, \rho_1 \to \infty, \rho_r = -q^{(m+1)/2}$ and divide both sides by $(q)_m$. The left-hand side is the desired one. The right-hand side is equal to

$$\frac{1}{(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{rj^2-ij+mrj-(m+1)j/2} \frac{1-q^{(m+2j)(i+1)}}{1-q^{m+2j}},$$

which by expanding the quotient yields

$$\frac{1}{(q)_\infty}\sum_{k=0}^{i} q^{mk}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+mrj+2kj-(m+1)j/2} = \frac{1}{(q)_\infty}\sum_{k=0}^{i} q^{mk}\sum_{j\in\mathbb{Z}}(-1)^j q^{2r\binom{j}{2}} q^{j((m+1)r-i+2k-(m+1)/2)}.$$

This gives the result by using the Jacobi triple product identity (2.3). □

Taking $m=0$ in (1.15) forces the index s_r to be 0. Next, shifting $q\to q^2$ and multiplying both sides by

$$(-q;q^2)_\infty = (-q;q^2)_{s_{r-1}}(-q^{1+2s_{r-1}};q^2)_\infty,$$

the left-hand side coincides with the one of (2.17). The right-hand side is also the one of (2.17): it is obvious for the even indices $2k$ on the right-hand side of (2.17) (for $0\le 2k\le i$), while the odd indices $2k+1$ correspond to indices $i-k$ on the right-hand side of (1.15). Taking $m=1$ in (1.15) also yields s_r to be 0, therefore the left-hand side is the one of (2.8) (in which i is replaced by $i+1$). Regarding the right-hand sides, the one of (2.8) (with i replaced by $i+1$) is given by the first term $k=0$ in (1.15), while the sum from 1 to i actually cancels, even though it is not immediate at first sight.

2.6.2. *m-version of* [Bre80, (3.7)]. First recall (3.7) in [Bre80]:

$$\sum_{s_1\ge\cdots\ge s_{r-1}\ge 0}\frac{q^{2(s_1^2+\cdots+s_{r-1}^2+s_{i+1}+\cdots+s_{r-1})}(-q^{3+2s_{r-1}};q^2)_\infty}{(q^2;q^2)_{s_1-s_2}\dots(q^2;q^2)_{s_{r-2}-s_{r-1}}(q^2;q^2)_{s_{r-1}}}$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{k=0}^{i}(-q)^k(q^{4r},q^{2i+1-2k},q^{4r-2i-1+2k};q^{4r})_\infty, \quad (2.18)$$

where $r\ge 2$ and $0\le i\le r-1$ are fixed integers. Note that the parameters in Bressoud's work are again renamed $(k,r,i)\to(r,i+1,k)$ to fit with our notation. Our m-version also extends (2.9) and reads as follows.

Theorem 2.8 (m-version of the Bressoud identities [Bre80, (3.7)])**.** *Let $m\ge 0$, $r\ge 2$, and $0\le i\le r-1$ be three integers. We have*

$$\sum_{s_1\ge\cdots\ge s_r\ge-\lfloor m/2\rfloor}\frac{q^{s_1^2+\cdots+s_r^2+m(s_1+\cdots+s_{r-1}+s_r/2)-s_1-\cdots-s_i}(-q^{m/2})_{s_r}}{(q)_{s_1-s_2}\dots(q)_{s_{r-1}-s_r}(-q^{1+m/2})_{s_{r-1}}}(-1)^{s_r}\begin{bmatrix}m+s_r\\ m+2s_r\end{bmatrix} = b_m, \quad (2.19)$$

where

$$b_{2m} = \frac{1+q^m}{2(q)_\infty}\sum_{k=0}^{i}\sum_{\ell=0}^{2m}(-1)^\ell q^{2mk+m\ell}(q^{2r},q^{2mr+r-i-m+2k+\ell},q^{r+i-2mr+m-2k-\ell};q^{2r})_\infty,$$

and

$$b_{2m+1} = (-1)^m q^{(1-r)m^2+(1+2i-2r)m/2}\frac{1+q^{(2m+1)/2}}{2(q)_\infty}\sum_{k=0}^{i}\sum_{\ell=0}^{m}q^{k+\ell}$$
$$\times\Big((q^{2r},q^{2r-i-m+2k+2\ell-1/2},q^{i+m-2k-2\ell+1/2};q^{2r})_\infty - q^{1/2}(q^{2r},q^{2r-i-m+2k+2\ell+1/2},q^{i+m-2k-2\ell-1/2};q^{2r})_\infty\Big).$$

Proof. We start from the bilateral Bailey pair (2.5) with $a=q^m$, to which we apply Corollary 2.3 with $a=q^m,\rho_1\to\infty,\rho_r=-q^{m/2}$ and divide both sides by $(q)_m$. The left-hand side is the desired one. The right-hand side is equal to

$$b_m = \frac{1+q^{m/2}}{(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+mrj-mj/2}\frac{1-q^{(m+2j)(i+1)}}{1-q^{m+2j}}\frac{1}{1+q^{(m+2j)/2}}. \quad (2.20)$$

As in the proof of Theorem 2.6, shifting the index j to $-j-m$ above and adding the result with (2.20) yields after rearranging:

$$b_m = \frac{1+q^{m/2}}{2(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+mrj-mj/2}\frac{1-q^{(m+2j)(i+1)}}{1-q^{m+2j}}\frac{1+(-1)^m q^{(m+2j)(m+1)/2}}{1+q^{(m+2j)/2}}. \quad (2.21)$$

This gives

$$b_{2m} = \frac{1+q^m}{2(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{rj^2-ij+2mrj-mj} \frac{1-q^{(2m+2j)(i+1)}}{1-q^{2m+2j}} \frac{1+q^{(m+j)(2m+1)}}{1+q^{m+j}},$$

in which we can expand both quotients and obtain the desired result by using (2.3). Equation (2.21) also yields

$$b_{2m+1} = \frac{1+q^{(2m+1)/2}}{2(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{rj^2-ij+(2m+1)rj-(2m+1)j/2} \frac{1-q^{(2m+2j+1)(i+1)}}{1-q^{2m+2j+1}} \frac{1-q^{(2m+2j+1)(m+1)}}{1+q^{(2m+2j+1)/2}}.$$

Using $(1+q^{(2m+2j+1)/2})(1-q^{(2m+2j+1)/2}) = 1-q^{2m+2j+1}$ and expanding both quotients yields

$$b_{2m+1} = \frac{1+q^{(2m+1)/2}}{2(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{rj^2-ij+(2m+1)rj-(2m+1)j/2} \times \left(1-q^{(2m+2j+1)/2}\right) \sum_{k=0}^{i} q^{(2m+2j+1)k} \sum_{\ell=0}^{m} q^{(2m+2j+1)\ell},$$

which, by using (2.3), yields

$$\begin{aligned} b_{2m+1} = \frac{1+q^{(2m+1)/2}}{2(q)_\infty} \sum_{k=0}^{i} \sum_{\ell=0}^{m} q^{(2m+1)(k+\ell)} &\times \Big((q^{2r}, q^{(2m+1)r-i+r-m+2k+2\ell-1/2}, q^{r+i-(2m+1)r+m-2k-2\ell+1/2}; q^{2r})_\infty \\ &\quad - q^{(2m+1)/2} (q^{2r}, q^{(2m+1)r-i+r-m+2k+2\ell+1/2}, q^{r+i-(2m+1)r+m-2k-2\ell-1/2}; q^{2r})_\infty \Big). \end{aligned}$$

The result follows after using manipulations similar to the ones for a_{2m+1} in the proof of Theorem 2.6. $\square$

Taking $m=0$ in (2.19) forces the index s_r to be 0, therefore we obtain (2.9). Taking $m=1$ in (2.19) also yields s_r to be 0. Next, shifting $q \to q^2$ and multiplying both sides by

$$(-q^3;q^2)_\infty = (-q^3;q^2)_{s_{r-1}} (-q^{3+2s_{r-1}};q^2)_\infty,$$

the left-hand side coincides with the one of (2.18). The right-hand side becomes

$$\frac{(-q;q^2)_\infty}{2(q^2;q^2)_\infty} \sum_{k=0}^{i} q^{2k} \left((q^{4r}, q^{4r-2i+4k-1}, q^{2i-4k+1}; q^{4r})_\infty - q (q^{4r}, q^{4r-2i+4k+1}, q^{2i-4k-1}; q^{4r})_\infty \right).$$

This sum is indeed twice the one on the right-hand side of (2.18): to see this, keep the terms k above for $0 \le 2k \le i$ and $0 \le 2k+1 \le i$, and replace k by $i-k$ for the terms k satisfying $i+1 \le 2k \le 2i$ and $i+1 \le 2k+1 \le 2i+1$.

2.6.3. *m-version of* [Bre80, (3.8)]. First recall (3.8) in [Bre80]:

$$\sum_{s_1 \ge \cdots \ge s_{r-1} \ge 0} \frac{q^{2(s_1^2+\cdots+s_{r-1}^2+s_{i+1}+\cdots+s_{r-1})} (-q^{1-2s_1};q^2)_{s_1}}{(q^2;q^2)_{s_1-s_2} \cdots (q^2;q^2)_{s_{r-2}-s_{r-1}} (q^2;q^2)_{s_{r-1}}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} (q^{4r}, q^{2i+1}, q^{4r-2i-1}; q^{4r})_\infty, \tag{2.22}$$

where $r \ge 2$ and $0 \le i \le r-1$ are fixed integers. Note that the parameters in Bressoud's work are again changed by $(k,r,i) \to (r,i+1,k)$ to fit with our notation. Our m-version reads as follows.

Theorem 2.9 (m-version of the Bressoud identities [Bre80, (3.8)]). *Let $m \ge 0$, $r \ge 2$, and $0 \le i \le r-1$ be three integers. We have*

$$\sum_{s_1 \ge \cdots \ge s_r \ge -\lfloor m/2 \rfloor} \frac{q^{s_1^2/2+s_2^2+\cdots+s_r^2+m(s_1/2+s_2+\cdots+s_{r-1})+s_1/2-(s_1+\cdots+s_i)} (-q^{m/2})_{s_1}}{(q)_{s_1-s_2} \cdots (q)_{s_{r-1}-s_r}} (-1)^{s_r} q^{\binom{s_r}{2}} \begin{bmatrix} m+s_r \\ m+2s_r \end{bmatrix} = c_m, \tag{2.23}$$

where

$$c_{2m}=\frac{(-q^m)_\infty}{2(q)_\infty}\sum_{k=0}^{2i}\sum_{\ell=0}^{2m}(-1)^\ell q^{mk+m\ell}(q^{2r},q^{2mr+r-i-m+k+\ell},q^{r+i-2mr+m-k-\ell};q^{2r})_\infty,$$

and

$$c_{2m+1}=(-1)^m q^{(1-r)m^2+(1+2i-2r)m/2}\frac{(-q^{(2m+1)/2})_\infty}{(q)_\infty}\sum_{\ell=0}^{m}q^\ell(q^{2r},q^{2r-i-m+2\ell-1/2},q^{i+m-2\ell+1/2};q^{2r})_\infty.$$

Proof. We start from the bilateral Bailey pair (2.5) with $a=q^m$, to which we apply Corollary 2.3 with $a=q^m,\rho_1=-q^{m/2},\rho_r\to\infty$ and divide both sides by $(q)_m$. The left-hand side is the desired one. The right-hand side is equal to

$$c_m=\frac{(-q^{m/2})_\infty}{(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+mrj-mj/2}\frac{1-q^{(m+2j)(2i+1)/2}}{1-q^{m+2j}}. \tag{2.24}$$

As in the proof of Theorem 2.6, shifting the index j to $-j-m$ above and adding the result with (2.24) yields after rearranging:

$$c_m=\frac{(-q^{m/2})_\infty}{2(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+mrj-mj/2}\frac{1-q^{(m+2j)(2i+1)/2}}{1-q^{m+2j}}\left(1+(-1)^m q^{(m+2j)(m+1)/2}\right). \tag{2.25}$$

This gives

$$c_{2m}=\frac{(-q^m)_\infty}{2(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+2mrj-mj}\frac{1-q^{(m+j)(2i+1)}}{1-q^{m+j}}\frac{1+q^{(m+j)(2m+1)}}{1+q^{m+j}},$$

in which we can expand both quotients and obtain the desired result by using (2.3). Equation (2.25) also yields

$$c_{2m+1}=\frac{(-q^{(2m+1)/2})_\infty}{2(q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{rj^2-ij+(2m+1)rj-(2m+1)j/2}\left(1-q^{(2m+2j+1)(2i+1)/2}\right)\frac{1-q^{(2m+2j+1)(m+1)}}{1-q^{2m+2j+1}},$$

and the result follows after expanding the quotient and using manipulations similar to the ones for a_{2m+1} in the proof of Theorem 2.6. □

Taking $m=0$ in (2.23) forces the index s_r to be 0, therefore we obtain the following identity, which seems to be new.

Corollary 2.10. *Let $r\geq 2$ and $0\leq i\leq r-1$ be two integers. We have*

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0}\frac{q^{s_1^2/2+s_2^2+\cdots+s_{r-1}^2+s_1/2-(s_1+\cdots+s_i)}(-1)_{s_1}}{(q)_{s_1-s_2}\dots(q)_{s_{r-2}-s_{r-1}}(q)_{s_{r-1}}}=\frac{(-q)_\infty}{(q)_\infty}\sum_{k=0}^{2i}(q^{2r},q^{r-i+k},q^{r+i-k};q^{2r})_\infty. \tag{2.26}$$

Taking $m=1$ in (2.23) also yields s_r to be 0. Replacing q by q^2 in the resulting formula therefore yields (2.22) by using

$$(-q;q^2)_{s_1}=q^{s_1^2}(-q^{1-2s_1};q^2)_{s_1}.$$

2.6.4. *m-version of* [Bre80, (3.9)]. First recall (3.9) in [Bre80]:

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0}\frac{q^{2(s_1^2+\cdots+s_{r-1}^2+s_{i+1}+\cdots+s_{r-1})}(-q^{1-2s_1};q^2)_{s_1}}{(q^2;q^2)_{s_1-s_2}\dots(q^2;q^2)_{s_{r-2}-s_{r-1}}(q^4;q^4)_{s_{r-1}}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{4r-2},q^{2i+1},q^{4r-2i-3};q^{4r-2})_\infty, \tag{2.27}$$

where $r\geq 2$ and $0\leq i\leq r-1$ are fixed integers. Note that the parameters in Bressoud's work are again renamed $(k,r,i)\to(r,i+1,k)$ to fit with our notation. Our m-version reads as follows.

Theorem 2.11 (m-version of the Bressoud identities [Bre80, (3.9)])**.** *Let $m\geq 0$, $r\geq 2$, and $0\leq i\leq r-1$ be three integers. We have*

$$\sum_{s_1\geq\cdots\geq s_r\geq -\lfloor m/2\rfloor}\frac{q^{s_1^2/2+s_2^2+\cdots+s_r^2+m(s_1/2+s_2+\cdots+s_{r-1})+s_1/2-(s_1+\cdots+s_i)}(-q^{m/2})_{s_1}(-q^{(m+1)/2})_{s_r}}{(q)_{s_1-s_2}\dots(q)_{s_{r-1}-s_r}(-q^{(m+1)/2})_{s_{r-1}}}$$

$$\times (-1)^{s_r} q^{-(m+1)s_r/2} \begin{bmatrix} m+s_r \\ m+2s_r \end{bmatrix} = d_m, \quad (2.28)$$

where

$$d_{2m} = \frac{(-q^m)_\infty}{2(q)_\infty} \sum_{k=0}^{2i} \sum_{\ell=0}^{2m} (-1)^\ell q^{mk+m\ell} (q^{2r-1}, q^{2mr+r-i-2m+k+\ell-1/2}, q^{r+i+2m-2mr-k-\ell-1/2}; q^{2r-1})_\infty,$$

and

$$d_{2m+1} = (-1)^m q^{(3-2r)m^2/2+(1+i-r)m} \frac{(-q^{(2m+1)/2})_\infty}{(q)_\infty} \sum_{\ell=0}^{m} q^\ell (q^{2r-1}, q^{2r-i-m+2\ell-3/2}, q^{i+m-2\ell+1/2}; q^{2r-1})_\infty.$$

Proof. We start from the bilateral Bailey pair (2.5) with $a = q^m$, to which we apply Corollary 2.3 with $a = q^m, \rho_1 = -q^{m/2}, \rho_r = q^{(m+1)/2}$ and divide both sides by $(q)_m$. The left-hand side is the desired one. The right-hand side is equal to

$$d_m = \frac{(-q^{m/2})_\infty}{(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{(r-1)j^2-ij+m(r-1)j+j^2/2} \frac{1-q^{(m+2j)(2i+1)/2}}{1-q^{m+2j}}. \quad (2.29)$$

As in the proof of Theorem 2.6, shifting the index j to $-j-m$ above and adding the result with (2.29) yields after rearranging:

$$d_m = \frac{(-q^{m/2})_\infty}{2(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{(r-1)j^2-ij+m(r-1)j+j^2/2} \frac{1-q^{(m+2j)(2i+1)/2}}{1-q^{m+2j}} \left(1+(-1)^m q^{(m+2j)(m+1)/2}\right). \quad (2.30)$$

This gives

$$d_{2m} = \frac{(-q^m)_\infty}{2(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{(r-1)j^2-ij+2m(r-1)j+j^2/2} \frac{1-q^{(m+j)(2i+1)}}{1-q^{m+j}} \frac{1+q^{(m+j)(2m+1)}}{1+q^{m+j}},$$

in which we can expand both quotients and obtain the desired result by using (2.3). Equation (2.25) also yields

$$d_{2m+1} = \frac{(-q^{(2m+1)/2})_\infty}{2(q)_\infty} \sum_{j\in\mathbb{Z}} (-1)^j q^{(r-1)j^2-ij+(2m+1)(r-1)j+j^2/2} \left(1-q^{(2m+2j+1)(2i+1)/2}\right) \frac{1-q^{(2m+2j+1)(m+1)}}{1-q^{2m+2j+1}},$$

and the result follows after expanding the quotient and using manipulations similar to the ones for a_{2m+1} in the proof of Theorem 2.6. ☐

Taking $m = 0$ in (2.28) forces the index s_r to be 0, therefore we obtain the following identity, which seems to be new.

Corollary 2.12. *Let $r \geq 2$ and $0 \leq i \leq r-1$ be two integers. We have*

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0} \frac{q^{s_1^2/2+s_2^2+\cdots+s_{r-1}^2+s_1/2-(s_1+\cdots+s_i)}(-1)_{s_1}}{(q)_{s_1-s_2}\cdots(q)_{s_{r-2}-s_{r-1}}(q)_{s_{r-1}}(-q^{1/2})_{s_{r-1}}} = \frac{(-q)_\infty}{(q)_\infty} \sum_{k=0}^{2i} (q^{2r-1}, q^{r-i+k-1/2}, q^{r+i-k-1/2}; q^{2r-1})_\infty. \quad (2.31)$$

Taking $m = 1$ in (2.28) also yields s_r to be 0. Replacing q by q^2 in the resulting formula therefore yields (2.27).

3. Bilateral N-extensions

3.1. **Results.** Using our new bilateral Bailey lattice given in Theorem 1.11 and Lemma 1.12, we were able to deduce Theorem 1.16, a very general bilateral N-Bailey lattice with parameters $b_1, \ldots, b_N$.

The proof of Theorem 1.16, quite technical, is left for the next subsection. However, some of its particular cases, which correspond to the two key lemmas 1.9 and 1.10, are much more simple to state (and to prove), and imply two bilateral N-extensions of the Bailey lattice found by Warnaar in [War01, Theorem 3.1 and Theorem 3.2]. Hence we state them separately here.

When $b_1 = \cdots = b_N = 0$ in Theorem 1.16, the only non-zero $e_M(b_1, \ldots, b_N)$ is $e_0(0, \ldots, 0) = 1$. Hence

$$f_{N,j,n}(0, \ldots, 0) = \sum_{u \in \mathbb{Z}} a^{j-u} q^{(0-u)(n-u)+(j-u)(n-N)} \begin{bmatrix} 0 \\ u \end{bmatrix} \begin{bmatrix} N \\ j-u \end{bmatrix} = a^j q^{j(n-N)} \begin{bmatrix} N \\ j \end{bmatrix},$$

and Theorem 1.16 reduces to the following.

Theorem 3.1 (First new N-Bailey lattice)**.** *Let (α_n, β_n) be a bilateral Bailey pair related to a. For all $N \geq 0$, define the pair $(\alpha_n^{(N)}, \beta_n^{(N)})$ by*

$$\alpha_n^{(N)} = (1 - aq^{2n-N})(aq^{1-N})_N \sum_{j \in \mathbb{Z}} (-1)^j \frac{a^j q^{(2n-N)j - j(j+1)/2}}{(aq^{2n-N-j})_{N+1}} \begin{bmatrix} N \\ j \end{bmatrix} \alpha_{n-j}, \tag{3.1}$$

and

$$\beta_n^{(N)} = \beta_n.$$

Then $(\alpha_n^{(N)}, \beta_n^{(N)})$ is a bilateral Bailey pair related to aq^{-N}.

Applying first Theorem 3.1 to a bilateral Bailey pair related to a, and then Theorem 1.7 with a replaced by aq^{-N} to the resulting Bailey pair, we immediately derive the following result, whose unilateral case is due to Warnaar [War01, Theorem 3.1].

Theorem 3.2 (Warnaar, bilateral version)**.** *Let (α_n, β_n) be a bilateral Bailey pair related to a, and $N \geq 0$ be a fixed integer. Then (α'_n, β'_n) is a bilateral Bailey pair related to aq^{-N}, where*

$$\alpha'_n = \frac{(\rho, \sigma)_n (aq^{1-N}/\rho\sigma)^n}{(aq^{1-N}/\rho, aq^{1-N}/\sigma)_n} (1 - aq^{2n-N})(aq^{1-N})_N \sum_{j=0}^{N} (-1)^j \frac{a^j q^{(2n-N)j - j(j+1)/2}}{(aq^{2n-N-j})_{N+1}} \begin{bmatrix} N \\ j \end{bmatrix} \alpha_{n-j},$$

and

$$\beta'_n = \sum_{j=0}^{n} \frac{(\rho, \sigma)_j (aq^{1-N}/\rho\sigma)_{n-j} (aq^{1-N}/\rho\sigma)^j}{(q)_{n-j} (aq^{1-N}/\rho, aq^{1-N}/\sigma)_n} \beta_j.$$

Note that the bilateral Bailey lattice given in Theorem 1.8 corresponds to the case $N = 1$ in Theorem 3.2.

On the other hand, applying first Theorem 1.7 to a bilateral Bailey pair related to a, and then Theorem 3.1 to the resulting bilateral Bailey pair, we derive the following second result, whose unilateral version is also due to Warnaar [War01, Theorem 3.2].

Theorem 3.3 (Warnaar, bilateral version)**.** *Let (α_n, β_n) be a bilateral Bailey pair related to a, and $N \geq 0$ be a fixed integer. Then (α'_n, β'_n) is a bilateral Bailey pair related to aq^{-N}, where*

$$\alpha'_n = (1 - aq^{2n-N})(aq^{1-N})_N \sum_{j=0}^{N} (-1)^j \frac{a^j q^{(2n-N)j - j(j+1)/2}}{(aq^{2n-N-j})_{N+1}} \begin{bmatrix} N \\ j \end{bmatrix} \frac{(\rho, \sigma)_{n-j} (aq/\rho\sigma)^{n-j}}{(aq/\rho, aq/\sigma)_{n-j}} \alpha_{n-j},$$

and

$$\beta'_n = \sum_{j=0}^{n} \frac{(\rho, \sigma)_j (aq/\rho\sigma)_{n-j} (aq/\rho\sigma)^j}{(q)_{n-j} (aq/\rho, aq/\sigma)_n} \beta_j.$$

When $b_1, \ldots, b_N \to \infty$, the only term $(1 - b_1)^{-1} \cdots (1 - b_N)^{-1} e_M(b_1, \ldots, b_N)$ which does not tend to zero in Theorem 1.16 is for $M = N$. Hence

$$\lim_{b_1, \ldots, b_N \to \infty} \frac{f_{N,j,n}(0, \ldots, 0)}{(1 - b_1) \cdots (1 - b_N)} = \sum_{u \in \mathbb{Z}} a^{j-u} q^{(N-u)(n-u)+(j-u)(n-N)} \begin{bmatrix} N \\ u \end{bmatrix} \times \begin{bmatrix} 0 \\ j-u \end{bmatrix} = q^{(N-j)(n-j)} \begin{bmatrix} N \\ j \end{bmatrix},$$

and Theorem 1.16 reduces to the following.

Theorem 3.4 (Second new N-Bailey lattice)**.** *Let (α_n, β_n) be a bilateral Bailey pair related to a. For all $N \geq 0$, define the pair $(\alpha_n^{(N)}, \beta_n^{(N)})$ by*

$$\alpha_n^{(N)} = (1 - aq^{2n-N})(aq^{1-N})_N \sum_{j \in \mathbb{Z}} (-1)^j \frac{q^{N(n-j) + j(j-1)/2}}{(aq^{2n-N-j})_{N+1}} \begin{bmatrix} N \\ j \end{bmatrix} \alpha_{n-j}, \tag{3.2}$$

and

$$\beta_n^{(N)} = q^{nN}\beta_n.$$

Then $(\alpha_n^{(N)},\, \beta_n^{(N)})$ *is a bilateral Bailey pair related to* aq^{-N}*.*

We give two new theorems similar to Warnaar's N-Bailey lattices, but coming from Theorem 3.4 instead of Theorem 3.1.

Applying first Theorem 3.4 to a Bailey pair related to a, and then Theorem 1.7 with a replaced by aq^{-N} to the resulting bilateral Bailey pair gives the following result.

Theorem 3.5. *Let* $(\alpha_n,\, \beta_n)$ *be a bilateral Bailey pair related to* a*, and* $N \geq 0$ *be a fixed integer. Then* $(\alpha'_n,\, \beta'_n)$ *is a bilateral Bailey pair related to* aq^{-N}*, where*

$$\alpha'_n = \frac{(\rho,\sigma)_n (aq^{1-N}/\rho\sigma)^n}{(aq^{1-N}/\rho, aq^{1-N}/\sigma)_n}(1-aq^{2n-N})(aq^{1-N})_N \sum_{j=0}^{N} (-1)^j \frac{q^{N(n-j)+j(j-1)/2}}{(aq^{2n-N-j})_{N+1}} \begin{bmatrix} N \\ j \end{bmatrix} \alpha_{n-j},$$

and

$$\beta'_n = \sum_{j=0}^{n} \frac{(\rho,\sigma)_j (aq^{1-N}/\rho\sigma)_{n-j} (aq^{1-N}/\rho\sigma)^j}{(q)_{n-j}(aq^{1-N}/\rho, aq^{1-N}/\sigma)_n} q^{jN}\beta_j.$$

Note that Theorem 1.11 corresponds to the case $N = 1$ of Theorem 3.5.

On the other hand, applying first Theorem 1.7 to a bilateral Bailey pair related to a, and then Theorem 3.4 to the resulting bilateral Bailey pair, we derive the following result.

Theorem 3.6. *Let* $(\alpha_n,\, \beta_n)$ *be a bilateral Bailey pair related to* a*, and* $N \geq 0$ *be a fixed integer. Then* $(\alpha'_n,\, \beta'_n)$ *is a bilateral Bailey pair related to* aq^{-N}*, where*

$$\alpha'_n = (1-aq^{2n-N})(aq^{1-N})_N \sum_{j=0}^{N} (-1)^j \frac{q^{N(n-j)+j(j-1)/2}}{(aq^{2n-N-j})_{N+1}} \begin{bmatrix} N \\ j \end{bmatrix} \frac{(\rho,\sigma)_{n-j}(aq/\rho\sigma)^{n-j}}{(aq/\rho, aq/\sigma)_{n-j}} \alpha_{n-j},$$

and

$$\beta'_n = q^{nN} \sum_{j=0}^{n} \frac{(\rho,\sigma)_j (aq/\rho\sigma)_{n-j} (aq/\rho\sigma)^j}{(q)_{n-j}(aq/\rho, aq/\sigma)_n} \beta_j.$$

3.2. **Proof of Theorem 1.16.** Now we can turn to the proof of Theorem 1.16. We will make crucial use of the two classical q-analogues of Pascal's triangle:

$$\begin{bmatrix} N+1 \\ j \end{bmatrix} = q^j \begin{bmatrix} N \\ j \end{bmatrix} + \begin{bmatrix} N \\ j-1 \end{bmatrix}, \tag{3.3}$$

and

$$\begin{bmatrix} N+1 \\ j \end{bmatrix} = \begin{bmatrix} N \\ j \end{bmatrix} + q^{N+1-j} \begin{bmatrix} N \\ j-1 \end{bmatrix}, \tag{3.4}$$

for all integers N, j with $N \geq 0$.

3.2.1. *Recurrence relation for* $f_{N,j,n}(b_1,\ldots,b_N)$. We first prove the following recurrence relation on $f_{N,j,n}(b_1,\ldots,b_N)$ defined in Theorem 1.16, which will play a central role in our proof.

Proposition 3.7 (Recurrence relation)**.** *For all* $0 \leq j \leq N+1$,

$$\begin{aligned}(1-aq^{2n-1-N}) f_{N+1,j,n}(b_1,\ldots,b_{N+1}) = (1-b_{N+1}q^n)(1-aq^{2n-1-N-j}) f_{N,j,n}(b_1,\ldots,b_N) \\ + (aq^{n-1-N} - b_{N+1})(1-aq^{2n-j}) f_{N,j-1,n-1}(b_1,\ldots,b_N).\end{aligned}$$

We start by proving two technical lemmas which rely on the q-analogues of Pascal's triangle.

Lemma 3.8. *For all* $M, j, u \in \mathbb{Z}$,

$$\begin{aligned}\begin{bmatrix} M \\ u \end{bmatrix}\begin{bmatrix} N+1-M \\ j-u \end{bmatrix} - q^{2u-M+1}\begin{bmatrix} M \\ u+1 \end{bmatrix}\begin{bmatrix} N+1-M \\ j-u-1 \end{bmatrix} = q^{j-u}\begin{bmatrix} M \\ u \end{bmatrix}\begin{bmatrix} N-M \\ j-u \end{bmatrix} \\ + q^{u-M}\begin{bmatrix} N-M \\ j-u-1 \end{bmatrix}\left(\begin{bmatrix} M \\ u \end{bmatrix} - \begin{bmatrix} M \\ u+1 \end{bmatrix}\right) - q^{3u-2M+3+N-j}\begin{bmatrix} M \\ u+1 \end{bmatrix}\begin{bmatrix} N-M \\ j-u-2 \end{bmatrix}.\end{aligned}$$

Proof.

$$\begin{aligned}
&\begin{bmatrix} M \\ u \end{bmatrix} \begin{bmatrix} N+1-M \\ j-u \end{bmatrix} - q^{2u-M+1} \begin{bmatrix} M \\ u+1 \end{bmatrix} \begin{bmatrix} N+1-M \\ j-u-1 \end{bmatrix} \\
&= q^{j-u} \begin{bmatrix} M \\ u \end{bmatrix} \begin{bmatrix} N-M \\ j-u \end{bmatrix} + \begin{bmatrix} M \\ u \end{bmatrix} \begin{bmatrix} N-M \\ j-u-1 \end{bmatrix} && \text{by (3.3)} \\
&\quad - q^{2u-M+1} \begin{bmatrix} M \\ u+1 \end{bmatrix} \begin{bmatrix} N-M \\ j-u-1 \end{bmatrix} - q^{3u-2M+3+N-j} \begin{bmatrix} M \\ u+1 \end{bmatrix} \begin{bmatrix} N-M \\ j-u-2 \end{bmatrix} && \text{by (3.4)} \\
&= q^{j-u} \begin{bmatrix} M \\ u \end{bmatrix} \begin{bmatrix} N-M \\ j-u \end{bmatrix} - q^{3u-2M+3+N-j} \begin{bmatrix} M \\ u+1 \end{bmatrix} \begin{bmatrix} N-M \\ j-u-2 \end{bmatrix} \\
&\quad + q^{u-M} \begin{bmatrix} N-M \\ j-u-1 \end{bmatrix} \left(q^{M-u} \begin{bmatrix} M \\ u \end{bmatrix} - q^{u+1} \begin{bmatrix} M \\ u+1 \end{bmatrix} \right) \\
&= q^{j-u} \begin{bmatrix} M \\ u \end{bmatrix} \begin{bmatrix} N-M \\ j-u \end{bmatrix} - q^{3u-2M+3+N-j} \begin{bmatrix} M \\ u+1 \end{bmatrix} \begin{bmatrix} N-M \\ j-u-2 \end{bmatrix} \\
&\quad + q^{u-M} \begin{bmatrix} N-M \\ j-u-1 \end{bmatrix} \left(\begin{bmatrix} M+1 \\ u+1 \end{bmatrix} - \begin{bmatrix} M \\ u+1 \end{bmatrix} - \begin{bmatrix} M+1 \\ u+1 \end{bmatrix} + \begin{bmatrix} M \\ u \end{bmatrix} \right) && \text{by (3.3) and (3.4)} \\
&= q^{j-u} \begin{bmatrix} M \\ u \end{bmatrix} \begin{bmatrix} N-M \\ j-u \end{bmatrix} - q^{3u-2M+3+N-j} \begin{bmatrix} M \\ u+1 \end{bmatrix} \begin{bmatrix} N-M \\ j-u-2 \end{bmatrix} \\
&\quad + q^{u-M} \begin{bmatrix} N-M \\ j-u-1 \end{bmatrix} \left(\begin{bmatrix} M \\ u \end{bmatrix} - \begin{bmatrix} M \\ u+1 \end{bmatrix} \right).
\end{aligned}$$

□

Note that when $M = u = 0$, Lemma 3.8 reduces to (3.3).

Lemma 3.9. *For all $M, j, u \in \mathbb{Z}$,*

$$\begin{aligned}
&\begin{bmatrix} M \\ u \end{bmatrix} \begin{bmatrix} N+1-M \\ j-u \end{bmatrix} - q^{2u-M+1} \begin{bmatrix} M \\ u+1 \end{bmatrix} \begin{bmatrix} N+1-M \\ j-u-1 \end{bmatrix} = q^{j} \begin{bmatrix} M-1 \\ u \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} \\
&- q^{2u-M+1} \begin{bmatrix} M-1 \\ u+1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix} + \begin{bmatrix} M-1 \\ u-1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} - q^{N-M-j+2u+2} \begin{bmatrix} M-1 \\ u \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix}.
\end{aligned}$$

Proof.

$$\begin{aligned}
&\begin{bmatrix} M \\ u \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} - q^{2u-M+1} \begin{bmatrix} M \\ u+1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix} \\
&= q^{u} \begin{bmatrix} M-1 \\ u \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} + \begin{bmatrix} M-1 \\ u-1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} && \text{by (3.3)} \\
&\quad - q^{2u-M+1} \begin{bmatrix} M-1 \\ u+1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix} - q^{u} \begin{bmatrix} M-1 \\ u \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix} && \text{by (3.4)} \\
&= q^{u} \begin{bmatrix} M-1 \\ u \end{bmatrix} \left(\begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} - \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix} \right) \\
&\quad + \begin{bmatrix} M-1 \\ u-1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} - q^{2u-M+1} \begin{bmatrix} M-1 \\ u+1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix} \\
&= q^{u} \begin{bmatrix} M-1 \\ u \end{bmatrix} \left(\begin{bmatrix} N-M+2 \\ j-u \end{bmatrix} - q^{N-M-j+u+2} \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix} \right. \\
&\quad \left. - \begin{bmatrix} N-M+2 \\ j-u \end{bmatrix} + q^{j-u} \begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} \right) \\
&\quad + \begin{bmatrix} M-1 \\ u-1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} - q^{2u-M+1} \begin{bmatrix} M-1 \\ u+1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix} && \text{by (3.4) and (3.3)} \\
&= q^{j} \begin{bmatrix} M-1 \\ u \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} - q^{2u-M+1} \begin{bmatrix} M-1 \\ u+1 \end{bmatrix} \begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix}
\end{aligned}$$

$$+\begin{bmatrix} M-1 \\ u-1 \end{bmatrix}\begin{bmatrix} N-M+1 \\ j-u \end{bmatrix} - q^{N-M-j+2u+2}\begin{bmatrix} M-1 \\ u \end{bmatrix}\begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix}.$$

□

Note that when $M=u=0$, Lemma (3.9) reduces to a combination of (3.3) and (3.4).
We can now prove our recurrence relation.

Proof of Proposition 3.7. Recall from Theorem 1.16 that

$$f_{N,j,n}(b_1,\dots,b_N) = \sum_{M\in\mathbb{Z}}\left((-1)^M\sum_{u\in\mathbb{Z}} a^{j-u}q^{(M-u)(n-u)+(j-u)(n-N)}\begin{bmatrix} M \\ u \end{bmatrix}\begin{bmatrix} N-M \\ j-u \end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N\}\\ s.t. |I|=M}}\prod_{i\in I} b_i.$$

Thus

$$\begin{aligned}
&(1-aq^{2n-1-N})f_{N+1,j,n}(b_1,\dots,b_{N+1}) = \\
&\sum_{M\in\mathbb{Z}}\left((-1)^M\sum_{u\in\mathbb{Z}} a^{j-u}q^{(M-u)(n-u)+(j-u)(n-N-1)}\begin{bmatrix} M \\ u \end{bmatrix}\begin{bmatrix} N-M+1 \\ j-u \end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M}}\prod_{i\in I} b_i \\
&+\sum_{M\in\mathbb{Z}}\left((-1)^{M+1}\sum_{u\in\mathbb{Z}} a^{j-u+1}q^{(M-u)(n-u)+(j-u+1)(n-N-1)+n}\begin{bmatrix} M \\ u \end{bmatrix}\begin{bmatrix} N-M+1 \\ j-u \end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M}}\prod_{i\in I} b_i \\
&=\sum_{M\in\mathbb{Z}}\left((-1)^M\sum_{u\in\mathbb{Z}} a^{j-u}q^{(M-u)(n-u)+(j-u)(n-N-1)}\begin{bmatrix} M \\ u \end{bmatrix}\begin{bmatrix} N-M+1 \\ j-u \end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M}}\prod_{i\in I} b_i \\
&+\sum_{M\in\mathbb{Z}}\left((-1)^{M+1}\sum_{u\in\mathbb{Z}} a^{j-u}q^{(M-u-1)(n-u-1)+(j-u)(n-N-1)+n}\begin{bmatrix} M \\ u+1 \end{bmatrix}\begin{bmatrix} N-M+1 \\ j-u-1 \end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M}}\prod_{i\in I} b_i \\
&=\sum_{M\in\mathbb{Z}}(-1)^M\sum_{u\in\mathbb{Z}} a^{j-u}q^{(M-u)(n-u)+(j-u)(n-N-1)} \\
&\quad\times\left(\begin{bmatrix} M \\ u \end{bmatrix}\begin{bmatrix} N+1-M \\ j-u \end{bmatrix} - q^{2u-M+1}\begin{bmatrix} M \\ u+1 \end{bmatrix}\begin{bmatrix} N+1-M \\ j-u-1 \end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M}}\prod_{i\in I} b_i.
\end{aligned}\tag{3.5}$$

Note that the quantity between parentheses in (3.5) is exactly the left-hand side of Lemmas 3.8 and 3.9. We will now show that the right-hand side of Proposition 3.7 can be written as the sum of two sums, containing the right hand side of Lemmas 3.8 and 3.9 respectively.

We expand, replace M by $M-1$ in the terms which are multiplied by b_{N+1}, and performe a change of the variable u to make all powers of a coincide. This yields:

$$\begin{aligned}
&(1-b_{N+1}q^n)(1-aq^{2n-1-N-j})f_{N,j,n}(b_1,\dots,b_N) + (aq^{n-1-N}-b_{N+1})(1-aq^{2n-j})f_{N,j-1,n-1}(b_1,\dots,b_N) \\
&=\sum_{M\in\mathbb{Z}}\left((-1)^M\sum_{u\in\mathbb{Z}} a^{j-u}q^{(M-u)(n-u)+(j-u)(n-N)}\begin{bmatrix} M \\ u \end{bmatrix}\begin{bmatrix} N-M \\ j-u \end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\notin I}}\prod_{i\in I} b_i \\
&+\sum_{M\in\mathbb{Z}}\left((-1)^{M+1}\sum_{u\in\mathbb{Z}} a^{j-u}q^{(M-u-1)(n-u-1)+(j-u-1)(n-N)+2n-1-N-j}\begin{bmatrix} M \\ u+1 \end{bmatrix}\begin{bmatrix} N-M \\ j-u-1 \end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\notin I}}\prod_{i\in I} b_i
\end{aligned}$$

$$+\sum_{M\in\mathbb{Z}}\left((-1)^M\sum_{u\in\mathbb{Z}}a^{j-u}q^{(M-u)(n-u-1)+(j-u-1)(n-N-1)+n-1-N}\begin{bmatrix}M\\u\end{bmatrix}\begin{bmatrix}N-M\\j-u-1\end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\notin I}}\prod_{i\in I}b_i$$

$$+\sum_{M\in\mathbb{Z}}\left((-1)^{M+1}\sum_{u\in\mathbb{Z}}a^{j-u}q^{(M-u-1)(n-u-2)+(j-u-2)(n-N-1)+3n-N-j-1}\begin{bmatrix}M\\u+1\end{bmatrix}\begin{bmatrix}N-M\\j-u-2\end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\notin I}}\prod_{i\in I}b_i$$

$$+\sum_{M\in\mathbb{Z}}\left((-1)^M\sum_{u\in\mathbb{Z}}a^{j-u}q^{(M-u-1)(n-u)+(j-u)(n-N)+n}\begin{bmatrix}M-1\\u\end{bmatrix}\begin{bmatrix}N-M+1\\j-u\end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\in I}}\prod_{i\in I}b_i$$

$$+\sum_{M\in\mathbb{Z}}\left((-1)^{M+1}\sum_{u\in\mathbb{Z}}a^{j-u}q^{(M-u-2)(n-u-1)+(j-u-1)(n-N)+3n-1-N-j}\begin{bmatrix}M-1\\u+1\end{bmatrix}\begin{bmatrix}N-M+1\\j-u-1\end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\in I}}\prod_{i\in I}b_i$$

$$+\sum_{M\in\mathbb{Z}}\left((-1)^M\sum_{u\in\mathbb{Z}}a^{j-u}q^{(M-u)(n-u)+(j-u)(n-N-1)}\begin{bmatrix}M-1\\u-1\end{bmatrix}\begin{bmatrix}N-M+1\\j-u\end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\in I}}\prod_{i\in I}b_i$$

$$+\sum_{M\in\mathbb{Z}}\left((-1)^{M+1}\sum_{u\in\mathbb{Z}}a^{j-u}q^{(M-u-1)(n-u-1)+(j-u-1)(n-N-1)+2n-j}\begin{bmatrix}M-1\\u\end{bmatrix}\begin{bmatrix}N-M+1\\j-u-1\end{bmatrix}\right)\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\in I}}\prod_{i\in I}b_i$$

$$=\sum_{M\in\mathbb{Z}}(-1)^M\sum_{u\in\mathbb{Z}}a^{j-u}q^{(M-u)(n-u)+(j-u)(n-N-1)}$$

$$\times\Bigg[\left(q^{j-u}\begin{bmatrix}M\\u\end{bmatrix}\begin{bmatrix}N-M\\j-u\end{bmatrix}+q^{u-M}\begin{bmatrix}N-M\\j-u-1\end{bmatrix}\left(\begin{bmatrix}M\\u\end{bmatrix}-\begin{bmatrix}M\\u+1\end{bmatrix}\right)\right.$$

$$\left.-q^{3u-2M+3+N-j}\begin{bmatrix}M\\u+1\end{bmatrix}\begin{bmatrix}N-M\\j-u-2\end{bmatrix}\right)\times\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\notin I}}\prod_{i\in I}b_i$$

$$+\left(q^j\begin{bmatrix}M-1\\u\end{bmatrix}\begin{bmatrix}N-M+1\\j-u\end{bmatrix}-q^{2u-M+1}\begin{bmatrix}M-1\\u+1\end{bmatrix}\begin{bmatrix}N-M+1\\j-u-1\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}M-1\\u-1\end{bmatrix}\begin{bmatrix}N-M+1\\j-u\end{bmatrix}-q^{N-M-j+2u+2}\begin{bmatrix}M-1\\u\end{bmatrix}\begin{bmatrix}N-M+1\\j-u-1\end{bmatrix}\right)\times\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t. |I|=M\\ \text{and } N+1\in I}}\prod_{i\in I}b_i\Bigg]$$

$$=\sum_{M\in\mathbb{Z}}(-1)^M\sum_{u\in\mathbb{Z}}a^{j-u}q^{(M-u)(n-u)+(j-u)(n-N-1)}\left(\begin{bmatrix}M\\u\end{bmatrix}\begin{bmatrix}N+1-M\\j-u\end{bmatrix}-q^{2u-M+1}\begin{bmatrix}M\\u+1\end{bmatrix}\begin{bmatrix}N+1-M\\j-u-1\end{bmatrix}\right)$$

$$\times\left(\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t.|I|=M\\ \text{and } N+1\in I}}\prod_{i\in I}b_i+\sum_{\substack{I\subseteq\{1,\dots,N+1\}\\ s.t.|I|=M\\ \text{and } N+1\notin I}}\prod_{i\in I}b_i\right)\qquad\text{by Lemmas 3.8 and 3.9.}$$

□

This is exactly (3.5), so Proposition 3.7 is proved.

3.2.2. *Proof of Theorem 1.16.* Now we use Proposition 3.7 and Lemma 1.12 to prove Theorem 1.16.

We proceed by induction on N. For $N=0$, by (1.12), $\beta_n^{(0)}=\beta_n$ and by (1.10),

$$\alpha_n^{(0)}=(1-aq^{2n})\frac{\alpha_n}{1-aq^{2n}}=\alpha_n.$$

Thus $(\alpha_n^{(0)},\beta_n^{(0)})$ is a bilateral Bailey pair related to a.

Now, assume that for some integer $N\ge 0$, $(\alpha_n^{(N)},\beta_n^{(N)})$ is a bilateral Bailey pair related to aq^{-N} and show that $(\alpha_n^{(N+1)},\beta_n^{(N+1)})$ is a bilateral Bailey pair related to aq^{-N-1}. By Lemma 1.12 with $b=b_{N+1}$, (α'_n,β'_n) is a bilateral Bailey pair related to aq^{-N-1}, where

$$\alpha'_n=(1-aq^{-N})\left(\frac{1-b_{N+1}q^n}{1-b_{N+1}}\frac{\alpha_n^{(N)}}{1-aq^{2n-N}}-\frac{q^{n-1}(aq^{n-N-1}-b_{N+1})}{1-b_{N+1}}\frac{\alpha_{n-1}^{(N)}}{1-aq^{2n-N-2}}\right),$$

and

$$\beta'_n=\frac{(b_{N+1}q)_n}{(b_{N+1})_n}\beta_n^{(N)}.$$

Now let us show that $(\alpha'_n,\beta'_n)=(\alpha_n^{(N+1)},\beta_n^{(N+1)})$.

We have

$$\beta'_n=\frac{(b_{N+1}q)_n}{(b_{N+1})_n}\beta_n^{(N)}=\frac{(b_1q,\dots,b_Nq,b_{N+1}q)_n}{(b_1,\dots,b_N,b_{N+1})_n}\beta_n=\beta_n^{(N+1)},$$

and by (1.10),

$$\begin{aligned}
\alpha'_n&=(1-aq^{-N})\left(\frac{1-b_{N+1}q^n}{1-b_{N+1}}\frac{\alpha_n^{(N)}}{1-aq^{2n-N}}-\frac{q^{n-1}(aq^{n-N-1}-b_{N+1})}{1-b_{N+1}}\frac{\alpha_{n-1}^{(N)}}{1-aq^{2n-N-2}}\right)\\
&=\frac{(aq^{-N})_{N+1}}{(1-b_1)\cdots(1-b_{N+1})}\left(\sum_{j\in\mathbb{Z}}(-1)^j\frac{q^{jn-j(j+1)/2}(1-b_{N+1}q^n)f_{N,j,n}(b_1,\dots,b_N)}{(aq^{2n-N-j})_{N+1}}\alpha_{n-j}\right.\\
&\qquad\left.+\sum_{j\in\mathbb{Z}}(-1)^{j+1}\frac{q^{(j+1)(n-1)-j(j+1)/2}(aq^{n-1-N}-b_{N+1})f_{N,j,n-1}(b_1,\dots,b_N)}{(aq^{2n-2-N-j})_{N+1}}\alpha_{n-1-j}\right)\\
&=\frac{(aq^{-N})_{N+1}}{(1-b_1)\cdots(1-b_{N+1})}\sum_{j\in\mathbb{Z}}(-1)^j\frac{q^{jn-j(j+1)/2}}{(aq^{2n-N-j-1})_{N+2}}\Big((1-b_{N+1}q^n)(1-aq^{2n-N-j-1})f_{N,j,n}(b_1,\dots,b_N)\\
&\qquad+(aq^{n-1-N}-b_{N+1})(1-aq^{2n-j})f_{N,j-1,n-1}(b_1,\dots,b_N)\Big)\alpha_{n-j}.
\end{aligned}$$

Now by Proposition 3.7, this equals

$$\begin{aligned}
\alpha'_n&=\frac{(aq^{-N})_{N+1}}{(1-b_1)\cdots(1-b_{N+1})}\sum_{j\in\mathbb{Z}}(-1)^j\frac{q^{jn-j(j+1)/2}}{(aq^{2n-N-j-1})_{N+2}}(1-aq^{2n-1-N})f_{N+1,j,n}(b_1,\dots,b_{N+1})\alpha_{n-j}\\
&=\alpha_n^{(N+1)}.
\end{aligned}$$

Thus the pair $(\alpha_n^{(N+1)},\beta_n^{(N+1)})$ is indeed a Bailey pair related to aq^{-N-1}.

4. A new proof of Bressoud's identity

In this last section, we show that the unilateral version of Theorem 1.11 can be used to give a simple proof of Bressoud's identity [Bre80] which is an analytic generalisation of the Rogers–Ramanujan identities.

Theorem 4.1 (Bressoud). *For integers $0 < r < k$ and parameters a, c_1, c_2, $b_1, \dots, b_{2r-1}$, we have:*

$$\sum_{s_1 \geq \cdots \geq s_{k-1} \geq 0} (-1)^{s_1} \frac{a^{s_1+\cdots+s_{k-1}} q^{s_1^2/2+s_{r+1}^2+\cdots+s_{k-1}^2-s_1/2+s_r}}{b_1^{s_1}(b_2 b_{2r-1})^{s_2}\dots(b_r b_{r+1})^{s_r}}$$
$$\times \frac{(aq/c_1c_2)_{s_{k-1}}}{(q, aq/c_1, aq/c_2)_{s_{k-1}}} \frac{(b_1)_{s_1}(b_2, b_{2r-1})_{s_2}\dots(b_r, b_{r+1})_{s_r}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-2}-s_{k-1}}} \frac{(a/b_2b_{2r-1})_{s_1-s_2}\dots(a/b_rb_{r+1})_{s_{r-1}-s_r}}{(a/b_2, a/b_{2r-1})_{s_1}\dots(a/b_r, a/b_{r+1})_{s_{r-1}}}$$
$$= \frac{(a/b_1)_\infty}{(a)_\infty} \sum_{j\geq 0} \frac{(b_1, \dots, b_{2r-1}, c_1, c_2, a)_j (b_1 \dots b_{2r-1}c_1c_2)^{-j} a^{kj} q^{(k-r)j^2+j}}{(a/b_1, \dots, a/b_{2r-1}, aq/c_1, aq/c_2, q)_j}$$
$$\times \left(1 + \frac{a^r q^j}{b_1 \dots b_{2r-1}} \frac{(1-b_1q^j)\dots(1-b_{2r-1}q^j)}{(1-aq^j/b_1)\dots(1-aq^j/b_{2r-1})}\right). \quad (4.1)$$

To see why the result below is actually equivalent to Bressoud's formula, see the following subsection.

The open problem of giving a combinatorial proof of Theorem 4.1 (when parameters $c_1, c_2 \to \infty$ and the other parameters have specific forms), known as Bressoud's conjecture, was solved by Bressoud himself in some cases; then the next big step towards its resolution was made by Kim and Yee [KY14], and the full problem has recently been settled by Kim [Kim18].

We give our proof of Theorem 4.1 in this section, by showing that it is a consequence of the unilateral version of our new bilateral Bailey lattice in Theorem 1.11. We also show that it does not seem to follow from the classical Bailey lattice of Theorem 1.4, which seems suprising at first sight.

4.1. Bressoud's result.

In the paper [Bre80], Bressoud defines two functions $F_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q)$ and $G_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q)$, where the integral parameters satisfy $k \geq r > \lambda/2 \geq 0$. This is equivalent to $2k-1 \geq 2r-1 \geq \lambda \geq 0$. Then Bressoud's main theorem in [Bre80] states on the one hand that for all $k > r > \lambda/2 \geq 0$, we have

$$F_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q) = G_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q), \quad (4.2)$$

and on the other hand that for all $k \geq r > \lambda/2 \geq 0$, we have

$$\lim_{c_1,c_2\to\infty} F_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q) = \lim_{c_1,c_2\to\infty} G_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q). \quad (4.3)$$

We want to prove that these identities are both special cases of our new Bailey lattice. To do that, first note that it is enough to prove them when λ takes its maximal value, that is $\lambda = 2r-1$. Indeed, by the definitions of Bressoud's functions, we have

$$\lim_{b_\lambda\to\infty} F_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q) = F_{\lambda-1,k,r}(c_1, c_2; b_1, \dots, b_{\lambda-1}; a; q),$$

and

$$\lim_{b_\lambda\to\infty} G_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q) = G_{\lambda-1,k,r}(c_1, c_2; b_1, \dots, b_{\lambda-1}; a; q).$$

Now for $\lambda = 2r-1$, one can define the first of these functions by

$$\frac{F_{2r-1,k,r}(c_1, c_2; b_1, \dots, b_{2r-1}; a; q)}{(a/b_2, \dots, a/b_{2r-1})_\infty} = \frac{(a/b_1)_\infty}{(a)_\infty} \sum_{j\geq 0} \frac{(b_1, \dots, b_{2r-1}, c_1, c_2, a)_j (b_1 \dots b_{2r-1}c_1c_2)^{-j} a^{kj} q^{(k-r)j^2+j}}{(a/b_1, \dots, a/b_{2r-1}, aq/c_1, aq/c_2, q)_j}$$
$$\times \left(1 + \frac{a^r q^j}{b_1 \dots b_{2r-1}} \frac{(1-b_1q^j)\dots(1-b_{2r-1}q^j)}{(1-aq^j/b_1)\dots(1-aq^j/b_{2r-1})}\right). \quad (4.4)$$

The second function of Bressoud can be defined for $\lambda = 2r-1$ as

$$G_{2r-1,k,r}(c_1, c_2; b_1, \dots, b_{2r-1}; a; q) = \sum_{s_1 \geq \cdots \geq s_{k-1} \geq 0} \frac{a^{s_1+\cdots+s_{k-1}} q^{s_1^2+\cdots+s_{k-1}^2-s_1-\cdots-s_{r-1}} (aq/c_1c_2)_{s_{k-1}}}{(q, aq/c_1, aq/c_2)_{s_{k-1}} (q)_{s_1-s_2}\dots(q)_{s_{k-2}-s_{k-1}}}$$

$$\times (q^{1-s_1}/b_1)_{s_1}(a/b_2b_{2r-1})_{s_1-s_2}\dots(a/b_rb_{r+1})_{s_{r-1}-s_r}$$
$$\times (q^{1-s_2}/b_2, q^{1-s_2}/b_{2r-1})_{s_2}\dots(q^{1-s_r}/b_r, q^{1-s_r}/b_{r+1})_{s_r}$$
$$\times (aq^{s_1}/b_2, aq^{s_1}/b_{2r-1},\dots,aq^{s_{r-1}}/b_r, aq^{s_{r-1}}/b_{r+1})_\infty.$$

Using

$$(q^{1-n}/b)_n = (-1)^n b^{-n} q^{-n(n-1)/2}(b)_n \quad \text{and} \quad (aq^n/b)_\infty = \frac{(a/b)_\infty}{(a/b)_n},$$

this gives

$$\frac{G_{2r-1,k,r}(c_1,c_2;b_1,\dots,b_{2r-1};a;q)}{(a/b_2,\dots,a/b_{2r-1})_\infty} = \sum_{s_1\geq\dots\geq s_{k-1}\geq 0}(-1)^{s_1}\frac{a^{s_1+\dots+s_{k-1}}q^{s_1^2/2+s_{r+1}^2\dots+s_{k-1}^2-s_1/2+s_r}}{b_1^{s_1}(b_2b_{2r-1})^{s_2}\dots(b_rb_{r+1})^{s_r}}$$
$$\times\frac{(aq/c_1c_2)_{s_{k-1}}}{(q,aq/c_1,aq/c_2)_{s_{k-1}}}\frac{(b_1)_{s_1}(b_2,b_{2r-1})_{s_2}\dots(b_r,b_{r+1})_{s_r}}{(q)_{s_1-s_2}\dots(q)_{s_{k-2}-s_{k-1}}}\frac{(a/b_2b_{2r-1})_{s_1-s_2}\dots(a/b_rb_{r+1})_{s_{r-1}-s_r}}{(a/b_2,a/b_{2r-1})_{s_1}\dots(a/b_r,a/b_{r+1})_{s_{r-1}}}. \quad (4.5)$$

Then identity (4.2) of Bressoud translates for $\lambda = 2r-1$ as

$$F_{2r-1,k,r}(c_1,c_2;b_1,\dots,b_{2r-1};a;q) = G_{2r-1,k,r}(c_1,c_2;b_1,\dots,b_{2r-1};a;q),$$

with $0<r<k$, which from (4.4) and (4.5) is equivalent to (4.1).

Finally Bressoud's second result (4.3) asserts that (4.1) is still valid for $r=k$ when $c_1,c_2\to\infty$.

4.2. **A proof through our new Bailey lattice.** Replacing the use of Theorem 1.4 by the unilateral version of our new Bailey lattice given in Theorem 1.11 gives the following sequence: iterate $r-i$ times Theorem 1.1, then use the unilateral version of Theorem 1.11, and finally $i-1$ times Theorem 1.1 with a replaced by a/q. This yields a final Bailey pair related to a/q to which we apply (1.1) with a replaced by a/q. This is summarised in the following result, to be compared with [AAB87, Theorem 3.1] (equivalently the unilateral version of Theorem 2.1).

Theorem 4.2. *If $(\alpha_n,\,\beta_n)$ is a Bailey pair related to a, then for all integers $0\leq i\leq r$ and $n\geq 0$, we have:*

$$\sum_{s_1\geq\dots\geq s_r\geq 0}\frac{a^{s_1+\dots+s_r}q^{s_i+\dots+s_r}\beta_{s_r}}{(\rho_1\sigma_1)^{s_1}\dots(\rho_r\sigma_r)^{s_r}}\frac{(\rho_1,\sigma_1)_{s_1}\dots(\rho_r,\sigma_r)_{s_r}}{(q)_{n-s_1}(q)_{s_1-s_2}\dots(q)_{s_{r-1}-s_r}}$$
$$\times\frac{(a/\rho_1\sigma_1)_{n-s_1}(a/\rho_2\sigma_2)_{s_1-s_2}\dots(a/\rho_i\sigma_i)_{s_{i-1}-s_i}}{(a/\rho_1,a/\sigma_1)_n(a/\rho_2,a/\sigma_2)_{s_1}\dots(a/\rho_i,a/\sigma_i)_{s_{i-1}}}\frac{(aq/\rho_{i+1}\sigma_{i+1})_{s_i-s_{i+1}}\dots(aq/\rho_r\sigma_r)_{s_{r-1}-s_r}}{(aq/\rho_{i+1},aq/\sigma_{i+1})_{s_i}\dots(aq/\rho_r,aq/\sigma_r)_{s_{r-1}}}$$
$$=\frac{\alpha_0}{(q)_n(a)_n}+\sum_{j=1}^{n}\frac{(\rho_1,\sigma_1,\dots,\rho_i,\sigma_i)_j(\rho_1\sigma_1\dots\rho_i\sigma_i)^{-j}a^{ij}(1-a)}{(q)_{n-j}(a)_{n+j}(a/\rho_1,a/\sigma_1,\dots,a/\rho_i,a/\sigma_i)_j}$$
$$\times\left(\frac{(\rho_{i+1},\sigma_{i+1},\dots,\rho_r,\sigma_r)_j(\rho_{i+1}\sigma_{i+1}\dots\rho_r\sigma_r)^{-j}(aq)^{(r-i)j}q^j\alpha_j}{(aq/\rho_{i+1},aq/\sigma_{i+1},\dots,aq/\rho_r,aq/\sigma_r)_j(1-aq^{2j})}\right.$$
$$\left.-\frac{(\rho_{i+1},\sigma_{i+1},\dots,\rho_r,\sigma_r)_{j-1}(\rho_{i+1}\sigma_{i+1}\dots\rho_r\sigma_r)^{-j+1}(aq)^{(r-i)(j-1)}q^{j-1}\alpha_{j-1}}{(aq/\rho_{i+1},aq/\sigma_{i+1},\dots,aq/\rho_r,aq/\sigma_r)_{j-1}(1-aq^{2j-2})}\right). \quad (4.6)$$

Now let $n\to\infty$ in (4.6) and simplify the factor $(q)_\infty^{-1}$ appearing on both sides, and rewrite the right-hand side by shifting the index j to $j+1$ in the summation involving α_{j-1}:

$$\sum_{s_1\geq\dots\geq s_r\geq 0}\frac{a^{s_1+\dots+s_r}q^{s_i+\dots+s_r}\beta_{s_r}}{(\rho_1\sigma_1)^{s_1}\dots(\rho_r\sigma_r)^{s_r}}\frac{(\rho_1,\sigma_1)_{s_1}\dots(\rho_r,\sigma_r)_{s_r}}{(q)_{s_1-s_2}\dots(q)_{s_{r-1}-s_r}}\frac{(a/\rho_2\sigma_2)_{s_1-s_2}\dots(a/\rho_i\sigma_i)_{s_{i-1}-s_i}}{(a/\rho_2,a/\sigma_2)_{s_1}\dots(a/\rho_i,a/\sigma_i)_{s_{i-1}}}$$
$$\times\frac{(aq/\rho_{i+1}\sigma_{i+1})_{s_i-s_{i+1}}\dots(aq/\rho_r\sigma_r)_{s_{r-1}-s_r}}{(aq/\rho_{i+1},aq/\sigma_{i+1})_{s_i}\dots(aq/\rho_r,aq/\sigma_r)_{s_{r-1}}}=\frac{(a/\rho_1,a/\sigma_1)_\infty}{(a,a/\rho_1\sigma_1)_\infty}$$
$$\times\sum_{j\geq 0}\frac{1-a}{1-aq^{2j}}\frac{(\rho_1,\sigma_1,\dots,\rho_r,\sigma_r)_j(\rho_1\sigma_1\dots\rho_r\sigma_r)^{-j}a^{rj}q^{(r-i+1)j}\alpha_j}{(a/\rho_1,a/\sigma_1,\dots,a/\rho_i,a/\sigma_i,aq/\rho_{i+1},aq/\sigma_{i+1},\dots,aq/\rho_r,aq/\sigma_r)_j}$$
$$\times\left(1-\frac{a^i}{\rho_1\sigma_1\dots\rho_i\sigma_i}\frac{(1-\rho_1q^j)(1-\sigma_1q^j)\dots(1-\rho_iq^j)(1-\sigma_iq^j)}{(1-aq^j/\rho_1)(1-aq^j/\sigma_1)\dots(1-aq^j/\rho_i)(1-aq^j/\sigma_i)}\right). \quad (4.7)$$

In (4.7), replace r by $r-1$, and use the Bailey pair obtained from the unit Bailey pair (1.2) by one iteration of the Bailey lemma given in Theorem 1.1. This yields for $0 \le i \le r-1$:

$$\sum_{s_1\geq\dots\geq s_{r-1}\geq 0} \frac{a^{s_1+\dots+s_{r-1}}q^{s_i+\dots+s_{r-1}}}{(\rho_1\sigma_1)^{s_1}\dots(\rho_{r-1}\sigma_{r-1})^{s_{r-1}}} \frac{(aq/\rho\sigma)_{s_{r-1}}}{(q,aq/\rho,aq/\sigma)_{s_{r-1}}} \frac{(\rho_1,\sigma_1)_{s_1}\dots(\rho_{r-1},\sigma_{r-1})_{s_{r-1}}}{(q)_{s_1-s_2}\dots(q)_{s_{r-2}-s_{r-1}}}$$
$$\times \frac{(a/\rho_2\sigma_2)_{s_1-s_2}\dots(a/\rho_i\sigma_i)_{s_{i-1}-s_i}}{(a/\rho_2,a/\sigma_2)_{s_1}\dots(a/\rho_i,a/\sigma_i)_{s_{i-1}}} \frac{(aq/\rho_{i+1}\sigma_{i+1})_{s_i-s_{i+1}}\dots(aq/\rho_{r-1}\sigma_{r-1})_{s_{r-2}-s_{r-1}}}{(aq/\rho_{i+1},aq/\sigma_{i+1})_{s_i}\dots(aq/\rho_{r-1},aq/\sigma_{r-1})_{s_{r-2}}}$$
$$= \frac{(a/\rho_1,a/\sigma_1)_\infty}{(a,a/\rho_1\sigma_1)_\infty} \sum_{j\geq 0} \frac{(-1)^j(\rho_1,\sigma_1,\dots,\rho_{r-1},\sigma_{r-1},\rho,\sigma,a)_j(\rho_1\sigma_1\dots\rho_{r-1}\sigma_{r-1}\rho\sigma)^{-j}a^{rj}q^{(r-i+1)j+j(j-1)/2}}{(a/\rho_1,a/\sigma_1,\dots,a/\rho_i,a/\sigma_i,aq/\rho_{i+1},aq/\sigma_{i+1},\dots,aq/\rho_{r-1},aq/\sigma_{r-1},aq/\rho,aq/\sigma,q)_j}$$
$$\times\left(1-\frac{a^i}{\rho_1\sigma_1\dots\rho_i\sigma_i}\frac{(1-\rho_1q^j)(1-\sigma_1q^j)\dots(1-\rho_iq^j)(1-\sigma_iq^j)}{(1-aq^j/\rho_1)(1-aq^j/\sigma_1)\dots(1-aq^j/\rho_i)(1-aq^j/\sigma_i)}\right). \quad (4.8)$$

Next, in (4.8), take $\sigma_1,\rho_j,\sigma_j\to\infty$ for $j=i+1,\dots,r-1$, which yields

$$\sum_{s_1\geq\dots\geq s_{r-1}\geq 0} (-1)^{s_1}\frac{a^{s_1+\dots+s_{r-1}}q^{s_1^2/2+s_{i+1}^2+\dots+s_{r-1}^2-s_1/2+s_i}}{(\rho_1)^{s_1}(\rho_2\sigma_2)^{s_2}\dots(\rho_i\sigma_i)^{s_i}} \frac{(aq/\rho\sigma)_{s_{r-1}}}{(q,aq/\rho,aq/\sigma)_{s_{r-1}}} \frac{(\rho_1)_{s_1}(\rho_2,\sigma_2)_{s_2}\dots(\rho_i,\sigma_i)_{s_i}}{(q)_{s_1-s_2}\dots(q)_{s_{r-2}-s_{r-1}}}$$
$$\times \frac{(a/\rho_2\sigma_2)_{s_1-s_2}\dots(a/\rho_i\sigma_i)_{s_{i-1}-s_i}}{(a/\rho_2,a/\sigma_2)_{s_1}\dots(a/\rho_i,a/\sigma_i)_{s_{i-1}}}$$
$$= \frac{(a/\rho_1)_\infty}{(a)_\infty}\sum_{j\geq 0}\frac{(\rho_1,\rho_2,\sigma_2,\dots,\rho_i,\sigma_i,\rho,\sigma,a)_j(\rho_1\rho_2\sigma_2\dots\rho_i\sigma_i\rho\sigma)^{-j}a^{rj}q^{(r-i)j^2+j}}{(a/\rho_1,a/\rho_2,a/\sigma_2,\dots,a/\rho_i,a/\sigma_i,aq/\rho,aq/\sigma,q)_j}$$
$$\times\left(1+\frac{a^iq^j}{\rho_1\rho_2\sigma_2\dots\rho_i\sigma_i}\frac{(1-\rho_1q^j)(1-\rho_2q^j)(1-\sigma_2q^j)\dots(1-\rho_iq^j)(1-\sigma_iq^j)}{(1-aq^j/\rho_1)(1-aq^j/\rho_2)(1-aq^j/\sigma_2)\dots(1-aq^j/\rho_i)(1-aq^j/\sigma_i)}\right). \quad (4.9)$$

Replacing k by r and r by i, Bressoud's formula (4.1) becomes

$$F_{2i-1,r,i}(c_1,c_2;b_1,\dots,b_{2i-1};a;q) = G_{2i-1,r,i}(c_1,c_2;b_1,\dots,b_{2i-1};a;q),$$

which corresponds to (4.9) by taking $c_1=\rho$, $c_2=\sigma$, $b_1=\rho_1$, $b_2=\rho_2, b_{2i-1}=\sigma_2,\dots,b_i=\rho_i,b_{i+1}=\sigma_i$. As (4.9) is valid for $0\le i\le r-1$, we conclude that Bressoud's theorem, that is both formulas (4.2) and (4.3), is a special case of our Bailey lattice.

In the first place we tried to use the classical Bailey lattice of [AAB87, Theorem 3.1] (or the unilateral version in Theorem 2.1) instead of Theorem 4.2, and saw that to recover Bressoud's formula (4.1), one has to follow the same lines as above. We came up with the following formula instead of (4.9):

$$\sum_{s_1\geq\dots\geq s_{r-1}\geq 0} (-1)^{s_1}\frac{a^{s_1+\dots+s_{r-1}}q^{s_1^2/2+s_{i+1}^2+\dots+s_{r-1}^2-s_1/2}}{(\rho_1)^{s_1}(\rho_2\sigma_2)^{s_2}\dots(\rho_i\sigma_i)^{s_i}} \frac{(aq/\rho\sigma)_{s_{r-1}}}{(q,aq/\rho,aq/\sigma)_{s_{r-1}}} \frac{(\rho_1)_{s_1}(\rho_2,\sigma_2)_{s_2}\dots(\rho_i,\sigma_i)_{s_i}}{(q)_{s_1-s_2}\dots(q)_{s_{r-2}-s_{r-1}}}$$
$$\times \frac{(a/\rho_2\sigma_2)_{s_1-s_2}\dots(a/\rho_i\sigma_i)_{s_{i-1}-s_i}}{(a/\rho_2,a/\sigma_2)_{s_1}\dots(a/\rho_i,a/\sigma_i)_{s_{i-1}}}$$
$$= \frac{(a/\rho_1)_\infty}{(a)_\infty}\sum_{j\geq 0}\frac{(\rho_1,\rho_2,\sigma_2,\dots,\rho_i,\sigma_i,\rho,\sigma,a)_j(\rho_1\rho_2\sigma_2\dots\rho_i\sigma_i\rho\sigma)^{-j}a^{rj}q^{(r-i)j^2}}{(a/\rho_1,a/\rho_2,a/\sigma_2,\dots,a/\rho_i,a/\sigma_i,aq/\rho,aq/\sigma,q)_j}$$
$$\times\left(1+\frac{a^{i+1}q^{3j}}{\rho_1\rho_2\sigma_2\dots\rho_i\sigma_i}\frac{(1-\rho_1q^j)(1-\rho_2q^j)(1-\sigma_2q^j)\dots(1-\rho_iq^j)(1-\sigma_iq^j)}{(1-aq^j/\rho_1)(1-aq^j/\rho_2)(1-aq^j/\sigma_2)\dots(1-aq^j/\rho_i)(1-aq^j/\sigma_i)}\right). \quad (4.10)$$

Therefore we could only prove the special case

$$\lim_{b_{i+1},b_{i+2}\to\infty} F_{2i+1,r,i+1}(c_1,c_2;b_1,\dots,b_{2i+1};a;q) = \lim_{b_{i+1},b_{i+2}\to\infty} G_{2i+1,r,i+1}(c_1,c_2;b_1,\dots,b_{2i+1};a;q)$$

of Bressoud's formula (4.1) in which one takes $k=r$, $r=i+1$, $c_1=\rho$, $c_2=\sigma$, $b_1=\rho_1$, $b_2=\rho_2, b_{2i+1}=\sigma_2,\dots,b_i=\rho_i,b_{i+3}=\sigma_i$.

But we could not derive the most general identity (4.1) of Bressoud in this way.

4.3. **Special cases.** The case $\lambda = 1$ obtained from (4.1) by taking $b_j \to \infty$ for all $j \geq 2$ (and replacing k by r and r by i), exactly corresponds to (4.10) in which one takes $\rho_j, \sigma_j \to \infty$ for all $j \geq 2$ and $\rho_1 = b_1, \rho = c_1$, $\sigma = c_2$:

$$\sum_{s_1 \geq \cdots \geq s_{r-1} \geq 0} (-1)^{s_1} \frac{a^{s_1+\cdots+s_{r-1}} q^{s_1^2/2+s_2^2+\cdots+s_{r-1}^2-s_1/2-s_2-\cdots-s_{i-1}}}{(q)_{s_1-s_2}\cdots(q)_{s_{r-2}-s_{r-1}}(q)_{s_{r-1}}} \frac{(b_1)_{s_1}}{b_1^{s_1}} \frac{(aq/c_1c_2)_{s_{r-1}}}{(aq/c_1, aq/c_2)_{s_{r-1}}}$$

$$= \frac{(a/b_1)_\infty}{(a)_\infty} \sum_{j \geq 0} a^{rj} q^{(r-1)j^2+(2-i)j} \left(1 + \frac{a^i q^{(2i-1)j}}{b_1} \frac{1-b_1q^j}{1-aq^j/b_1}\right) \frac{(b_1, c_1, c_2, a)_j (b_1c_1c_2)^{-j}}{(a/b_1, aq/c_1, aq/c_2, q)_j}. \quad (4.11)$$

Note that we obtain the exact same formula with i replaced by $i+1$ by taking similar limits in (4.10).

Obviously, the case $\lambda = 0$ of Bressoud's result is obtained from (4.11) by taking $b_1 \to \infty$. Moreover all special case (3.2)–(3.7) in [Bre80] are consequences of the latter $\lambda = 0$ case, with the choices ($c_1 \to \infty, c_2 \to \infty, a = q$), ($c_1 \to \infty, c_2 \to \infty, a = 1$), ($c_1 = -q, c_2 \to \infty, a = q$), ($c_1 = -1, c_2 \to \infty, a = 1$), ($q \to q^2, c_1 = -q, c_2 \to \infty, a = 1$), and ($q \to q^2, c_1 = -q, c_2 \to \infty, a = q^2$) respectively. The two other special cases (3.8) and (3.9) of [Bre80] are obtained from (4.11) with ($q \to q^2, c_1 \to \infty, c_2 \to \infty, b_1 = -q, a = q^2$) and ($q \to q^2, c_1 = -q^2, c_2 \to \infty, b_1 = -q, a = q^2$), respectively.

Therefore we can conclude that all special cases of Bressoud's theorem exhibited in [Bre80] (giving Andrews–Gordon, Bressoud, and Göllnitz–Gordon type identities) are consequences of both unilateral Bailey lattices, the classical one and the new one. This is not surprising by Remark 2.4.

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References

[ADJM23] P. Afsharijoo, J. Dousse, F. Jouhet, and H. Mourtada, *Andrews–Gordon identities and commutative algebra*, Adv. Math. **417** (2023), paper No 108946, 28 pp.

[AAB87] A. Agarwal, G. E. Andrews, and D. Bressoud *The Bailey Lattice*, J. Indian Math. Soc., **51** (1987), 57–73.

[And74] G. E. Andrews, *An analytic generalization of the Rogers–Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci. USA **71** (1974), 4082–4085.

[And84] G. E. Andrews, *Multiple series Rogers–Ramanujan type identities*, Pac. J. Math. **114** (1984), 267–283.

[And86] G. E. Andrews, *q-series: their development and application in analysis, number theory, combinatorics, physics and computer algebra*, CBMS Regional Conference Series in Mathematics, **66**, AMS, Providence, 1986.

[AAR99] G. E. Andrews, R. Askey, and M. Roy, *Special functions*, Encyclopedia of Mathematics And Its Applications 71, Cambridge University Press, Cambridge, 1999.

[ASW99] G. E. Andrews, A. Schilling, and S.O. Warnaar, *An A_2 Bailey lemma and Rogers–Ramanujan-type identities*, J. Amer. Math. Soc. **12** (1999), 677–702.

[Bai49] W. N. Bailey, *Identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) **50** (1949), 1–10.

[BMS96] A. Berkovich, B. M. McCoy and A. Schilling, *$N = 2$ Supersymmetry and Bailey Pairs*, Physica A **228** (1996), 33-62.

[Bre79] D. Bressoud, *A generalization of the Rogers–Ramanujan identities for all moduli*, J. Comb. Th. A **27** (1979), 64–68.

[Bre80] D. Bressoud, *Analytic and combinatorial generalization of the Rogers–Ramanujan identities*, Mem. Amer. Math. Soc. **24** (1980), no 227, 54 pp.

[BIS00] D. Bressoud, M. Ismail, and D. Stanton, *Change of Base in Bailey Pairs*, The Ramanujan J. **4** (2000), 435–453.

[DJK23+] J. Dousse, F. Jouhet, and I. Konan, *Combinatorial approach to Andrews–Gordon and Bressoud-type identities*, in preparation, 2023.

[GR04] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Second Edition, Encyclopedia of Mathematics and its Applications 96, Cambridge University Press, Cambridge, 2004.

[Go67] H. Göllnitz, *Partitionen mit differenzenbedingungen*, J. reine angew. Math. **225** (1967), 154–190.

[Gor61] B. Gordon, *A Combinatorial Generalization of the Rogers–Ramanujan Identities*, Amer. J. Math. **83** (1961), 393–399.

[Gor65] B. Gordon, *Some continued fractions of the Rogers-Ramanujan type*, Duke Math. J. **31** (1965), 741–748.

[HZ23] T.Y. He and A. X. H. Zhao, *New companions to the generalizations of the Göllnitz–Gordon identities*, The Ramanujan J. **61**, 1077–1120 (2023).

[Jou10] F. Jouhet, *Shifted versions of the Bailey and well-poised Bailey lemmas*, The Ramanujan J. **23** (2010), 315–333, special issue of the conference "Combinatory Analysis 2008" dedicated to George Andrews.

[Kim18] S. Kim, *Bressoud's conjecture*, Adv. Math. **325** (2018), 770–813.

[KY14] S. Kim and A. J. Yee, *Partitions with part difference conditions and Bressoud's conjecture*, J. Combin. Theory Ser. A **126** (2014), 35–69.

[Lov04] J. Lovejoy, *A Bailey lattice*, Proc. Amer. Math. Soc. **132** (2004), 1507–1516.

[Lov22] J. Lovejoy, *Bailey pairs and indefinite quadratic forms, II. False indefinite theta functions*, Res. Number Theory **8**:24 (2022).

[McL18] J. McLaughlin, *Topics and methods in q-series*, Monographs in Number Theory, 8, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.

[RR19] L. J. Rogers and S. Ramanujan, *Proof of certain identities in combinatory analysis*, Cambr. Phil. Soc. Proc. **19** (1919), 211–216.

[War01] S. O. Warnaar, 50 *Years of Bailey's lemma*, in Algebraic Combinatorics and Applications, pp. 333–347, A. Betten et al. eds. (Springer, Berlin, 2001).

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