NEW CURIOUS BILATERAL q-SERIES IDENTITIES

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Abstract. By applying a classical method, already employed by Cauchy, to a terminating curious summation by one of the authors, a new curious bilateral q-series identity is derived. We also apply the same method to a quadratic summation by Gessel and Stanton, and to a cubic summation by Gasper, respectively, to derive a bilateral quadratic and a bilateral cubic summation formula.

1. Introduction

In [11, Thm. 7.29] one of the authors derived the following curious summation:

$$\begin{aligned}
&(qa/b^2c;q)_n \\
&= \sum_{k=0}^{n} \frac{(c+1-(a+b))}{(c+1-(a+bq^{-k}))} \frac{(bc+a+b)}{(bc+a+bq^{-k})} \frac{(ac-(a+bq^{-k})^2)}{(ac-(a+b)(a+bq^{-k}))} \\
&\quad \times \frac{\big(\frac{aq}{b^2c}(a+bq^{-k}-c);q\big)_n}{\big(aq\frac{c-(a+bq^{-k})}{b(a+bq^{-k})};q\big)_n} \Big(-\frac{aq}{b(a+bq^{-k})};q\Big)_n \\
&\quad \times \frac{(q^{-n};q)_k\,(a+bq^{-k}-c;q)_k\,\big(-\frac{1}{bc}(a+bq^{-k});q\big)_k}{(q;q)_k\,\big(-\frac{bq^{-n}}{a}(a+bq^{-k});q\big)_k\,\big(\frac{aq}{b^2c}(a+bq^{-k}-c);q\big)_k} q^k.
\end{aligned} \tag{1}$$

Here $(\alpha;q)_k$ denotes the *q-shifted factorial*, defined by

$$(\alpha;q)_\infty := \prod_{j\geq 0}(1-\alpha q^j), \qquad \text{and} \qquad (\alpha;q)_k := \frac{(\alpha;q)_\infty}{(\alpha q^k;q)_\infty},$$

where q (the *base*) is a fixed complex parameter with $0<|q|<1$, α is a complex parameter and k is any integer. For brevity, we shall also use the compact notation

$$(\alpha_1,\cdots,\alpha_m;q)_k := (\alpha_1;q)_k\cdots(\alpha_m;q)_k.$$

The summation in (1) was derived by application of *inverse relations* to the q-Pfaff–Saalschütz summation (cf. [6, Appendix (II.12)]). In [11] also several other "curious summations" (involving series which themselves do *not* belong to the respective hierarchies of hypergeometric and basic hypergeometric series) were derived by utilizing various summation formulae for hypergeometric and basic hypergeometric series. Similar identities were derived by the same means in [8]. Special

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cases of two of the summations were even extended there to *bilateral* summations by means of analytic continuation.

Another method to obtain bilateral summations from terminating ones was employed in [12] to give a new proof of Ramanujan's ${}_1\psi_1$ summation formula and to derive (for the first time) Abel–Rothe type extensions of Jacobi's triple product identity. Actually, the method of [12] was already utilized by Cauchy [4] in his second proof of Jacobi's [9] triple product identity. The very same method (which we shall refer to as "Cauchy's method of bilateralization") had also been exploited by Bailey [1, Secs. 3 and 6], [2], and Slater [13, Sec. 6.2]. In [10] the current authors used a variant of Cauchy's method to give a new derivation of Bailey's [1, Eq. (4.7)] very-well-poised ${}_6\psi_6$ summation (cf. [6, Appendix (II.33)]),

$$\sum_{k=-\infty}^{\infty}\frac{(1-aq^{2k})}{(1-a)}\frac{(b,c,d,e;q)_k}{(aq/b,aq/c,aq/d,aq/e;q)_k}\left(\frac{qa^2}{bcde}\right)^k$$
$$=\frac{(q,aq,q/a,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de;q)_\infty}{(q/b,q/c,q/d,q/e,aq/b,aq/c,aq/d,aq/e,a^2q/bcde;q)_\infty}, \tag{2}$$

where $|q|<1$ and $|qa^2/bcde|<1$.

In Section 2, we apply Cauchy's method of bilateralization to the curious summation in (1). (This possibility, which appears to be applicable to (1) but, to the best of our knowledge, not to any of the other curious summations of [11, Sec. 7], was missed so far.) As a result, we obtain the new curious bilateral summation in Proposition 2.1. In the same section we explicitly display some noteworthy special cases of the new curious bilateral identity. In Section 3 we apply Cauchy's method to a terminating quadratic summation by Gessel and Stanton [7], and to a terminating cubic summation by Gasper [5]. Hereby we obtain a bilateral quadratic and a bilateral cubic summation, both which evaluate to zero, see Propositions 3.1 and 3.2, respectively.

For a comprehensive treatise on basic hypergeometric series, see Gasper and Rahman's text [6]. Several of the computations in this paper rely on various elementary identities for q-shifted factorials, listed in [6, Appendix I].

2. A new curious bilateral summation

To apply Cauchy's method to the terminating summation in (1), we first replace n by $2n$ and then shift the index of summation by n such that the new sum runs from $-n$ to n. Further, we replace b by bq^n. In total, we thus obtain

$$(q^{1-2n}a/b^2c;q)_{2n}$$
$$=\sum_{k=-n}^{n}\frac{(c+1-(a+bq^n))}{(c+1-(a+bq^{-k}))}\frac{(bcq^n+a+bq^n)}{(bcq^n+a+bq^{-k})}\frac{(ac-(a+bq^{-k})^2)}{(ac-(a+bq^n)(a+bq^{-k}))}$$
$$\times(-1)^{n+k}q^{\binom{n}{2}+\binom{k}{2}-nk-2n^2+n+k}\frac{(q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}}$$
$$\times\frac{\left(\frac{aq^{1-2n}}{b^2c}(a+bq^{-k}-c);q\right)_{2n}}{\left(aq^{1-n}\frac{c-(a+bq^{-k})}{b(a+bq^{-k})};q\right)_{2n}}\left(-\frac{aq^{1-n}}{b(a+bq^{-k})};q\right)_{2n}$$
$$\times\frac{(a+bq^{-k}-c;q)_{n+k}\left(-\frac{q^{-n}}{bc}(a+bq^{-k});q\right)_{n+k}}{\left(-\frac{bq^{-n}}{a}(a+bq^{-k});q\right)_{n+k}\left(\frac{aq^{1-2n}}{b^2c}(a+bq^{-k}-c);q\right)_{n+k}}.$$

Now, after multiplying both sides by $(q;q)_n(b^2c/a)^{2n}q^{\binom{2n}{2}}$ we may let $n\to\infty$, assuming $\max(|q|,|b^2/a|,|a-c|)<1$, while appealing to Tannery's theorem [3] for being allowed to interchange limit and summation. This results, after some elementary manipulations of q-shifted factorials, in the following curious bilateral summation:

Proposition 2.1. *Let a, b, c be indeterminates, let $|q|<1$ and $\max(|q|,|b^2/a|,|a-c|)<1$. Then*

$$
\begin{aligned}
(q,b^2c/a;q)_\infty = \sum_{k=-\infty}^{\infty} & \frac{(c+1-a)}{(c+1-(a+bq^{-k}))}\,\frac{a}{(a+bq^{-k})}\,\frac{(ac-(a+bq^{-k})^2)}{(ac-a(a+bq^{-k}))}\\
&\times\frac{\big(\frac{b^2c}{a(a+bq^{-k}-c)};q\big)_\infty\big(-\frac{bcq}{a+bq^{-k}};q\big)_\infty}{\big(aq\frac{c-(a+bq^{-k})}{b(a+bq^{-k})};q\big)_\infty\big(\frac{b(a+bq^{-k})}{a(c-(a+bq^{-k}))};q\big)_\infty}\\
&\times(a+bq^{-k}-c;q)_\infty\Big(-\frac{b}{a}(a+bq^{-k})q^k;q\Big)_\infty\\
&\times\Big(-\frac{1}{bc}(a+bq^{-k});q\Big)_k\left(\frac{b^2c}{a(a+bq^{-k}-c)}\right)^k. \qquad (3)
\end{aligned}
$$

Remark 2.2. We checked the validity of the identity in (3) by Mathematica. In particular, by replacing a,b,c by aq,bq,cq, respectively, the identity can be interpreted as a power series identity in q (valid for $|q|<\min\big(\big|\frac{a}{b^2}\big|,\big|\frac{1}{a-c}\big|,1\big)$, in particular, for q around zero). Only a finite number of terms contribute to the coefficient of q^n for each $n\geq 0$.

We write out some noteworthy special cases of Proposition 2.1. The first one is obtained by replacing (a,b,c) by (at,bt,ct) and then taking the limit $t\to 0$ (which, again, is justified by Tannery's theorem [3]).

Corollary 2.3. *Let a, b and c be indeterminates and $|q|<1$. Then*

$$
\begin{aligned}
(q;q)_\infty = \sum_{k=-\infty}^{\infty} & \frac{(ac-(a+bq^{-k})^2)}{(ac-a(a+bq^{-k}))}\left(\frac{a+bq^{-k}}{a}\right)^{k-1}\left(\frac{b}{a+bq^{-k}-c}\right)^k\\
&\times q^{\binom{k}{2}}\Big(aq\frac{c-(a+bq^{-k})}{b(a+bq^{-k})};q\Big)_\infty^{-1}\Big(\frac{b(a+bq^{-k})}{a(c-(a+bq^{-k}))};q\Big)_\infty^{-1}. \qquad (4)
\end{aligned}
$$

This turns out to be a generalization of Jacobi's triple product identity (the $c\to 0$, $b\to -az$ case of (4)).

If instead, we directly take $c\to 0$ in (3), then we obtain another generalization of Jacobi's triple product identity, a special case of a curious bilateral summation considered in [12].

It is also interesting to take the $c\to a$ case of (3). The result, after some elementary manipulations, is

Corollary 2.4. *Let a and b be indeterminates, let $|q|<1$ and $|b^2/a|<1$. Then*

$$
\begin{aligned}
\frac{(q,b^2;q)_\infty}{(b,bq;q)_\infty} = \sum_{k=-\infty}^{\infty} & \frac{(2a+bq^{-k})}{(a+bq^{-k})}\,\frac{(1/b;q)_k\big(-\frac{1}{ab}(a+bq^{-k});q\big)_k}{(b;q)_k}(-1)^kb^{2k}\\
&\times q^{\binom{k+1}{2}}\frac{\big(-\frac{bq^k}{a}(a+bq^{-k});q\big)_\infty\big(-\frac{abq}{a+bq^{-k}};q\big)_\infty}{\big(-\frac{aq^{1-k}}{a+bq^{-k}};q\big)_\infty\big(-q^k\frac{a+bq^{-k}}{a};q\big)_\infty}. \qquad (5)
\end{aligned}
$$

If we now let $a \to \infty$, we obtain after some elementary manipulations of q-shifted factorials the following summation for a bilateral ${}_1\psi_2$ series.

Corollary 2.5. *Let b be an indeterminate and $|q|<1$. Then*

$$\frac{(q^2, b^2q; q^2)_\infty}{(q, b^2q^2; q^2)_\infty} = \sum_{k=-\infty}^{\infty} \frac{(1/b^2;q^2)_k}{(b^2;q^2)_k} (-1)^k q^{k^2} b^{2k}. \tag{6}$$

As a matter of fact, the identity in (6) is *not* a special case of the bilateral q-Kummer summation [6, Appendix (II.30)], the latter which is an easy consequence of Bailey's ${}_6\psi_6$ summation formula (2). Nevertheless, also Corollary 2.5 can be derived from Bailey's ${}_6\psi_6$ summation formula. Indeed, note that by replacing the summation index k by $1-k$ in (6), the right-hand side becomes

$$\sum_{k=-\infty}^{\infty} \frac{(1/b^2;q^2)_k}{(b^2;q^2)_k} (-1)^k q^{k^2-2k+1} b^{2k}.$$

It follows that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{(1/b^2;q^2)_k}{(b^2;q^2)_k} (-1)^k q^{k^2} b^{2k} \\
&\qquad = \frac{1+q}{2} \sum_{k=-\infty}^{\infty} \frac{1+q^{2k-1}}{1+q^{-1}} \frac{(1/b^2;q^2)_k}{(b^2;q^2)_k} (-1)^k q^{k^2-2k} b^{2k}.
\end{aligned}$$

But this can be evaluated by the $(a,b,c,d,e,q) \to (-q^{-1}, 1/b, -1/b, \infty, \infty, q)$ limit case of (2), after which one readily obtains the product side of (6).

3. A bilateral quadratic and a bilateral cubic summation

First we apply Cauchy's method of bilateralization to the following quadratic summation formula due to Gessel and Stanton [7, Equation (1.4), $q \to q^2$]:

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(1-aq^{3k})}{(1-a)} \frac{(a, b, q/b; q)_k \, (d, a^2q^{1+2n}/d, q^{-2n}; q^2)_k}{(aq/d, dq^{-2n}/a, aq^{1+2n}; q)_k \, (q^2, aq^2/b, abq; q^2)_k} q^k \\
&\qquad\qquad = \frac{(aq;q)_{2n}}{(aq/d;q)_{2n}} \frac{(abq/d, aq^2/bd; q^2)_n}{(aq^2/b, abq; q^2)_n}.
\end{aligned} \tag{7}$$

We replace n by $2n$ and then shift the index of summation by n such that the new sum runs from $-n$ to n. We also replace a by aq^{-3n}, and b by bq^{-n}, respectively. After some elementary manipulations of q-shifted factorials we thus obtain the identity

$$\begin{aligned}
&\sum_{k=-n}^{n} \frac{(1-aq^{3k})}{(1-a)} \frac{(aq^{-2n}, b, q^{1+2n}/b; q)_k \, (dq^{2n}, a^2q/d, q^{-2n}; q^2)_k}{(aq^{1-2n}/d, d/a, aq^{1+2n}; q)_k \, (q^{2+2n}, aq^2/b, abq^{1-2n}; q^2)_k} q^k \\
&\quad = \frac{(aq, q/a; q)_{2n}}{(q/b, d/a; q)_{2n}} \frac{(q^2, q^2, aq^2/bd, bd/a; q^2)_n}{(d, aq^2/b, q/ab, dq/a^2; q^2)_n} \frac{(dq/ab; q^2)_{2n}}{(q^2;q^2)_{2n}} \left(\frac{d}{abq}\right)^n.
\end{aligned}$$

Now, under the assumption $|q|<1$ and $|d/abq|<1$ we may let $n \to \infty$, while appealing to Tannery's theorem for being allowed to interchange limit and summation. Finally, we perform the substitution $d \mapsto a^2q/c$ and arrive at the following bilateral quadratic summation formula:

Proposition 3.1. *Let a, b, c be indeterminates, let $|q| < 1$ and $|a/bc| < 1$. Then*

$$\sum_{k=-\infty}^{\infty} \frac{(1-aq^{3k})}{(1-a)} \frac{(b;q)_k}{(aq/c;q)_k} \frac{(c;q^2)_k}{(aq^2/b;q^2)_k} \left(\frac{a}{bc}\right)^k = 0. \tag{8}$$

Next, we apply Cauchy's method of bilateralization to the following cubic summation formula due to Gasper [5, Equation (5.22), $c \to q^{-3n}$]:

$$\sum_{k=0}^{n} \frac{(1-aq^{4k})}{(1-a)} \frac{(a,b;q)_k\,(q/b;q)_{2k}\,(a^2bq^{3n},q^{-3n};q^3)_k}{(aq^{1+3n},q^{1-3n}/ab;q)_k\,(ab;q)_{2k}\,(q^3,aq^3/b;q^3)_k} q^k = \frac{(aq;q)_{3n}}{(ab;q)_{3n}} \frac{(ab^2;q^3)_n}{(aq^3/b;q^3)_n}. \tag{9}$$

We replace n by $2n$ and then shift the index of summation by n such that the new sum runs from $-n$ to n. We also replace a by aq^{-4n}, and b by bq^{-n}, respectively. Then, under the assumption $|q| < 1$ and $|1/ab^2| < 1$ we let $n \to \infty$, while appealing to Tannery's theorem for being allowed to interchange limit and summation. We eventually arrive at the following bilateral cubic summation formula:

Proposition 3.2. *Let a, b be indeterminates, let $|q| < 1$ and $|1/ab^2| < 1$. Then*

$$\sum_{k=-\infty}^{\infty} \frac{(1-aq^{4k})}{(1-a)} \frac{(b;q)_k}{(q/ab;q)_k} \frac{(a^2b;q^3)_k}{(aq^3/b;q^3)_k} \left(\frac{1}{ab^2}\right)^k = 0. \tag{10}$$

It seems worth checking whether Cauchy's method could be applied to other quadratic, cubic and quartic summation formulae appearing in the literature, leading to other interesting bilateral summations.

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