Arithmetic properties for q-zeta values at positive integers

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Hypergeometric series and their generalizations in algebra, geometry, number theory and physics

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Dirichlet series

The Dirichlet series associated to the character χ is defined for $Re(s) > 1$ by :

$$L_\chi(s) := \sum_{n \geq 1} \frac{\chi(n)}{n^s}$$

For the trivial character $\chi = 1$, $L_1 = \zeta$ is the Riemann zeta function

At even integers $2m \geq 2$: if $B_{2m} \in \mathbb{Q}$ are the Bernoulli numbers, then

$$\zeta(2m) = (-1)^{m-1} 2^{2m-1} B_{2m} \frac{\pi^{2m}}{(2m)!}$$ transcendantal numbers (Lindemann)

Arithmetic properties of zeta values at odd positive integers

Apéry (1979) : $\zeta(3) \notin \mathbb{Q}$

Rivoal, Ball-Rivoal (2000) : there are among the values $\zeta(2m+1)$ infinitely many irrational numbers

Zudilin (2004) : at least one of the values $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is an irrational number

Rivoal (2002) : at least one of the values $\zeta(5), \zeta(7), \dots, \zeta(21)$ is an irrational number

Krattenthaler-Rivoal (2007) : by using limit cases of a basic hypergeometric series transformation formula due to Andrews, resolution of the so-called denominators conjecture, which implies that at least one of the values $\zeta(5), \zeta(7), \dots, \zeta(19)$ is an irrational number

q-zeta values at positive integers

For $s \in \mathbb{N}^*$ and $|q| < 1$, Kaneko-Kurokawa-Wakayama (2003) defined :

$$\zeta_q(s) := \sum_{k \geq 1} q^k \sum_{d|k} d^{s-1} = \sum_{k \geq 1} k^{s-1} \frac{q^k}{1-q^k}$$

We have for $s \in \mathbb{N}^* \setminus \{1\}$:

$$\lim_{q \to 1} (1-q)^s \zeta_q(s) = (s-1)! \zeta(s)$$

For $s = 2m$, we have :

$$\zeta_q(2m) = \frac{B_{2m}}{4m}(1 - E_{2m}(q))$$

where $E_{2m}(q)$ are Eisenstein series and B_{2m} are still Bernoulli numbers

Values at even positive integers

By the structure of the space of modular forms on $SL_2(\mathbb{Z})$, it is well-known that for $m \geq 2$:

$$E_{2m}(q) = \sum_{4a+6b=2m} c_{a,b} E_4(q)^a E_6(q)^b, \quad c_{a,b} \in \mathbb{Q}$$

Theorem (Nesterenko, 1996)

For $q \in \mathbb{C}$ such that $0 < |q| < 1$, at least three of the numbers q, $E_2(q)$, $E_4(q)$, and $E_6(q)$ are algebraically independant over $\mathbb{Q}$

Consequence : for $m \geq 1$ and q algebraic, $\zeta_q(2m)$ is a transcendantal number

and the role played by π for $\zeta(2m)$ seems here to be played by $E_4(q)$ and $E_6(q)$

Values at odd positive integers (1)

Borwein (1992) : $\zeta_q(1)$ is an irrational number for some values of q

Postelmans-Van Assche (2007) : linear independance of 1, $\zeta_q(1)$, and $\zeta_q(2)$ for q integer, $|q| > 1$

Consequence of an older result of Tachiya (2004)

Bundschuh-Vn (2005) and independently Zudilin (2006) : linear independance measures for these numbers

Values at odd positive integers (2)

By using basic hypergeometric series and a linear independance criterion due to Nesterenko (1985), Krattenthaler-Rivoal-Zudilin (2006) proved that for $1/q \in \mathbb{Z} \setminus \{-1;1\}$ and a positive even integer A :

$$\dim_{\mathbb{Q}} \left(\mathbb{Q} + \mathbb{Q}\zeta_q(3) + \cdots + \mathbb{Q}\zeta_q(A-1)\right) \geq f(A),$$

where

$$f(A) = \max_{\substack{r\in\mathbb{N} \\ 1\leq r\leq A/2}} f(r;A) \quad \text{and} \quad f(r;A) := \frac{4rA + A - 4r^2}{\left(\frac{24}{\pi^2}+2\right)A + 8r^2}$$

Consequences : for $1/q \in \mathbb{Z} \setminus \{-1;1\}$,

at least one of the values $\zeta_q(3), \zeta_q(5), \zeta_q(7), \zeta_q(9), \zeta_q(11)$ is an irrational number (when $A = 12$, we have $f(12) > 1$)

there are among the values $\zeta_q(2m+1)$ infinitely many irrational numbers (when $A \to \infty$, we have $f(A) \sim \frac{\pi}{2\sqrt{\pi^2+12}}\sqrt{A}$)

Refinement of the lower bound

Theorem (J-Mosaki, 2009)

For $1/q \in \mathbb{Z} \setminus \{-1; 1\}$ and a positive even integer A :

$$\dim_{\mathbb{Q}} \left(\mathbb{Q} + \mathbb{Q}\zeta_q(3) + \cdots + \mathbb{Q}\zeta_q(A-1) \right) \geq g(A),$$

where

$$g(A) = \max_{\substack{r \in \mathbb{N} \\ 1 \leq r \leq A/2}} g(r; A) \quad \text{and} \quad g(r; A) := \frac{4rA + A - 4r^2}{\left(\frac{24}{\pi^2} + 2\right) A - \frac{24}{\pi^2} + 8r^2}$$

The estimation $g(2; 10) \geq 1,001...$ implies :

Corollary

For $1/q \in \mathbb{Z} \setminus \{-1; 1\}$, at least one of the values $\zeta_q(3), \zeta_q(5), \zeta_q(7), \zeta_q(9)$ is an irrational number

Other results

We have more precisely :

$$f(10) < 1 < g(10) = g(10;2) \simeq 1,001 \quad (1)$$

$$f(38) < g(38) < 2 < f(40) < g(40) \quad (2)$$

$$f(86) < 3 < g(86) \quad (3)$$

From (1), we get the corollary

From (2), K-R-Z and we get : for $1/q \in \mathbb{Z} \setminus \{-1;1\}$, there exist two odd integers j_1 and j_2 such that $3 \leq j_1 < j_2 \leq 39$ and the numbers 1, $\zeta_q(j_1)$, and $\zeta_q(j_2)$ are linearly independant over $\mathbb{Q}$

From (3), we get : for $1/q \in \mathbb{Z} \setminus \{-1;1\}$, there exist three odd integers j_1, j_2 and j_3 such that $3 \leq j_1 < j_2 < j_3 \leq 85$ and the numbers 1, $\zeta_q(j_1)$, $\zeta_q(j_2)$, and $\zeta_q(j_3)$ are linearly independant over $\mathbb{Q}$

Fischler-Zudilin (2010) : new proof and refinement of Nesterenko's linear independance criterion, which implies that there exist four odd integers $1 < i_0 < i_1 < i_2 < i_3$ such that $i_0 \leq 9$, $i_1 \leq 37$, $i_2 \leq 83$, $i_3 \leq 145$ and the numbers 1, $\zeta_q(i_0)$, $\zeta_q(i_1)$, $\zeta_q(i_2)$ and $\zeta_q(i_3)$ are linearly independant over $\mathbb{Q}$

Nesterenko's criterion

Here is a formulation of a special case that we will need

Proposition (Nesterenko's linear independance criterion, 1985)

Let $N \geq 2$ be an integer and $v_1, \ldots, v_N$ be real numbers. Assume that there exist N integer sequences $(p_{j,n})_{n\geq 0}$ and real numbers α_1 and α_2 with $\alpha_2 > 0$ such that :

i) $\lim_{n\to+\infty} \frac{1}{n^2} \log |p_{1,n}v_1 + \cdots + p_{N,n}v_N| = -\alpha_1$,

ii) for all $j \in \{1, \ldots, N\}$, we have $\limsup_{n\to+\infty} \frac{1}{n^2} \log |p_{j,n}| \leq \alpha_2$.

Then :

$$\dim_{\mathbb{Q}} (\mathbb{Q}v_1 + \cdots + \mathbb{Q}v_N) \geq 1 + \frac{\alpha_1}{\alpha_2}$$

A basic hypergeometric series

Recall the q-rising factorial :

$$(a;q)_n \equiv (a)_n := \begin{cases} 1, & n=0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2,\ldots \end{cases}$$

and for practical reasons : $(a,b)_n := (a)_n(b)_n$

Consider, for positive integers A and r with $A-2r>0$ and A even, the series :

$$\tilde{S}_n(q) := (q)_n^{A-2r} \sum_{k\geq 1}(1-q^{2k+n}) \frac{(q^{k-rn}, q^{k+n+1})_{rn}}{(q^k)_{n+1}^A} q^{k(A-2r)n/2+kA/2-k}$$

In terms of very-well poised basic hypergeometric series : with $a=q^{(2r+1)n+2}$,

$$\tilde{S}_n(q) = q^{(rn+1)((A-2r)n/2+A/2-1)}(1-q^{n+2rn+2})(q)_n^{A-2r} \frac{(q, q^{n+rn+2})_{rn}}{(q^{rn+1})_{n+1}^A}$$
$$\times {}_{A+4}\phi_{A+3}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{rn+1}, \ldots, q^{rn+1} \\ \sqrt{a}, -\sqrt{a}, q^{(r+1)n+2}, \ldots, q^{(r+1)n+2} \end{matrix}; q, q^{(A-2r)n/2+A/2-1}\right]$$

Linear forms in odd q-zeta values

By partial fraction expansion, we get the linear forms :

$$\tilde{S}_n(q) = \hat{P}_{0,n}(q) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{A-1} \hat{P}_{j,n}(q)\zeta_q(j) \quad \text{where} \quad \hat{P}_{j,n}(q) \in \mathbb{Q}(q)$$

We prove that : $D_n(q)\hat{P}_{j,n}(q) \in \mathbb{Z}\left[\frac{1}{q}\right] \quad \forall j \in \{0, 3, 5, \ldots, A-1\}$

where $D_n(q) = (A-1)!\, q^{\lfloor \alpha n^2 + \beta n + \gamma \rfloor} d_n(1/q)^A$, $\alpha = -A/8 - r^2/2$, and $d_n(q) = \operatorname{lcm}(q-1, \ldots, q^n - 1)$

The asymptotics of $\tilde{S}_n(q)$, $D_n(q)$, and $\hat{P}_{j,n}(q)$, together with Nesterenko's criterion, gives back Krattenthaler-Rivoal-Zudilin's result

We need to prove a q-denominators conjecture :

Theorem (J-Mosaki, 2009)

We have $\tilde{D}_n(q)\hat{P}_{j,n}(q) \in \mathbb{Z}\left[\frac{1}{q}\right] \quad \forall j \in \{0, 3, 5, \ldots, A-1\}$

where $\tilde{D}_n(q) = (A-1)!\, q^{\lfloor \alpha n^2 + \beta n + \gamma \rfloor} d_n(1/q)^{A-1}$

Bailey's lemma

(α_n, β_n) is a Bailey pair related to a if

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}} \qquad \forall\ n \geq 0$$

Lemma (Bailey, 1950)

If (α_n, β_n) is a Bailey pair related to a, then so is (α'_n, β'_n), where

$$\alpha'_n = \frac{(b,c)_n}{(aq/b, aq/c)_n}(aq/bc)^n \alpha_n$$

$$\beta'_n = \sum_{k=0}^{n} \frac{(b,c)_k (aq/bc)_{n-k}}{(q)_{n-k}(aq/b, aq/c)_n}(aq/bc)^k \beta_k$$

Iterating : Bailey chain (Andrews, 1984) :

$$(\alpha_n, \beta_n) \longrightarrow (\alpha'_n, \beta'_n) \longrightarrow (\alpha''_n, \beta''_n) \longrightarrow \cdots$$

Proving q-series identities

The unit Bailey pair :

$$\alpha_n = (-1)^n q^{\binom{n}{2}} \frac{1-aq^{2n}}{1-a} \frac{(a)_n}{(q)_n}, \qquad \beta_n = \delta_{n,0}$$

One iteration $\Rightarrow (\alpha'_n, \beta'_n)$, which gives the finite summation :

$$\sum_{k=0}^{n} \frac{1-aq^{2k}}{1-a} \frac{(a,b,c,q^{-n})_k}{(q,aq/b,aq/c,aq^{n+1})_k} \left(\frac{aq^{1+n}}{bc}\right)^k = \frac{(aq,aq/bc)_n}{(aq/b,aq/c)_n}$$

Two iterations $\Rightarrow (\alpha''_n, \beta''_n)$, which gives Watson's finite transformation :

$$\sum_{k=0}^{n} \frac{1-aq^{2k}}{1-a} \frac{(a,b,c,d,e,q^{-n})_k}{(q,aq/b,aq/c,aq/d,aq/e,aq^{n+1})_k} \left(\frac{a^2q^{2+n}}{bcde}\right)^k$$

$$= \frac{(aq,aq/de)_n}{(aq/d,aq/e)_n} \sum_{k=0}^{n} \frac{(aq/bc,d,e,q^{-n})_k}{(q,aq/b,aq/c,deq^{-n}/a)_k} q^k$$

Six parameters finite extension of the famous Rogers-Ramanujan identities

Andrews' formula

Iterating $m+1$ times Bailey's lemma, Andrews (1975, 1986) proves that for integers $m \geq 0$ and $n \geq 0$:

$$\sum_{k=0}^{n} \frac{1-aq^{2k}}{1-a} \frac{(a, b_1, c_1, \ldots, b_{m+1}, c_{m+1}, q^{-n})_k}{(q, aq/b_1, aq/c_1, \ldots, aq/b_{m+1}, aq/c_{m+1}, aq^{n+1})_k} \times \left(\frac{a^{m+1} q^{m+1+n}}{b_1 c_1 \ldots b_{m+1} c_{m+1}} \right)^k$$

$$= \frac{(aq, aq/b_{m+1}c_{m+1})_n}{(aq/b_{m+1}, aq/c_{m+1})_n} \sum_{0 \leq l_1 \leq \cdots \leq l_m \leq n} \frac{a^{l_1+\cdots+l_{m-1}} q^{l_1+\cdots+l_m}}{(b_2 c_2)^{l_1} \ldots (b_m c_m)^{l_{m-1}}} \times \frac{(q^{-n})_{l_m}}{(b_{m+1} c_{m+1} q^{-N}/a)_{l_m}} \prod_{i=1}^{m} \frac{(b_{i+1}, c_{i+1})_{l_i}}{(aq/b_i, aq/c_i)_{l_i}} \frac{(aq/b_i c_i)_{l_i - l_{i-1}}}{(q)_{l_i - l_{i-1}}}$$

Using Andrews' formula to prove the q-denominators conjecture

We have to prove that :

$$\frac{1}{1-q^{-k}} \, \frac{d_n(1/q)^{A-s}}{(A-s)!} \left[\frac{d^{A-s}}{du^{A-s}} \sum_{j=k}^{n} (1-q^{n-2j}u^2) e_j(u) \right]_{u=1} \in \mathbb{Z}\left[q; \frac{1}{q}\right]$$

By Andrews' formula :

$$\sum_{j=k}^{n} (1-q^{n-2j}u^2) e_j(u) = \sum_{\underline{j}} v_{\underline{j}}(u)$$

and with an arithmetical study :

$$\frac{1}{1-q^{-k}} \, \frac{d_n(1/q)^{A-s}}{(A-s)!} \left[\frac{d^{A-s}}{du^{A-s}} v_{\underline{j}}(u) \right]_{u=1} \in \mathbb{Z}\left[q; \frac{1}{q}\right]$$

Illustration for $s = A$

Set $p = 1/q$, the sum $\sum_{j=k}^{n}(1 - q^{n-2j})e_j(1)$ is for $(A, r) = (0, 0)$:

$$\sum_{j=k}^{n}(1 - q^{n-2j})q^{j} = (1 - p^{k})p^{-n}\frac{1 - p^{n-k+1}}{1 - p}$$

for $(A, r) = (2, 0)$:

$$\sum_{j=k}^{n}(1 - q^{n-2j})q^{j(j-n+1)}\begin{bmatrix} n \\ j \end{bmatrix}_q^2 = (1 - p^{k})p^{-k(n-k+1)}\begin{bmatrix} n \\ k \end{bmatrix}_p\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_p$$

and for $(A, r) = (2, 1)$:

$$\sum_{j=k}^{n}(1 - q^{n-2j})q^{j^2+j-nj-n(n+1)/2}\begin{bmatrix} n+j \\ n \end{bmatrix}_q\begin{bmatrix} 2n-j \\ n \end{bmatrix}_q\begin{bmatrix} n \\ j \end{bmatrix}_q^2 = (1 - p^{k})$$

$$\times p^{k^2-kn-k-n(n-1)/2}\sum_{l=0}^{n-k}(-1)^{l}p^{l(2k+l-1)/2}\begin{bmatrix} n+k \\ k+l \end{bmatrix}_p\begin{bmatrix} n-l-1 \\ k-1 \end{bmatrix}_p\begin{bmatrix} 2n-k-l \\ k, n-k, n-k-l \end{bmatrix}_p .$$

Dirichlet's beta function

For the non trivial character modulo 4 defined by $\chi_4(2n+1) = (-1)^n$ and $\chi_4(2n) = 0$, we have Dirichlet's beta function :

$$\beta(s) := L_{\chi_4}(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}$$

At odd integers $2m+1 \geq 1$: let $E_{2m} \in \mathbb{Q}$ be Euler numbers, then

$$\beta(2m+1) = \frac{(-1)^m E_{2m}}{2^{2m+2}(2m)!}\pi^{2m+1}$$ transcendantal numbers (Lindemann)

Rivoal-Zudilin (2003) :
- there are among the values $\beta(2m)$ infinitely many irrational numbers
- at least one of the values $\beta(2), \beta(4), \ldots, \beta(12)$ is an irrational number

Catalan's constant $G = \beta(2) = \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)^2}$ irrational number ?

Still open despite the resolution of a denominators conjecture by :
- Rivoal (2006), Padé approximants
- Rivoal-Krattenthaler (2008), special cases of Andrews' formula

q-analogues of Dirichlet series at positive integers

We can define for $|q| < 1$ and $\mathrm{Re}(s) > 1$: $L_\chi(s,q) := \sum_{k\geq 1} q^k \sum_{d|k} d^{s-1}\chi(k/d)$

satisfying $\lim_{q\to 1}(1-q)^s L_\chi(s,q) = (s-1)! L_\chi(s)$

For χ_4, we define (J-Mosaki, 2010) :

$$\beta_q(s) := L_{\chi_4}(s,q) = \sum_{k\geq 1}\sum_{d|k}\chi_4(k/d)d^{s-1}q^k = \sum_{k\geq 1} k^{s-1}\frac{q^k}{1+q^{2k}}$$

One can rewrite $\beta_q(1) = (\pi_{1/q}-1)/4$, where π_q is the q-analogue of π considered by Bundschuh-Zudilin (2007), who found an upper bound for its irrationality exponant

The theta function $\theta(q) := \sum_{n\in\mathbb{Z}} q^{n^2}$ gives, through Jacobi's triple product identity and Ramanujan's ${}_1\Psi_1$ summation :

$$\sum_{n\geq 0} r_2(n)q^n = \theta^2(q) = 1 + 4\sum_{k\geq 1} q^k \sum_{d|k}\chi_4(k/d),$$

where $r_2(n)$ is the number of ways to write n as a sum of two squares

Values at odd positive integers

Therefore we have : $\beta_q(1) = (\theta^2(q) - 1)/4$ weight 1 modular form on $\Gamma_1(4)$

For $s \geq 1$ and $q = e^{2i\pi z}$:

$$\beta_q(2s+1) = i(-1)^{s+1} \frac{E_{2s}}{2\beta(2s+1)} G_{2s+1}^{(1,0)}(4z)$$

where $G_{2s+1}^{(1,0)}$ is Eisenstein series of level 4

Thus for $s \geq 0$, $\beta_q(2s+1)$ is a weight $2s+1$ modular form on $\Gamma_1(4)$

Setting $\phi_s(q) := \beta_q(2s+1)/\theta^{4s+2}(q)$, and through Nesterenko's algebraic independance theorem, we derive the transcendance of the values $\beta_q(2s+1)$ when q is algebraic

Values at even positive integers

Theorem (J-Mosaki, 2010)

For $1/q \in \mathbb{Z} \setminus \{-1; 1\}$ *and any odd integer* $A \geq 3$, *we have the lower bound :*

$$\dim_{\mathbb{Q}} (\mathbb{Q} + \mathbb{Q}\beta_q(2) + \cdots + \mathbb{Q}\beta_q(A-1)) \geq h(A),$$

where

$$h(A) = \max_{\substack{r \in \mathbb{N} \\ 1 \leq r < A/2}} h(r; A) \quad \text{and} \quad h(r; A) := \frac{4rA + A - 4r^2}{\left(\frac{48}{\pi^2} + 2\right) A + 8r^2 - \frac{16}{\pi^2} + \frac{16r}{3}},$$

$h(A)$ *satisfying* $h(A) \sim \dfrac{\pi}{2\sqrt{\pi^2 + 24}} \sqrt{A}$ *when* $A \to +\infty$.

Corollary

For $1/q \in \mathbb{Z} \setminus \{-1; 1\}$, *there are infinitely many irrational numbers among the values* $\beta_q(2)$, $\beta_q(4)$, $\beta_q(6)$,...

Corollary

For $1/q \in \mathbb{Z} \setminus \{-1; 1\}$, *at least one of the values* $\beta_q(2), \beta_q(4)$, $\beta_q(6), \ldots, \beta_q(20)$ *is an irrational number*

Another basic hypergeometric series

For an integer A, $r \in \mathbb{N}^*$ such that $A - 2r > 0$:

$$S_n(q) := (q)_n^{A-2r} \sum_{k \geq 1} (-1)^{k+1} (1 - q^{2k+n-1}) \times q^{(k-1/2)((A-2r)n/2+A/2-1)} \frac{(q^{k-rn})_{rn} (q^{k+n})_{rn}}{(q^{k-1/2})_{n+1}^{A}}$$

Linear forms for n odd :

$$S_n(q^2) = \hat{P}_{0,n}(q^2) + \sum_{\substack{j=2 \\ j \text{ even}}}^{A-1} \hat{P}_{j,n}(q^2) \beta_q(j)$$

Arithmetics of cyclotomic polynomials

We prove that :

$$D_n(q)\hat{P}_{j,n}(q^2) \in \mathbb{Z}\left[\frac{1}{q}\right] \quad \forall j \in \{0, 2, 4, \dots, A-1\},$$

where

$$D_n(q) := (A-1)!\, q^{\lfloor \alpha n^2 + \beta n + \gamma \rfloor} \varphi_n(1/q)^{2r}\, d_{2n}(1/q)^{A-1}\, \Delta_n(1/q),$$

and

$$\begin{aligned} d_n(x) &:= \prod_{t=1}^{n} \phi_t(x) = \mathrm{lcm}(x-1, \dots, x^n - 1) \\ \Delta_n(x) &:= \prod_{\substack{t=1 \\ t \text{ odd}}}^{2n-1} \phi_t(x) \\ \varphi_n(x) &:= \phi_2(x)^n \phi_4(x)^{\lfloor n/2 \rfloor} \dots \phi_{2n}(x) \end{aligned}$$

recalling the cyclotomic polynomials $\phi_t(x) := \prod_{\substack{k \wedge t = 1 \\ k \leq t}} (x - e^{2ik\pi/t}) \in \mathbb{Z}[x]$

Asymptotics

We also prove :

$$\begin{aligned}
\lim_{n\to+\infty} \frac{1}{n^2} \log|S_n(q)| &= -\frac{1}{2} r(A-2r)\log|1/q|,\\
\limsup_{\substack{n\to+\infty\\ n\,\text{impair}}} \frac{1}{n^2} \log|\hat{P}_{j,n}(q)| &\leq \frac{1}{8}(A+4r^2)\log|1/q|,\\
\lim_{n\to+\infty} \frac{1}{n^2} \log|D_n(q)| &= \left(\frac{A}{4}+r^2+\frac{12}{\pi^2}(A-1)+\frac{4r}{3}+\frac{8}{\pi^2}\right)\log|1/q|
\end{aligned}$$

Then we apply for odd n Nesterenko's criterion to :

$$D_n(q)\times S_n(q^2) = D_n(q)\times \hat{P}_{0,n}(q^2) + \sum_{\substack{j=2\\ j\,\text{even}}}^{A-1} D_n(q)\times \hat{P}_{j,n}(q^2)\times \beta_q(j),$$

with $1/q \in \mathbb{Z}\setminus\{-1;1\}$