TWO NEW TRIANGLES OF q-INTEGERS VIA q-EULERIAN POLYNOMIALS OF TYPE A AND B

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ABSTRACT. The classical Eulerian polynomials can be expanded in the basis $t^{k-1}(1+t)^{n+1-2k}$ $(1 \leq k \leq \lfloor (n+1)/2 \rfloor)$ with positive integral coefficients. This formula implies both the symmetry and the unimodality of the Eulerian polynomials. In this paper, we prove a q-analogue of this expansion for Carlitz's q-Eulerian polynomials as well as a similar formula for Chow-Gessel's q-Eulerian polynomials of type B. We shall give some applications of these two formulae, which involve two new sequences of polynomials in the variable q with positive integral coefficients. An open problem is to give a combinatorial interpretation for these polynomials.

1. Introduction

The *Eulerian polynomials* $A_n(t) := \sum_{k=1}^{n} A_{n,k} t^{k-1}$ (see [FS70, Fo09, St97]) may be defined by

$$\sum_{k\geq 1} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}} \qquad (n \in \mathbb{N}).$$

It is well known (see [FS70]) that there are nonnegative integers $a_{n,k}$ such that

$$A_n(t) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a_{n,k} t^{k-1}(1+t)^{n+1-2k}. \tag{1.1}$$

For example, for $n = 1, \ldots, 4$, the identity reads

$$A_1(t) = 1, \quad A_2(t) = 1+t, \quad A_3(t) = (1+t)^2 + 2t^2, \quad A_4(t) = (1+t)^3 + 8t(1+t).$$

In particular, this formula implies both the *symmetry* and the *unimodality* (see for instance [Br08] for the definitions) of the Eulerian numbers $(A_{n,k})_{1\leq k\leq n}$ for any fixed n. The coefficients $a_{n,k}$ defined by (1.1) satisfy the following recurrence relation:

$$a_{n,k} = k a_{n-1,k} + 2(n+2-2k) a_{n-1,k-1} \tag{1.2}$$

for $n \geq 2$ and $1 \leq k \leq \lfloor (n+1)/2 \rfloor$, with $a_{1,1} = 1$, and $a_{n,k} = 0$ for $k \leq 0$ or $k > \lfloor (n+1)/2 \rfloor$.

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$n\backslash k$	1	2	3	4
1	1			
2	1			
3	1	2		
4	1	8		
5	1	22	16	
6	1	52	136	

$n\backslash k$	0	1	2	3
1	1			
2	1	4		
3	1	20		
4	1	72	80	
5	1	232	976	
6	1	716	766	3904

Table 1. The first values of $(a_{n,k})$ and $(b_{n,k})$

The classical Eulerian polynomials are the descent polynomials of the symmetric group or *Coxeter group of type A*. Analogues of Eulerian polynomials for other Coxeter groups were introduced and studied from a combinatorial point of view in the last three decades. Recall that for instance the Eulerian polynomials of type B are defined by

$$\sum_{n\geq 0}(2k+1)^n t^n = \frac{B_n(t)}{(1-t)^{n+1}}. \tag{1.3}$$

The type B version of (1.1) appeared quite recently (see [Pe07, St08, Ch08]) and reads as follows

$$B_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k} t^k (1+t)^{n-2k}, \tag{1.4}$$

where $b_{n,k}$ are positive integers satisfying the recurrence relation

$$b_{n,k} = (2k+1)b_{n-1,k} + 4(n+1-2k)b_{n-1,k-1}, \tag{1.5}$$

for $n \geq 2$ and $0 \leq k \leq \lfloor n/2 \rfloor$, with $b_{1,0} = 1$, and $b_{n,k} = 0$ for $k \leq 0$ or $k > \lfloor n/2 \rfloor$.

The numbers $a_{n,k}$ and $4^{-k} b_{n,k}$ appear as $A101280$ and $A008971$ in The *On-Line Encyclopedia of Integer Sequences* : http://oeis.org.

The aim of this paper is to prove a q-analogue of (1.1) with a refinement of the triangle $(a_{n,k})$ for Carlitz's q-Eulerian polynomials [Ca75], and also a q-analogue of (1.4) with a refinement of the triangle $(b_{n,k})$ for Chow-Gessel's q-Eulerian polynomials of type B [CG07]. Note that some other extensions of (1.1) are discussed in [Br08, SW10, SZ10].

This paper is organized as follows: we derive in Section 2 a q-analogue of (1.1) using Carlitz's q-Eulerian polynomials and derive some results about the q-tangent number $T_{2n+1}(q)$ studied in [FH09]. In Section 3, we give a q-analogue of (1.4) using Chow-Gessel's q-Eulerian polynomials of type B, which yields new q-analogues of the secant numbers. In Section 4, we apply our constructions to some conjectures on the unimodality from [CG07]. Finally, we will briefly give some concluding remarks in the fifth and last section.

2. A q-analogue for type A

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \qquad n \geq k \geq 0,$$

where $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and $(x;q)_0 = 1$. Recall [Ca54] that Carlitz's q-Eulerian polynomials $A_n(t,q) := \sum_{k=1}^{n-1} A_{n,k}(q) t^k$ can be defined by

$$\sum_{k\geq 0} [k+1]_q^n t^k = \frac{A_n(t,q)}{(t;q)_{n+1}}, \tag{2.1}$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$. It is easy to see that $A_{n,k}(q)$ satisfy the recurrence:

$$A_{n,k}(q) = [k]_q A_{n-1,k}(q) + q^{k-1}[n+1-k]_q A_{n-1,k-1}(q) \qquad (1 \leq k \leq n). \tag{2.2}$$

The following is our q-analogue of (1.1).

Theorem 1. *For any positive integer n, there are polynomials $a_{n,k}(q) \in \mathbb{N}[q]$ such that the q-Eulerian polynomials $A_n(t,q)$ can be written as follows:*

$$A_n(t,q) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a_{n,k}(q) t^{k-1} (-tq^k; q)_{n+1-2k}. \tag{2.3}$$

Moreover, the polynomials $a_{n,k}(q)$ satisfy the following recurrence relation

$$a_{n,k}(q) = [k]_q a_{n-1,k}(q) + (1+q^{k-1}) q^{k-1} [n+2-2k]_q a_{n-1,k-1}(q) \tag{2.4}$$

for $n \geq 2$ and $1 \leq k \leq \lfloor (n+1)/2 \rfloor$, with $a_{1,1}(q) = 1$ and $a_{n,k}(q) = 0$ for $k \leq 0$ or $k > \lfloor (n+1)/2 \rfloor$.

Proof. Assume that $a_{n,k}(q)$ are coefficients satisfying (2.4). Then, by the q-binomial formula (cf. [An98, Theorem 3.3]),

$$(z;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_q (-z)^j q^{j(j-1)/2}, \tag{2.5}$$

we see that (2.3) is equivalent to:

$$A_{n,k}(q) = \sum_{s\geq 1} \begin{bmatrix} n+1-2s \\ k-s \end{bmatrix}_q q^{(k-s)s + \binom{k-s}{2}} a_{n,s}(q). \tag{2.6}$$

Substituting (2.6) in (2.2), and using (2.4), we derive:

$$\sum_{s\geq 1}\begin{bmatrix} n+1-2s \\ k-s \end{bmatrix}_q q^{(k-s)s+\binom{k-s}{2}}\left([s]_q a_{n-1,s}(q)+(1+q^{s-1})q^{s-1}[n+2-2s]_q a_{n-1,s-1}(q)\right)$$

$$=\sum_{s\geq 1} q^{(k-s)s+\binom{k-s}{2}}\left([k]_q\begin{bmatrix} n-2s \\ k-s \end{bmatrix}_q+[n+1-k]_q\begin{bmatrix} n-2s \\ k-1-s \end{bmatrix}_q\right)a_{n-1,s}(q).$$

Extracting the coefficents of $a_{n-1,s}(q)$ we obtain:

$$\begin{bmatrix} n+1-2s \\ k-s \end{bmatrix}_q [s]_q+\begin{bmatrix} n-1-2s \\ k-s-1 \end{bmatrix}_q (1+q^s)[n-2s]_q$$
$$=[k]_q\begin{bmatrix} n-2s \\ k-s \end{bmatrix}_q+[n+1-k]_q\begin{bmatrix} n-2s \\ k-1-s \end{bmatrix}_q.$$

Canceling the common factors we get:

$$[n+1-2s]_q[s]_q+[n-k-s+1]_q(1+q^s)[k-s]_q=[k]_q[n-k-s+1]_q+[n+1-k]_q[k-s]_q.$$

The last identity is easy to verify, and this shows that (2.3) is satisfied. □

The first values of the coefficients $a_{n,k}(q)$ read as follows:

$n\backslash k$	1	2	3
1	1		
2	1		
3	1	$q+q^2$	
4	1	$2q(1+q)^2$	
5	1	$q(1+q)(3+5q+3q^2)$	$2q^3(1+q)^2(1+q^2)$
6	1	$q(1+q)^2(4+5q+4q^2)$	$q^3(1+q)^2(1+q^2)(5+7q+5q^2)$

In [FH09] Foata and Han defined a new sequence of q-tangent numbers $T_{2n+1}(q)$ by

$$T_{2n+1}(q)=(-1)^n q^{\binom{n}{2}} A_{2n+1}(-q^{-n},q). \tag{2.7}$$

We derive easily the following result from Theorem 1, which is the most difficult part of the main result in [FH09, Theorem 1.1].

Corollary 2. *The q-tangent number $T_{2n+1}(q)$ is a polynomial with positive integral coefficients.*

Proof. Let $a^*_{n,k}(q)=q^{-k(k+1)/2}a_{n,k}(q)$. Then (2.4) becomes

$$a^*_{n,k}(q)=[k]_q a^*_{n-1,k}(q)+(1+q^{k-1})[n+2-2k]_q a^*_{n-1,k-1}(q)$$

with the same initial conditions as $a_{n,k}(q)$. This proves that $a^*_{n,k}(q)$ is a polynomial in q with nonnegative integral coefficients. Now we show that $T_{2n+1}(q)=a^*_{2n+1,n+1}(q)$, which is sufficient to conclude. Replacing n by $2n+1$, k by $n+1$, and t by $-q^{-n}$ in (2.3), we get

$$A_{2n+1}(-q^{-n},q)=\sum_{k=1}^{n+1} a_{2n+1,k}(q)(-q^{-n})^{k-1}(q^{k-n};q)_{2n+2-2k}=a_{2n+1,n+1}(q)(-q^{-n})^n,$$

since $(q^{k-n};q)_{2n+2-2k}=0$ for $k=1,2,\ldots,n$. The result follows then from (2.7). □

We can also derive straightforwardly the following result, which was proved in [FH09] using combinatorics of the so-called *doubloons*.

Corollary 3. *The quotient $A_{2n}(t,q)/(1+tq^n)$ is a polynomial in t and q with positive integral coefficients.*

Proof. Note that

$$A_{2n}(t,q)=\sum_{k=1}^{n} a_{2n,k}(q)t^{k-1}(-tq^k;q)_{2n+1-2k}.$$

The result follows then from the fact that for $k=1,\ldots,n$, the coefficient $(-tq^k;q)_{2n+1-2k}=(1+tq^k)\cdots(1+tq^{2n-k})$ contains the factor $1+tq^n$. □

For any nonnegative integer n, set

$$f_n(q):=\sum_{k=0}^{2n+1}\binom{2n+1}{k}\frac{(-1)^k}{1+q^{k-n}}. \tag{2.8}$$

Using the doubloon model, Foata-Han [FH09] proved that

$$d_n(q):=\frac{T_{2n+1}(q)}{(1+q)(1+q^2)\ldots(1+q^n)}=\frac{(-1)^{n+1}(-1;q)_{n+2}}{(1-q)^{2n+1}}f_n(q)$$

is a polynomial in $\mathbb{N}[q]$. Actually we can prove the integrality of $d_n(q)$ without using the combinatorial device.

Proposition 4. *We have $d_n(q)\in\mathbb{Z}[q]$.*

Proof. Let $g_n(q)=(-1)^{n+1}(-1;q)_{n+2}$. Then $f_n(q)g_n(q)$ is clearly a polynomial in $\mathbb{Z}[q]$. We must show that 1 is a zero of order $2n+1$ of the polynomial $f_n(q)g_n(q)$ or

$$d^p(f_n(q)g_n(q))/dq^n|_{q=1}=0 \quad \text{for} \quad p=0,\ldots,2n.$$

By Leibniz's rule it suffices to show that $f_n^{(p)}(1)=0$ for $p=0,\ldots,2n$.

For any $k\in\mathbb{Z}$ and $m\in\mathbb{N}$, we define the Laurent polynomial $P_{m,k}(x)$ by the relation:

$$h_k^{(m)}(x)=\left(\frac{d}{dx}\right)^m(1+x^k)^{-1}=\frac{P_{m,k}(x)}{(1+x^k)^{m+1}}.$$

Thus $P_{0,k}=1$, $P_{1,k}=-kx^{k-1}$, and for $m\geq 0$, we have

$$P_{m+1,k}(x)=(1+x^k)P'_{m,k}(x)-k(m+1)x^{k-1}P_{m,k}(x).$$

Therefore the $P_{m,k}$ can, for $m \geq 1$, be written as follows:

$$P_{m,k}(x) = \sum_{l=1}^{m} \alpha_{l,m} x^{lk-m},$$

where $\alpha_{1,1} = -k$ and for $m \geq 1$, $\alpha_{1,m+1} = (k-m)\alpha_{1,m}$, $\alpha_{m+1,m+1} = (m-k)\alpha_{m,m}$,

$$\alpha_{l,m+1} = (lk-m)\alpha_{l,m} + (lk-mk-2k-m)\alpha_{l-1,m}, \quad 2 \leq l \leq m.$$

This shows that for $m \geq 1$ and $1 \leq l \leq m$, the coefficient $\alpha_{l,m}$ is a polynomial in the variable k, with degree less than or equal to m. We deduce that $P_{m,k}(1) = \sum_{l=1}^{m} \alpha_{l,m}$ is also a polynomial in the variable k, with degree less than or equal to m, therefore we can write for some rational coefficients $a_j(m)$ only depending on m:

$$h_k^{(m)}(1) = \frac{P_{m,k}(1)}{2^{m+1}} = \sum_{j=0}^{m} a_j(m) k^j.$$

Thus, differentiating (2.8) m times ($m \geq 0$) and then setting $q = 1$, we get

$$\begin{aligned} f_n^{(m)}(1) &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k \sum_{j=0}^{m} a_j(m)(k-n)^j \\ &= \sum_{j=0}^{2n} a_j(m) \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k (k-n)^j. \end{aligned}$$

Now, applying $2n+1$ times the finite difference operator Δ (defined by $\Delta f(x) := f(x+1) - f(x)$) to the polynomial $(n+1-x)^j$ $(0 \leq j \leq 2n)$ and setting $x = 0$ we get

$$\Delta^{2n+1}(n+1-x)^j\big|_{x=0} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k (k-n)^j,$$

which should vanish because $(n+1-x)^j$ is a polynomial in x of degree $j < 2n+1$. □

3. A q-analogue for type B

A B_n-analogue of Carlitz's q-Eulerian polynomials are introduced by Chow and Gessel [CG07]. These polynomials $B(t,q)$ are defined by

$$\sum_{k \geq 0} [2k+1]_q^n t^k = \frac{B_n(t,q)}{(t;q^2)_{n+1}}. \tag{3.1}$$

Let $B(t,q) := \sum_{k=0}^{n} B_{n,k}(q) t^k$. Then, the coefficients $B_{n,k}(q)$ satisfy the recurrence relation [CG07, Prop. 3.2]:

$$B_{n,k}(q) = [2k+1]_q B_{n-1,k}(q) + q^{2k-1}[2n-2k+1]_q B_{n-1,k-1}(q) \qquad 1 \leq k \leq n. \tag{3.2}$$

We have the following B_n-analogue of (2.3).

Theorem 5. *For any positive integer n, there are polynomials $b_{n,k}(q) \in \mathbb{N}[q]$ such that the q-Eulerian polynomials of type B can be written as follows:*

$$B_n(t,q) = \sum_{k=0}^{n} B_{n,k}(q)t^k = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k}(q)t^k(-tq^{2k+1};q^2)_{n-2k}. \tag{3.3}$$

Moreover, the coefficients $b_{n,k}(q)$ satisfy the following recurrence relation:

$$b_{n,k}(q) = [2k+1]_q b_{n-1,k}(q) + (1+q)(1+q^{2k-1})q^{2k-1}[n+1-2k]_{q^2} b_{n-1,k-1}(q) \tag{3.4}$$

for $n \geq 2$ and $0 \leq k \leq \lfloor n/2 \rfloor$, with $b_{1,0}(q) = 1$, and $b_{n,k}(q) = 0$ for $k < 0$ or $k > \lfloor n/2 \rfloor$.

Proof. Assume that $b_{n,k}(q)$ are coefficients satisfying (3.4). Then, by applying (2.5) with the substitution $q \leftarrow q^2$, we derive that (3.3) is equivalent to:

$$B_{n,k}(q) = \sum_{s \geq 0} \begin{bmatrix} n-2s \\ k-s \end{bmatrix}_{q^2} q^{k^2-s^2} b_{n,s}(q). \tag{3.5}$$

Substituting (3.5) in (3.2), and using (3.4), we get:

$$\sum_{s\geq 0} \begin{bmatrix} n-2s \\ k-s \end{bmatrix}_{q^2} q^{k^2-s^2} \left([2s+1]_q b_{n-1,s}(q) + (1+q)(1+q^{2s-1})q^{2s-1}[n+1-2s]_{q^2} b_{n-1,s-1}(q)\right)$$

$$= \sum_{s\geq 0} q^{k^2-s^2} \left([2k+1]_q \begin{bmatrix} n-1-2s \\ k-s \end{bmatrix}_{q^2} + [2n+1-2k]_q \begin{bmatrix} n-1-2s \\ k-1-s \end{bmatrix}_{q^2} \right) b_{n-1,s}(q).$$

Extracting the coefficents of $b_{n-1,s}(q)$ we obtain:

$$\begin{bmatrix} n-2s \\ k-s \end{bmatrix}_{q^2} [2s+1]_q + \begin{bmatrix} n-2-2s \\ k-s-1 \end{bmatrix}_{q^2} (1+q)(1+q^{2s+1})[n-1-2s]_{q^2}$$
$$= [2k+1]_q \begin{bmatrix} n-1-2s \\ k-s \end{bmatrix}_{q^2} + [2n+1-2k]_q \begin{bmatrix} n-1-2s \\ k-1-s \end{bmatrix}_{q^2}.$$

Canceling the common factors yields:

$$[n-2s]_{q^2}[2s+1]_q + [n-k-s]_{q^2}(1+q)(1+q^{2s+1})[k-s]_{q^2}$$
$$= [2k+1]_q[n-k-s]_{q^2} + [2n+1-2k]_q[k-s]_{q^2}.$$

The last identity is easy to verify, and this proves (3.3). □

For $n = 1, \ldots, 4$, equation (3.3) reads:

$B_1(t,q) = 1 + qt;$

$B_2(t,q) = (-tq;q^2)_2 + (q + 2q^2 + q^3)t;$

$B_3(t,q) = (-tq;q^2)_3 + (2q + 5q^2 + 6q^3 + 5q^4 + 2q^5)t(1 + tq^3);$

$$B_4(t,q) = (-tq;q^2)_4 + (3q + 9q^2 + 15q^3 + 18q^4 + 15q^5 + 9q^6 + 3q^7)t(-tq^3;q^2)_2$$
$$+ (2q^4 + 7q^5 + 11q^6 + 13q^7 + 14q^8 + 13q^9 + 11q^{10} + 7q^{11} + 2q^{12})t^2.$$

Theorem 5 implies immediately the following result, of which the first was derived in [FH10', Theorem 1.1 (d)] with more work.

Corollary 6. *For $n \geq 0$, we have*

$$B_{2n+1}(-q^{-2n-1}, q) = 0, \tag{3.6}$$

$$B_{2n}(-q^{-2n-1}, q) = (-1)^n q^{-n(2n+1)} b_{2n,n}(q). \tag{3.7}$$

Proof. By (3.3) we get

$$B_{2n+1}(-q^{-2n-1}, q) = \sum_{k=0}^{n} b_{2n+1,k}(q)(-q^{-2n-1})^k (q^{-2n+2k}; q^2)_{2n+1-2k} = 0.$$

Substituting n by $2n$ and t by $-q^{-2n-1}$ in (3.3) yields

$$\begin{aligned} B_{2n}(-q^{-2n-1}, q) &= \sum_{k=0}^{n} b_{2n,k}(q)(-q^{-2n-1})^k (q^{-2n+2k}; q^2)_{2n-2k} \\ &= (-1)^n q^{-n(2n+1)} b_{2n,n}(q). \end{aligned}$$

□

The above result leads to define a q-analogue of $B_{2n}(-1) = (-1)^n 4^n E_{2n}$ (where the E_{2n}'s are the famous *secant numbers*) by

$$E^*_{2n}(q) := (-1)^n q^{n(n+1)} B_{2n}(-q^{-2n-1}, q). \tag{3.8}$$

Theorem 7. *There is a polynomial $G^*_{2n}(q) \in \mathbb{Z}[q]$ such that $G^*_{2n}(1) = E_{2n}$ and*

$$E^*_{2n}(q) = (1+q)(1+q^3)(1+q^5)\cdots(1+q^{2n-1}) \cdot (1+q)^n \cdot G^*_{2n}(q).$$

Proof. Recall that $E^*_{2n}(q) = (-1)^n q^{n(n+1)} B_{2n}(-q^{-2n-1}, q)$. From (3.1) we derive

$$\begin{aligned} \frac{B_n(t,q)}{(t;q^2)_{n+1}} &= (1-q)^{-n} \sum_{j\geq 0} (1-q^{2j+1})^n t^j \\ &= (1-q)^{-n} \sum_{j\geq 0} t^j \sum_{k=0}^{n} \binom{n}{k} (-q^{2j+1})^k \\ &= (1-q)^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-q)^k}{1-tq^{2k}}. \end{aligned}$$

Substituting n by $2n$ and setting $t = -q^{-2n-1}$ we obtain

$$E^*_{2n}(q) = (-1)^n q^{n(n+1)} \frac{(-q^{-2n-1}; q^2)_{2n+1}}{(1-q)^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} \frac{(-q)^k}{1+q^{2k-2n-1}}.$$

Let

$$G^*_{2n}(q) := \frac{E^*_{2n}(q)}{(1+q)(1+q^3)\dots(1+q^{2n-1})(1+q)^n}$$
$$= (-1)^n q^{-n-1} \frac{(-q;q^2)_{n+1}}{(1+q)^n(1-q)^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} \frac{(-q)^k}{1+q^{2k-2n-1}}.$$

For any nonnegative integer n, set

$$f^*_n(q) := \sum_{k=0}^{2n} \binom{2n}{k} \frac{(-q)^k}{1+q^{2k-2n-1}}. \tag{3.9}$$

Let $g^*_n(q) = (-1)^n q^{-n-1}(-q;q^2)_{n+1}/(1+q)^n$. Then $f^*_n(q)g^*_n(q)$ is clearly a polynomial in $\mathbb{Z}[q]$. We must show that 1 is a zero of order $2n$ of the polynomial $f^*_n(q)g^*_n(q)$ or

$$d^p(f^*_n(q)g^*_n(q))/dq^n|_{q=1} = 0 \quad \text{for} \quad p = 0,\dots,2n-1.$$

By Leibniz's rule it suffices to show that $d^p(f^*_n(q))/dq^p|_{q=1} = 0$ for $p = 0,\dots,2n-1$. The rest of the proof is almost the same as that of Proposition 4, and is left to the reader. □

Conjecture 8. *All the coefficients of the polynomials $G^*_{2n}(q)$ are positive.*

Since $G^*_{2n}(1) = E_{2n}$, the above conjecture would yield a new refinement of the secant number.

4. Application to unimodal problems

A sequence $\{\alpha_0,\dots,\alpha_d\}$ is *unimodal* if there exists an index $0 \le j \le d$ such that $\alpha_i \le \alpha_{i+1}$ for $i = 1,\dots,j-1$ and $\alpha_i \ge \alpha_{i+1}$ for $i = j,\dots,d$. Chow and Gessel [CG07] studied a kind of unimodality property of the q-Eulerian numbers assuming that q is a real number. In this section, we derive some unimodal properties of the sequence $(A_{n,k}(q))_{1\le k\le n}$ and $(B_{n,k}(q))_{1\le k\le n}$ from our previous results. From Theorem 1, we are able to deduce the following corollary, which provides a further support to Conjecture 4.8 in [CG07].

Proposition 9. *Let $n \ge 2$ be an integer and $j = \lfloor (n+1)/2 \rfloor$. Then for $k = 1,\dots,j-1$, we have $A_{n,k+1}(q) > A_{n,k}(q)$ if $q > 1$ and $A_{n,n-k+1}(q) < A_{n,n-k}(q)$ if $q < 1$.*

Proof. We start from (2.6), which can be rewritten

$$A_{n,k}(q) = \sum_{s=1}^{k} \begin{bmatrix} n+1-2s \\ k-s \end{bmatrix}_q q^{(k-s)(k+s-1)/2} a_{n,s}(q),$$

for $k = 1, \dots, n$, where we assume $a_{n,s}(q) = 0$ for $s > j$. Thus we can write for $k = 1, \dots, j-1$:

$$A_{n,k+1}(q) - A_{n,k}(q) = a_{n,k+1}(q) + \sum_{s=1}^{k} \begin{bmatrix} n+1-2s \\ k+1-s \end{bmatrix}_q q^{(k+1-s)(k+s)/2} a_{n,s}(q) \left(1 - q^{-k} \frac{1-q^{k+1-s}}{1-q^{n+1-k-s}}\right).$$

We know that the q-binomial coefficient is a polynomial in q with nonnegative integer coefficients, and from Theorem 1 that this is also true for $a_{n,s}(q)$, $s = 1, \dots, k+1$. Therefore it is enough to show that the coefficient between brackets is nonegative for $1 \le s \le k \le j-1$. This coefficient can be rewritten as:

$$\frac{q^{n+1} - q^{k+s} + q^s - q^{k+1}}{q^{n+1} - q^{k+s}}.$$

Assume first that $q > 1$. As $k+s \le 2j-2 \le n-1 < n+1$, the denominator of this fraction is positive. Moreover, it is not difficult to see that under the conditions $1 \le s \le k \le j-1$, and by using $(n-1)/2 \le j \le (n+1)/2$, we have the following inequalities:

$$\begin{aligned} q^{n+1} - q^{k+s} + q^s - q^{k+1} &\ge q^{n+1} - q^{2k} + q^k - q^{k+1} \\ &\ge q^{n+1} - q^{2j-2} + q^{j-1} - q^j \\ &\ge q^{n+1} - q^{n-1} + q^{(n-3)/2} - q^{(n+1)/2}. \end{aligned}$$

This last expression can be rewritten $(q^{(n+1)/2} - 1)(q^{(n+1)/2} - q^{(n-3)/2})$ and is nonnegative, which shows that $A_{n,k+1}(q) \ge A_{n,k}(q)$ for $k = 1, \dots, j-1$.

In the case $0 < q < 1$, we only need to use the well-known relation $A_{n,n-k+1}(q) = q^{n(n-1)/2} A_{n,k}(1/q)$ for any $k = 1, \dots, n$, and the result is obvious from the case $q > 1$. □

In the type B case, it is conjectured in [CG07, Conjecture 4.6] that the sequence $(B_{n,k}(q))_{0 \le k \le n}$ is unimodal. By Theorem 5, we are able to confirm partially this conjecture.

Proposition 10. *Let $n \ge 2$ be an integer and $j = \lfloor n/2 \rfloor$. Then for $k = 1, \dots, j-1$, we have $B_{n,k+1}(q) > B_{n,k}(q)$ if $q > 1$ and $B_{n,n-k}(q) < B_{n,n-k-1}(q)$ if $q < 1$.*

Proof. We start from (3.5), which can be rewritten

$$B_{n,k}(q) = \sum_{s=0}^{k} \begin{bmatrix} n-2s \\ k-s \end{bmatrix}_{q^2} q^{k^2-s^2} b_{n,s}(q),$$

for $k = 0, \ldots, n$, where we assume $b_{n,s}(q) = 0$ for $s > j$. Thus we can write for $k = 0, \ldots, j-1$:

$$B_{n,k+1}(q) - B_{n,k}(q) = b_{n,k+1}(q) + \sum_{s=0}^{k} \begin{bmatrix} n-2s \\ k+1-s \end{bmatrix}_{q^2} q^{(k+1)^2-s^2} b_{n,s}(q) \left(1 - q^{-2k-1} \frac{1-q^{2(k+1-s)}}{1-q^{2(n-k-s)}}\right).$$

We know that the q-binomial coefficient is a polynomial in q with nonnegative integer coefficients, and from Theorem 5 that this is also true for $b_{n,s}(q)$, $s = 0, \ldots, k+1$. Therefore it is enough to show that the coefficient between brackets is nonegative for $0 \leq s \leq k \leq j-1$. This coefficient can be rewritten as:

$$\frac{q^{2n} - q^{2s+2k} + q^{2s-1} - q^{2k+1}}{q^{2n} - q^{2k+2s}}.$$

Assume first that $q > 1$. As $k + s \leq 2j - 2 \leq n - 2 < n$, the denominator of this fraction is positive. Moreover, it is not difficult to see that under the conditions $0 \leq s \leq k \leq j-1$, and by using $n/2 - 1 \leq j \leq n/2$, we have the following inequalities:

$$\begin{aligned} q^{2n} - q^{2s+2k} + q^{2s-1} - q^{2k+1} &\geq q^{2n} - q^{4k} + q^{2k-1} - q^{2k+1} \\ &\geq q^{2n} - q^{4j-4} + q^{2j-3} - q^{2j-1} \\ &\geq q^{2n} - q^{2n-4} + q^{n-5} - q^{n-1}. \end{aligned}$$

This last expression can be rewritten $(q^{2n} - q^{n-1})(1 - q^{-4})$ and is nonnegative, which shows that $B_{n,k+1}(q) \geq B_{n,k}(q)$ for $k = 0, \ldots, j-1$.

In the case $0 < q < 1$, we only need to use the well-known relation $B_{n,n-k}(q) = q^{n^2} B_{n,k}(1/q)$ for any $k = 0, \ldots, n$, and the result is obvious from the case $q > 1$. □

5. An open problem on the combinatorial interpretations

By Theorems 1 and 5, the polynomials $a_{n,k}(q)$ and $b_{n,k}(q)$ have positive integral coefficients. It is then natural to ask the following question.

Problem 11. *What are the combinatorial interpretations for $a_{n,k}(q)$ and $b_{n,k}(q)$?*

We can give a combinatorial interpretation for the *odd central terms* $a_{2n+1,n+1}(q)$ by using the *doubloon* model. Recall [FH09] that a *doubloon* of order $(2n+1)$ is defined to be a permutation of the word $012\cdots(2n+1)$, represented as a $2\times(n+1)$-matrix $\delta = \binom{a_0 \cdots a_n}{b_0 \cdots b_n}$. Define

$$\mathrm{cmaj}'\,\delta := \mathrm{maj}(a_0 \cdots a_n b_n \cdots b_0) - (n+1)\mathrm{des}(a_0 \cdots a_n b_n \cdots b_0) + n^2,$$

where "des" and "maj" are the usual *number of descents* and *major index* defined for words. A doubloon $\delta = \binom{a_0 \cdots a_n}{b_0 \cdots b_n}$ is said to be *interlaced*, if for every $k = 1, 2, \ldots, n$ the sequence $(a_{k-1}, a_k, b_{k-1}, b_k)$ or one of its three *cyclic rearrangements* is monotonic increasing or decreasing. By Theorem 1.5 in [FH09] we have the following result.

Proposition 12. *The polynomial $a_{2n+1,n+1}(q)$ is the generating function for the set of interlaced doubloons of order $2n+1$ by the statistic* cmaj'.

Another sequence of q-secant numbers is introduced in [FH10'] by

$$E_{2n}(q) = (-1)^n q^{n^2} B_{2n}(-q^{-2n}, q).$$

Unfortunately, it seems not easy to relate our coefficients $b_{n,k}(q)$ from Section 3 to the doubloons of type B, even for the central cases.

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