

LENGTH ENUMERATION OF FULLY COMMUTATIVE ELEMENTS IN FINITE AND AFFINE COXETER GROUPS

RICCARDO BIAGIOLI, MIREILLE BOUSQUET-MÉLOU, FRÉDÉRIC JOUHET,
AND PHILIPPE NADEAU

ABSTRACT. An element w of a Coxeter group W is said to be *fully commutative* if any reduced expression of w can be obtained from any other by a sequence of transpositions of adjacent commuting generators. These elements were described in 1996 by Stembridge in the case of finite irreducible groups, and more recently by Biagioli, Jouhet and Nadeau (BJN) in the affine cases. We focus here on the length enumeration of these elements. Using a recursive description, BJN established systems of non-linear q -equations for the associated generating functions. Here, we show that an alternative recursive description leads to explicit expressions for these generating functions.

1. INTRODUCTION

Let (W, S) be a Coxeter system. An element w of W is said to be *fully commutative* (or fc for short) if any reduced expression of w can be obtained from any other by a sequence of transpositions of adjacent commuting generators. For instance, in the finite or affine symmetric group, fc elements coincide with 321-avoiding permutations [7, 22]. The description and enumeration of fully commutative elements has been of interest in the algebraic combinatorics literature for about 20 years, starting with the work of Stembridge [31, 32, 33]. We refer to the above papers and to [5] for motivations of this topic. Stembridge classified Coxeter groups having finitely many fc elements, and was able to count those elements in each case [31, 33].

More recently, several authors got interested, not only in the number of such elements, but also in their q -enumeration, where the variable q records their Coxeter length [5, 23]. For instance, in the symmetric group A_2 , all elements except the maximal permutation are fc, and their length enumeration yields the polynomial

$$A_2^{FC}(q) = 1 + 2q + 2q^2.$$

This point of view naturally extends the study to arbitrary Coxeter groups W (having possibly infinitely many fc elements), since they still have finitely many elements of given length. In this case the length enumeration of fc elements in W gives rise to a power series rather than a polynomial. We denote this series by $W^{FC}(q)$.

In particular, three of the authors of the present paper (BJN) were able to characterize, for all families of classical finite or affine Coxeter groups $(A_n, \tilde{A}_n, B_n, \text{etc.})$, the series $W_n^{FC}(q)$, and in fact, the bivariate generating function

$$W(x, q) := \sum_n W_n^{FC}(q)x^n, \tag{1}$$

by systems of non-linear q -equations [5]. For instance, in the A -case, the system can be reduced to a single quadratic q -equation for a series denoted $M^*(x) \equiv M^*(x, q)$:

$$M^*(x) = 1 + xM^*(x) + xq(M^*(x) - 1)M^*(xq).$$

Using these systems of equations, BJN were able to prove that $W_n^{FC}(q)$ is always a rational function of q , with simple poles at roots of unity: this means that the coefficients of these series are ultimately periodic [5, 25]. This was first proved in the \tilde{A} -case by Hanusa and Jones [23].

Subsequently, one of us (N) showed that $W^{FC}(q)$ is in fact rational for any Coxeter group W , and determined when its coefficients are ultimately periodic [27].

The aim of this paper is to provide closed form expressions for generating functions of the form (1) for all classical Coxeter groups. Some of them turn out to be particularly elegant. Here are, for instance, our results in the A - and \tilde{A} -cases. For $n \geq 0$, we denote

$$(x)_n \equiv (x; q)_n = (1-x)(1-xq) \cdots (1-xq^{n-1}). \quad (2)$$

Theorem 1.1. *Let $A(x, q) \equiv A$ and $\tilde{A}(x, q) \equiv \tilde{A}$ be the generating functions of fully commutative elements of type A and \tilde{A} , respectively defined by*

$$A = \sum_{n \geq 0} A_n^{FC}(q)x^n \quad \text{and} \quad \tilde{A} = \sum_{n \geq 1} \tilde{A}_{n-1}^{FC}(q)x^n.$$

Then

$$A = \frac{1}{1-xq} \frac{J(xq)}{J(x)} \quad \text{and} \quad \tilde{A} = -x \frac{J'(x)}{J(x)} - \sum_{n \geq 1} \frac{x^n q^n}{1-q^n},$$

where $J(x)$ is the following series:

$$J(x) = \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n}.$$

The A -result was already obtained by Barucci *et al.* in terms of pattern avoiding permutations [3]. The \tilde{A} -result is new, though another (much more complicated) expression of this series was given by Hanusa and Jones [23].

We obtain similar results for fc involutions, already considered by Stembridge in 1998 [33], and for which non-linear q -equations were given in [4]. We denote by $\mathcal{W}^{FC}(q)$ (with a calligraphic \mathcal{W}) the length generating function of fc involutions in a Coxeter group W , and for a family W_n of such groups, we consider the generating function

$$\mathcal{W}(x, q) = \sum_n \mathcal{W}_n^{FC}(q)x^n.$$

For $n \geq 0$, we denote

$$((x))_n \equiv (x; q^2)_n = (1-x)(1-xq^2) \cdots (1-xq^{2n-2}). \quad (3)$$

Theorem 1.2. *Let $\mathcal{A}(x, q) \equiv \mathcal{A}$ and $\tilde{\mathcal{A}}(x, q) \equiv \tilde{\mathcal{A}}$ be the generating functions of fully commutative involutions of type A and \tilde{A} , respectively defined by*

$$\mathcal{A} = \sum_{n \geq 0} \mathcal{A}_n^{FC}(q)x^n \quad \text{and} \quad \tilde{\mathcal{A}} = \sum_{n \geq 1} \tilde{\mathcal{A}}_{n-1}^{FC}(q)x^n.$$

Then

$$\mathcal{A} = \frac{\mathcal{J}(-xq)}{\mathcal{J}(x)} \quad \text{and} \quad \tilde{\mathcal{A}} = -x \frac{\mathcal{J}'(x)}{\mathcal{J}(x)},$$

with

$$\mathcal{J}(x) = \sum_{n \geq 0} \frac{(-1)^{\lceil n/2 \rceil} x^n q^{\binom{n}{2}}}{((q^2))_{\lfloor n/2 \rfloor}}.$$

The story of this paper, and of the tools that we use, parallels the story of the area enumeration of *convex polyominoes* (examples of such objects appear in Figure 5): early results on this topic involved systems of non-linear q -equations arising from a certain recursive description of these objects [9, 17, 16]. These equations were solved — when they were solved — by guessing and checking [17, 11]. Then came a method that could linearize some of them [29]. Finally, an alternative recursive description, pioneered as early as 1974 but overlooked for two decades [26], led directly, in a constructive fashion, to closed form expressions [10, 28]. This approach had been previously used to obtain *perimeter* (rather than area) generating functions of polyominoes [34].

As we shall see, fc elements of type A are in fact very directly related to a simple class of convex polyominoes, called staircase (or: parallelogram) polyominoes.

Here is now an outline of the paper. In Section 2, we recall Stembridge's description of fc elements in terms of *heaps* (or partially commutative words), in the sense of Viennot [35] (see Figure 2 for an illustration). Then the key point in the enumeration of fc elements in families of irreducible finite or affine Coxeter groups becomes the enumeration of the so-called *alternating heaps* over a path or a cycle (Figure 2).

In Section 3 we state our results for the finite groups A_n , B_n and D_n . The forms of the B - and D -series are similar to those of the A -series shown in Theorems 1.1 and 1.2. To obtain these results, the key problem is to count alternating heaps over a path *having at most one piece in the rightmost column*. We solve this problem in Section 4 using a recursive approach, and this yields the finite type results stated in Section 3.

In Section 5 we address the \tilde{A} -case, which boils down to counting alternating heaps over a cycle. There, our recursive approach seems harder to implement. Instead, we start from the q -equations of [5] describing the A - and \tilde{A} -cases, and use our results for the A -case to solve them. We thus obtain the nice expressions of Theorems 1.1 and 1.2.

The other affine cases (\tilde{B} , \tilde{C} and \tilde{D}) require to count alternating heaps over a path, this time with no condition on the rightmost column. The recursive approach of Section 4 would yield very heavy expressions. We use instead an alternative one, described in Section 6: it yields simpler expressions, but they involve positive *and negative* powers of q . This phenomenon has in fact been witnessed in polyomino enumeration already (see [18, 19] and [21, Ex. 5.5.2]). We then derive from our results on alternating heaps the generating functions of fc elements in the affine cases \tilde{B} , \tilde{C} and \tilde{D} (Section 7).

We have checked all our expressions using the package GAP [20]. Details on this procedure are given after Theorem 3.1.

2. FULLY COMMUTATIVE ELEMENTS AND HEAPS OF GENERATORS

Let M be a square symmetric matrix indexed by a finite set S , satisfying $m_{ss} = 1$ and, for $s \neq t$, $m_{st} = m_{ts} \in \{2, 3, \dots\} \cup \{\infty\}$. The *Coxeter group* W associated with the matrix M is defined by its set S of generators and by the following relations: all generators are reflections ($s^2 = \text{id}$ for all $s \in S$), and they satisfy *braid relations*:

$$\underbrace{sts \cdots}_{m_{st}} = \underbrace{tst \cdots}_{m_{st}} \quad \text{if } m_{st} < \infty. \quad (4)$$

When $m_{st} = 2$, the braid relation reduces to a *commutation relation* $st = ts$. Note that what we mean here by ‘‘Coxeter group’’ is W plus its presentation in terms of S and M . We refer to [8, 24] for classical textbooks on this topic.

The *Coxeter graph* associated to W is the graph Γ with vertex set S and, for each pair $\{s, t\}$ with $m_{st} \geq 3$, an edge between s and t . This edge is labeled by m_{st} if $m_{st} > 3$ and left unlabeled if $m_{st} = 3$. Two generators that are not joined by an edge commute. Note that the graph completely characterizes the group. We will consider here the main infinite families of finite and affine Coxeter groups, whose Coxeter graphs are shown in Figure 1.

We consider *words* on the alphabet S (that is, finite sequences of elements of S), which we denote between brackets to distinguish them from group elements. For instance, $[ss]$ is a 2-letter word, different from the empty word, while ss is the identity in W . For $w \in W$, the *length* of w , denoted by $\ell(w)$, is the minimum length l of any word $[v_1 \cdots v_l]$, with $v_j \in S$ for all j , such that $w = v_1 \cdots v_l$. Such a word is then called a *reduced expression* of w . A fundamental result in Coxeter group theory, sometimes called the *Matsumoto property*, is that any reduced expression of w can be obtained from any other using only the braid relations (4) on the generators v_j .

Definition 2.1. *An element $w \in W$ is fully commutative (fc) if any reduced expression of w can be obtained from any other by using only commutation relations.*

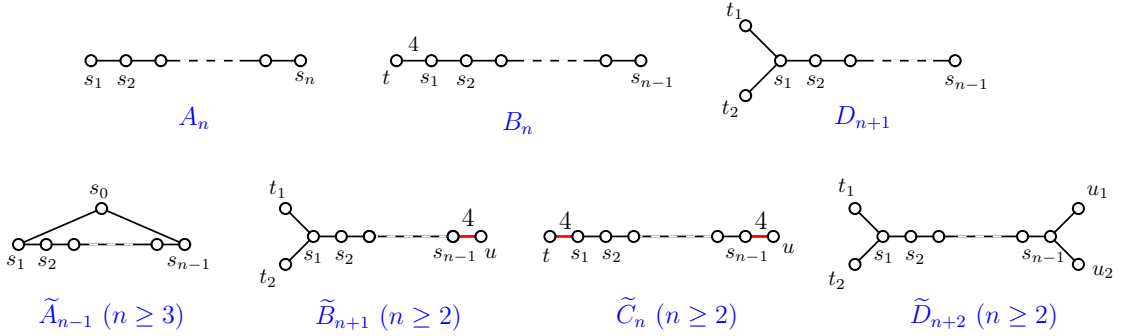


FIGURE 1. Coxeter graphs for classical types.

The set of words on the alphabet S , quotiented by the commutation relations $st = ts$ for s and t such that $m_{st} = 2$, forms a *partially commutative monoid* in the sense of Cartier and Foata [14]. Its elements, which are the commutation classes, can be represented as special posets called, in Viennot's terminology [35], *heaps over the Coxeter graph* Γ . For instance, the first heap of Figure 2 represents the commutation class of the word $[s_4 s_3 s_1 s_2 s_1 s_2 s_5 s_5 s_5 s_4 s_3 s_5 s_2 s_4 s_6 s_1]$. The correspondence between the word and the poset is rather intuitive, but let us still recall the definition of these heaps [35, Def. 2.1].

Definition 2.2. A heap (H, \prec, ϵ) over the graph Γ is a partially ordered set (poset) (H, \prec) and a labeling function $\epsilon : H \rightarrow S$ such that elements of H labeled s (resp. labeled s or t) form a chain for any $s \in S$ (resp. for any s, t that are adjacent in Γ). This chain is denoted H_s (resp. H_{st}). Moreover these chains must induce the ordering \prec of H .

We consider heaps up to isomorphism, that is to say poset isomorphism that preserves the labeling function.

We will call *points* the elements of H , each point being labeled with a generator s of S . In our figures, all points with label s are placed on a vertical line above s (Figure 2). We let $|H|_s$ denote the number of points labeled s , and $|H| = \sum_{s \in S} |H|_s$ the total number of points, also called *size* of H . Of course, this is the length of the associated word on S .

In order to discuss fc involutions, we will need to define *self-dual heaps*: a heap (H, \prec, ϵ) is self-dual if it is isomorphic to (H, \succ, ϵ) . An example is shown on Figure 4.

Now take w in the group W . Saying that w is fc means precisely that all its reduced expressions correspond to the same heap H , see [31, Sec. 2.2]. We will say that H itself is fully commutative, and identify fc elements with these heaps.

A characterization of fc heaps for an arbitrary Coxeter graph was given by Stembridge [31, Prop. 3.3]. Underlying this characterization is the notion of *alternating heaps*.

Definition 2.3. A heap H over Γ is alternating if, for all s and t that are adjacent in Γ , the points of the chain H_{st} are alternatingly labeled s and t .

In this paper we will consider alternating heaps over a path and over a cycle (Figure 2, center and right). Then the chains H_{st} form increasing zigzag paths between the points of neighboring columns. In the following sections, we do not draw these paths, since they can be read off from the positions of points; see for instance Figure 3.

We finish this section with some standard algebraic notation. For a ring R , we denote by $R[x]$ the ring of polynomials in x with coefficients in R . If R is a field, then $R(x)$ stands for the field of rational functions in x . This notation is generalized to several variables in the usual way.

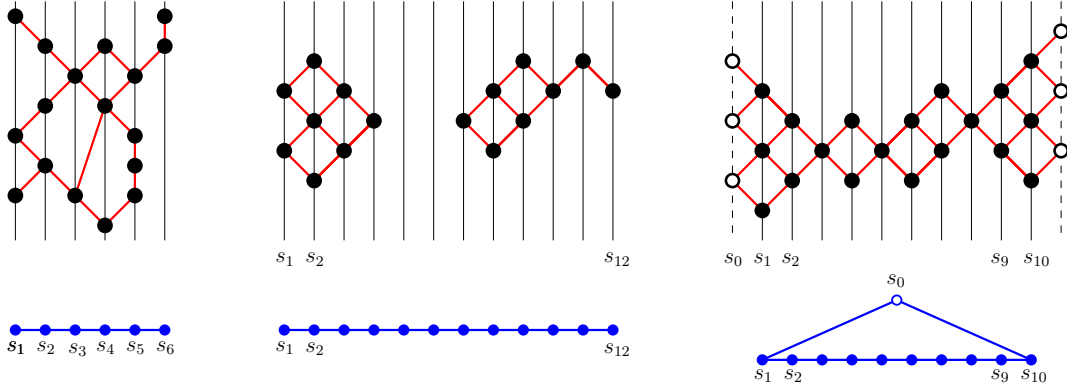


FIGURE 2. Left: A heap over the 6-point path. Center: An alternating heap over the 12-point path. Right: an alternating heap over the 11-point cycle.

3. GENERATING FUNCTIONS FOR FINITE TYPES

In this section, we state our results for fully commutative elements, and then for fully commutative involutions, in Coxeter groups of finite types. They will be proved in the next section.

3.1. ALL FULLY COMMUTATIVE ELEMENTS

The generating functions of fully commutative elements will be expressed in terms of two series $J(x)$ and $K(x)$, defined by

$$J(x) = \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n}, \quad K(x) = \sum_{n \geq 0} \frac{x^n q^{\binom{n+1}{2}}}{(xq)_n} \sum_{k=0}^n \frac{(-1)^k}{(q)_k}, \quad (5)$$

where the notation $(x)_n$ is defined by (2).

Theorem 3.1. *Let $A(x, q) \equiv A$, $B(x, q) \equiv B$, and $D(x, q) \equiv D$ be the generating functions of fully commutative elements of types A , B and D , defined respectively by:*

$$A = \sum_{n \geq 0} A_n^{FC}(q) x^n, \quad B = \sum_{n \geq 0} B_n^{FC}(q) x^n, \quad D = \sum_{n \geq 0} D_{n+1}^{FC}(q) x^n.$$

Then:

$$A = \frac{1}{1-xq} \frac{J(xq)}{J(x)}, \quad (6)$$

$$B = \frac{K(x)}{J(x)} + \frac{xq^2(1-x)}{(1-xq)(1-xq^2)} \frac{J(xq)}{J(x)} - \frac{xq^2}{1-xq^2},$$

and

$$D = \frac{2K(x)}{J(x)} + \frac{q - (1+q)x + x^2q^2}{(1-xq)(1-xq^2)} \frac{J(xq)}{J(x)} - \frac{q}{1-xq^2} - 1,$$

where the series J and K are defined by (5).

Remarks

1. Above, we have adopted the convention that $B_0 = A_0 = \{\text{id}\}$ and $B_1 \cong A_1 = \{\text{id}, s_1\}$, so that $B_0^{FC}(q) = A_0^{FC}(q) = 1$ and $B_1^{FC}(q) = A_1^{FC}(q) = 1 + q$. For $n = 0$ or 1 , the group D_{n+1} is not well defined, and the coefficients of x^0 and x^1 in $D(x, q)$ are irrelevant. We could thus drop the term -1 in the expression of D . The only reason why it is there is to fit with the series D^{FC} of [5, Prop. 4.6]. With this convention, $D_1^{FC}(q) = 1$ and $D_2^{FC}(q) = (1+q)^2$.

2. One can feed a computer algebra system with the above expressions and expand them in x to compute the polynomials $W_n^{FC}(q)$ for small values of n . This gives for instance the values of Table 1.

This also allows us to check our results, since we can compute independently the series $W_n^{FC}(q)$ (or at least its first coefficients) for any given Coxeter group W_n : according to [5, Sec. 6], fc elements of W_n index a basis of the so-called *nil-Temperley-Lieb algebra* associated with W_n . This algebra is graded, and in fact $W_n^{FC}(q)$ is its Hilbert series. For small values of n and ℓ , the package GBNP of GAP can compute a basis of the grade- ℓ component of the algebra: this is nothing but the list of fc elements of length ℓ in W_n , and the number of such elements is the coefficient of q^ℓ in $W_n^{FC}(q)$. We can then select the involutions among these elements to determine the first terms of $\mathcal{W}_n^{FC}(q)$.

n	2	3	4
A_n^{FC}	[1, 2, 2]	[1, 3, 5, 4, 1]	[1, 4, 9, 12, 10, 4, 2]
B_n^{FC}	[1, 2, 2, 2]	[1, 3, 5, 6, 5, 3, 1]	[1, 4, 9, 14, 16, 15, 11, 7, 3, 2, 1]
D_{n+1}^{FC}	[1, 3, 5, 4, 1]	[1, 4, 9, 13, 11, 7, 3]	[1, 5, 14, 26, 34, 32, 25, 17, 7, 4, 2]

TABLE 1. Length generating functions of fc elements in finite Coxeter groups. The list $[a_0, \dots, a_k]$ stands for the polynomial $a_0 + a_1q + \dots + a_kq^k$.

3. In the series $J(x)$ and $K(x)$, the coefficient of x^n is a *rational function* of q whose poles are roots of unity, whereas in A, B and D , the coefficient of x^n is a *polynomial* in q (because there are finitely many fc elements in each group A_n, B_n and D_n). In Section 6 we derive alternative expressions of A, B and C , in which the coefficient of x^n appears as a *Laurent polynomial* in q . See for instance Theorem 6.1 for the A -case.

The next proposition clarifies the algebraic properties of the series J and K . Both satisfy linear q -equations, of respective orders 2 and 3. By analogy with the theory of D-finite series [30, Chap. 6], we could say that J and K are *q-finite*. We refer to [15] for an algebraic treatment of such series in terms of Ore algebras, where the *q-shift* $F(x) \mapsto F(xq)$ plays the role of a derivation. Thanks to the following proposition, one can conclude that, if $G(x)$ is any of the above generating function of fc elements, $G(x)J(x)$ belongs to a 3-dimensional vector space over $\mathbb{Q}(x, q)$ closed under the q -shift.

To study the series J and K , it is convenient to introduce another series H , closely related to J :

$$H(x) = \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (x)_n}. \quad (7)$$

Proposition 3.2. *The following identities hold:*

$$J(x) = (1-x)H(x) + xH(xq), \quad (8)$$

and

$$J(x) + \frac{x}{1-xq}J(xq) = H(xq). \quad (9)$$

Consequently,

$$H(x) = J(x) - \frac{x^2 J(xq)}{(1-x)(1-xq)}, \quad (10)$$

and the series $H(x)$ and $J(x)$ satisfy second order linear q -equations:

$$(1-x)H(x) - (1-2x)H(xq) + \frac{x^2 q}{1-xq}H(xq^2) = 0, \quad (11)$$

$$(1-xq)J(x) - (1-x(1+q))J(xq) + \frac{x^2 q^2}{1-xq^2}J(xq^2) = 0. \quad (12)$$

The vector space (over $\mathbb{Q}(x, q)$) spanned by $J(x)$ and its q -shifts $J(xq), J(xq^2), \dots$ has dimension 2, and coincides with the space spanned by all series $H(xq^i)$.

We also have

$$K(x) - \frac{xq}{1-xq}K(xq) = H(xq), \quad (13)$$

so that $K(x)$ satisfies a linear q -equation of order 3:

$$K(x) - K(xq) + \frac{xq^2}{1-xq^2}K(xq^2) - \frac{x^3q^6}{(1-xq)(1-xq^2)(1-xq^3)}K(xq^3) = 0.$$

The vector space spanned by the series $K(xq^i)$ over $\mathbb{Q}(x, q)$ has dimension 3, and contains all q -shifts of J and H by (13).

Proof. We begin with the first two identities:

$$\begin{aligned} J(x) - (1-x)H(x) &= \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n} - \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}} (1-x)}{(q)_n (x)_n} \\ &= \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n} (1 - (1-xq^n)) = xH(xq), \end{aligned}$$

and

$$\begin{aligned} J(x) + \frac{x}{1-xq}J(xq) &= \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n} - \sum_{n \geq 0} \frac{(-x)^{n+1} q^{\binom{n+1}{2}}}{(q)_n (xq)_{n+1}} \\ &= \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n} - \sum_{n \geq 1} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_{n-1} (xq)_n} \\ &= \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n} (1 - (1-q^n)) = H(xq). \end{aligned}$$

By eliminating $H(xq)$ between (8) and (9) we obtain (10). Also, by eliminating $J(x)$ and $J(xq)$ between (8), (9), and the shifted version of (8), we obtain the second order equation (11) satisfied by H . By eliminating $H(xq)$ between (9) and the shifted version of (10), we obtain the linear equation (12) satisfied by J .

Let us now prove (13):

$$\begin{aligned} K(x) - \frac{xq}{1-xq}K(xq) &= \sum_{n \geq 0} \frac{x^n q^{\binom{n+1}{2}}}{(xq)_n} \sum_{k=0}^n \frac{(-1)^k}{(q)_k} - \sum_{n \geq 1} \frac{x^n q^{\binom{n+1}{2}}}{(xq)_n} \sum_{k=0}^{n-1} \frac{(-1)^k}{(q)_k} \\ &= \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n+1}{2}}}{(xq)_n (q)_n} = H(xq). \end{aligned}$$

Finally, combining (13) with the shifted version of (11) gives the third order equation satisfied by K .

It remains to prove our results about dimensions. If the space spanned by the q -shifts of $J(x)$ had dimension 1 only, J would satisfy a first order linear equation, so that $J(x)/J(xq)$ would be a rational function. Assume this is the case, and write

$$(1-xq) \frac{J(x)}{J(xq)} = \frac{N(x)}{D(x)},$$

where N and D are polynomials in x with coefficients in $\mathbb{Q}(q)$, with no common factor. Dividing (12) by $J(xq)$ then gives

$$\frac{N(x)}{D(x)} + x^2 q^2 \frac{D(xq)}{N(xq)} = 1 - x(1+q).$$

Given that $J(x) = 1 + O(x)$, we can normalize N and D by fixing $N(0) = D(0) = 1$. As N and D are relatively prime, the above identity then implies that $D(x) = N(xq)$. Thus the polynomial $N(x)$ must satisfy

$$N(x) + x^2q^2N(xq^2) = (1 - x(1 + q))N(xq),$$

but considering the degree in x shows that this equation has no polynomial solution, except $N(x) = 0$. Hence $J(x)$ and its q -shifts span a 2-dimensional space, and the same holds for $H(x)$ thanks to (10) and (8).

Finally, let us prove that the space spanned by the shifts of K cannot have dimension less than 3. In that case, it would have dimension 2 (since it contains $H(x)$ and its q -shifts, by (13)) and there would exist rational functions α and β such that

$$K(x) = \alpha(x)H(x) + \beta(x)H(xq).$$

By combining (13) and (11), this would imply

$$H(x) \left(\alpha(x) + \frac{\beta(xq)(1-x)}{x} \right) + H(xq) \left(\beta(x) - \frac{xq\alpha(xq)}{1-xq} - \frac{\beta(xq)(1-2x)}{x} - 1 \right) = 0,$$

from which we derive $\alpha(x) = -(1-x)\beta(xq)/x$ and

$$\beta(x) + \beta(xq^2) - \frac{\beta(xq)(1-2x)}{x} - 1 = 0.$$

It remains to apply Abramov's algorithm [1], which determines all rational solutions of a linear q -equation, to conclude that such a β does not exist (we have used the MAPLE implementation of Abramov's algorithm, via the `RationalSolution` command of the `QDifferenceEquations` package). ■

3.2. FULLY COMMUTATIVE INVOLUTIONS

We now state analogous results for fc involutions. With the notation (3), the main two series are now:

$$\mathcal{J}(x) = \sum_{n \geq 0} \frac{(-1)^{\lceil n/2 \rceil} x^n q^{\binom{n}{2}}}{((q^2))_{\lfloor n/2 \rfloor}}, \quad (14)$$

and

$$\mathcal{K}(x) = \sum_{n \geq 0} x^n q^{\binom{n+1}{2}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{((q^2))_k}. \quad (15)$$

For any series $F(x)$, we denote by $F_e(x)$ and $F_o(x)$ its even and odd parts in x :

$$F_e(x) = \frac{1}{2}(F(x) + F(-x)), \quad F_o(x) = \frac{1}{2}(F(x) - F(-x)). \quad (16)$$

Theorem 3.3. *Let $\mathcal{A}(x, q) \equiv \mathcal{A}$, $\mathcal{B}(x, q) \equiv \mathcal{B}$, and $\mathcal{D}(x, q) \equiv \mathcal{D}$ be the generating functions of fully commutative involutions of types A, B and D, defined respectively by:*

$$\mathcal{A} = \sum_{n \geq 0} \mathcal{A}_n^{FC}(q)x^n, \quad \mathcal{B} = \sum_{n \geq 0} \mathcal{B}_n^{FC}(q)x^n, \quad \mathcal{D} = \sum_{n \geq 0} \mathcal{D}_{n+1}^{FC}(q)x^n.$$

Then:

$$\mathcal{A} = \frac{\mathcal{J}(-xq)}{\mathcal{J}(x)}, \quad (17)$$

$$\mathcal{B} = \frac{\mathcal{K}(x)}{\mathcal{J}(x)} + \frac{xq^2(1-x)}{1-xq^2} \frac{\mathcal{J}(-xq)}{\mathcal{J}(x)} - \frac{xq^2}{1-xq^2},$$

and

$$\mathcal{D} = \frac{2xq\mathcal{K}_e(xq)}{\mathcal{J}(x)} + \frac{q + x(1-q) - x^2q^2}{1-xq^2} \frac{\mathcal{J}(-xq)}{\mathcal{J}(x)} - \frac{q}{1-xq^2} + 1,$$

where the series \mathcal{J} and \mathcal{K} are defined by (14) and (15).

Remark. Again, the coefficients of x^0 and x^1 in $\mathcal{D}(x, q)$ are irrelevant, but in agreement with the convention of [4]. They read $\mathcal{D}_1^{FC}(q) = 1$ and $\mathcal{D}_2^{FC}(q) = (1 + q)^2$. Table 2 lists the values of $\mathcal{W}_n^{FC}(q)$ for small values of n .

n	2	3	4
\mathcal{A}_n^{FC}	[1, 2]	[1, 3, 1, 0, 1]	[1, 4, 3, 0, 2]
\mathcal{B}_n^{FC}	[1, 2, 0, 2]	[1, 3, 1, 2, 1, 1, 1]	[1, 4, 3, 2, 4, 1, 3, 1, 1, 0, 1]
\mathcal{D}_{n+1}^{FC}	[1, 3, 1, 0, 1]	[1, 4, 3, 1, 3, 1, 3]	[1, 5, 6, 2, 4, 2, 3, 1, 1]

TABLE 2. Length generating functions of fc involutions in finite Coxeter groups. The list $[a_0, \dots, a_k]$ stands for the polynomial $a_0 + a_1q + \dots + a_kq^k$.

Proposition 3.4. *The series \mathcal{J} satisfies a linear q -equation of order 2:*

$$(1 - 2xq)\mathcal{J}(x) - (1 - x(1 + q))\mathcal{J}(xq) + x^2q^2(1 - 2x)\mathcal{J}(xq^2) = 0. \quad (18)$$

We also have

$$(1 - 2x)\mathcal{J}(-xq) = 2\mathcal{J}(x) - \mathcal{J}(xq), \quad (19)$$

$$\mathcal{J}(x) = \mathcal{J}_e(x) - x\mathcal{J}_e(xq), \quad (20)$$

and

$$\mathcal{J}_e(x) - \mathcal{J}_e(xq) + x^2q\mathcal{J}_e(xq^2) = 0. \quad (21)$$

The vector space (over $\mathbb{Q}(x, q)$) spanned by $\mathcal{J}(x)$ and its q -shifts $\mathcal{J}(xq), \mathcal{J}(xq^2), \dots$ has dimension 2, and coincides with the space spanned by all series $\mathcal{J}(-xq^i)$, and with the space spanned by all series $\mathcal{J}_e(xq^i)$.

The series \mathcal{K} is related to \mathcal{J} by:

$$\mathcal{K}(x) - xq\mathcal{K}(xq) = \frac{\mathcal{J}(x) - x\mathcal{J}(xq)}{1 - 2x} = \mathcal{J}(x) + x\mathcal{J}(-xq), \quad (22)$$

and thus satisfies a q -equation of order 3:

$$\mathcal{K}(x) - (1 + xq)\mathcal{K}(xq) + xq^2(1 + xq)\mathcal{K}(xq^2) - x^3q^6\mathcal{K}(xq^3) = 0. \quad (23)$$

The series $\mathcal{K}(xq^i)$ span a 3-dimensional vector space, which contains $\mathcal{J}(x)$ since

$$\mathcal{J}(x) = (1 - x)\mathcal{K}(x) - qx\mathcal{K}(xq) + x^3q^3\mathcal{K}(xq^2). \quad (24)$$

Finally,

$$\mathcal{K}(x) = \mathcal{K}_e(x) + xq\mathcal{K}_e(xq), \quad (25)$$

and the series $\mathcal{K}_e(xq^i)$ span a 4-dimensional vector space, which contains $\mathcal{K}(x)$ and $\mathcal{J}(x)$.

Proof. Since the expansions in x of the series \mathcal{J} and \mathcal{K} are given explicitly, all identities above are readily proved by extracting the coefficient of x^n , for $n \in \mathbb{N}$.

It remains to prove our results about dimensions. The arguments are the same as in the last part of the proof of Proposition 3.2. If the space spanned by the $\mathcal{J}(xq^i)$ had dimension 1, then $\mathcal{J}(x)/\mathcal{J}(xq)$ would be a rational function $N(x)/D(x)$, with $N(x)$ and $D(x)$ coprime in $\mathbb{Q}(q)[x]$ and $N(0) = D(0) = 1$. Then dividing (18) by $\mathcal{J}(xq)$ would give

$$(1 - 2xq)\frac{N(x)}{D(x)} + x^2q^2(1 - 2x)\frac{D(xq)}{N(xq)} = 1 - x(1 + q). \quad (26)$$

This gives four possible relations between the polynomials N and D :

$$D(x) = N(xq), \quad D(x) = (1 - 2xq)N(xq), \quad D(x) = \frac{N(xq)}{1 - 2x}, \quad D(x) = \frac{(1 - 2xq)N(xq)}{1 - 2x}.$$

Reporting each of these possibilities in (26) gives a linear q -equation for N , and a degree argument shows that this equation has no non-zero solution (in the third case, the degree argument does

not suffice, and we have to extract the dominant coefficient in x to conclude). Hence $\mathcal{J}(x)$ and its q -shifts span a 2-dimensional space. The same space is spanned by the series $\mathcal{J}(-xq^i)$, thanks to (19), and finally by the series $\mathcal{J}_e(xq^i)$, thanks to (20) and (21).

Now assume that the space spanned by the series $\mathcal{K}(xq^i)$ has dimension less than 3. Then its dimension is 2, since it contains $\mathcal{J}(x)$ by (24), and there exist rational functions α and β such that

$$\mathcal{K}(x) = \alpha(x)\mathcal{J}(x) + \beta(x)\mathcal{J}(xq).$$

Then (22) and (18) give

$$\begin{aligned} & \left(xq(1-2x)\alpha(x) + (1-2xq)\beta(xq) - xq \right) \mathcal{J}(x) \\ & + \left(x^2q^2(2x-1)\alpha(xq) - xq(2x-1)\beta(x) + (xq+x-1)\beta(xq) + x^2q \right) \mathcal{J}(xq) = 0, \end{aligned}$$

from which it follows that

$$xq(1-2x)(2xq-1)\beta(x) + (xq+x-1)(2xq-1)\beta(xq) - x(2x-1)(2xq^2-1)\beta(xq^2) = x^2q(1-q).$$

But applying Abramov's algorithm proves that this equation has no rational solution.

Finally, assume that the space spanned by the series $\mathcal{K}_e(xq^i)$ has dimension less than 4. Then it has dimension 3 (since it contains $\mathcal{K}(x)$ by (25)) and there exist rational functions α , β and γ such that

$$\mathcal{K}_e(x) = \alpha(x)\mathcal{K}(x) + \beta(x)\mathcal{K}(xq) + \gamma(x)\mathcal{K}(xq^2).$$

Then we derive from (25) and (23) a linear q -equation for $\gamma(x)$:

$$q^6x\gamma(x) + q^3(xq+1)\gamma(xq) + (xq^2+1)\gamma(xq^2) + x\gamma(xq^3) = x^3q^9.$$

Applying again Abramov's algorithm proves that this equation has no rational solution. \blacksquare

4. FIRST RECURSIVE APPROACH: PEELING A DIAGONAL

In this section, we prove the finite type results stated in the previous section. We use the description of fc elements in terms of heaps given in [5, 31]. Then the central question is to count alternating heaps over a path, having at most one point in the rightmost column. Our approach is recursive, and consists in peeling the rightmost NW-SE diagonal of alternating heaps. A precise description is given below, but we refer to Figure 3 for a quick intuition. This is an adaptation of a classical method used to count convex polyominoes [26, 10].

4.1. TYPE A

It is known from [31] that fc elements in A_n are in bijection with alternating heaps over the n -point path, having at most one point in their first (i.e. leftmost) and last (rightmost) columns. We count them by recording three parameters: the number n of generators of the group (variable x), the length of the fc element (variable q), and the size i of its largest right factor $s_{n-i+1} \cdots s_n$ (variable s). In graphical terms, i is the size of the rightmost NW-SE diagonal of the heap (Figure 3). We denote by $A(s) \equiv A(s; x, q)$ the corresponding generating function, and by $A_i \equiv A_i(x, q)$ the coefficient of s^i in this series:

$$A(s) \equiv A(s; x, q) = \sum_{i \geq 0} A_i s^i.$$

Thus the series $A(x, q)$ of Theorem 1.1 is now $A(1; x, q) \equiv A(1)$. We hope that this will not cause any confusion.

We count separately four types of heaps.

- The trivial group ($n = 0$) contributes 1; we assume in what follows that $n \geq 1$.
- If the last column is empty, that is, $i = 0$, the heap encodes an fc element of A_{n-1} . Thus the generating function for this type is simply $xA(1)$.

- If the last column contains a point, and this point is lower than any point in the next-to-last column, removing this rightmost point leaves an fc element of A_{n-1} (Figure 3, left; this includes the case where the next-to-last column is empty). The size of the rightmost diagonal decreases by 1. Hence the generating function for this type is $xsqA(s)$.
- Finally, if the last column contains a point, and this point is higher than the lowest point of the next-to-last column, we remove the top point in each of the i rightmost columns (Figure 3, right). In other words, we peel off the rightmost NW-SE diagonal. This leaves an fc element of A_{n-1} , with a rightmost diagonal of size $j \geq i$. This shows that the generating function for this type is

$$x \sum_{j \geq 1} A_j \sum_{i=1}^j (sq)^i = x \sum_{j \geq 1} A_j \frac{sq - (sq)^{j+1}}{1 - sq} = \frac{xsq}{1 - sq} (A(1) - A(sq)).$$

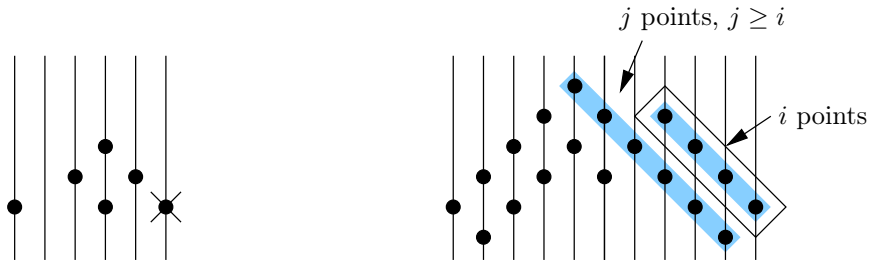


FIGURE 3. Two types of alternating heaps with one point in the last column.

Putting together the four cases gives:

$$A(s) = 1 + xA(1) + xsqA(s) + \frac{xsq}{1 - sq} (A(1) - A(sq)). \quad (27)$$

Grouping the terms in $A(s)$ and in $A(1)$ and dividing by $(1 - xsq)$ yields:

$$A(s) = \frac{1}{1 - xsq} + \frac{x}{(1 - sq)(1 - xsq)} A(1) - \frac{xsq}{(1 - sq)(1 - xsq)} A(sq).$$

Iterating this equation m times provides an expression of $A(s)$ in terms of $A(sq^{m+1})$:

$$A(s) = \sum_{n=0}^m \frac{(-xs)^n q^{\binom{n+1}{2}}}{(sq)_n (xsq)_n} \left(\frac{1}{1 - xsq^{n+1}} + \frac{x}{(1 - sq^{n+1})(1 - xsq^{n+1})} A(1) \right) + \frac{(-xs)^{m+1} q^{\binom{m+2}{2}}}{(sq)_{m+1} (xsq)_{m+1}} A(sq^{m+1}).$$

As a series in x , the rightmost term tends to 0 as m tends to infinity. Hence:

$$A(s) = \sum_{n \geq 0} \frac{(-xs)^n q^{\binom{n+1}{2}}}{(sq)_n (xsq)_n} \left(\frac{1}{1 - xsq^{n+1}} + \frac{x}{(1 - sq^{n+1})(1 - xsq^{n+1})} A(1) \right).$$

Setting $s = 1$ gives

$$\begin{aligned} A(1) &= \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n+1}{2}}}{(q)_n (xq)_{n+1}} - A(1) \sum_{n \geq 0} \frac{(-x)^{n+1} q^{\binom{n+1}{2}}}{(q)_{n+1} (xq)_{n+1}} \\ &= \frac{J(xq)}{1 - xq} - A(1) (J(x) - 1), \end{aligned}$$

with $J(x)$ defined by (5). Solving for $A(1)$ gives the first result of Theorem 3.1. ■

We now adapt this to fc involutions of A_n . They are in bijection with self-dual alternating heaps over the n -point path in which the first and last columns contain at most one point. In order to preserve self-duality, the peeling procedure must now remove the rightmost NW-SE diagonal *and the rightmost SW-NE diagonal* (Figure 4). We denote by $\mathcal{A}(s; x, q) = \sum_{i \geq 0} \mathcal{A}_i s^i$ the generating function of fc involutions, refined by the same parameter as above. We count separately the same four types of heaps, but there are no self-dual heaps of the third type.

- The empty group ($n = 0$) contributes 1; we assume from now on that $n \geq 1$.
- If the last column is empty, that is, $i = 0$, the generating function is $x\mathcal{A}(1)$.
- If the last column contains one point, that is, $i \geq 1$, there are two cases: either $n = 1$, and then the contribution is xsq , or $n \geq 2$. In this case, removing the rightmost NW-SE and SW-NE diagonals leaves an fc involution of A_{n-2} , with a rightmost diagonal of size $j \geq i - 1$ (Figure 4). Hence the generating function for this type reads:

$$xsq + x^2 \sum_{j \geq 0} \mathcal{A}_j \sum_{i=1}^{j+1} s^i q^{2i-1} = xsq + \frac{x^2 sq}{1 - sq^2} (\mathcal{A}(1) - sq^2 \mathcal{A}(sq^2)).$$

Putting all contributions together and gathering the terms $\mathcal{A}(1)$ gives:

$$\mathcal{A}(s) = 1 + xsq + x \left(1 + \frac{xsq}{1 - sq^2} \right) \mathcal{A}(1) - \frac{x^2 s^2 q^3}{1 - sq^2} \mathcal{A}(sq^2). \quad (28)$$

Iterating m times this equation and then letting m tends to infinity yields

$$\mathcal{A}(s) = \sum_{n \geq 0} \frac{(-x^2 s^2)^n q^{n(2n+1)}}{((sq^2))_n} \left(1 + xsq^{2n+1} + x \left(1 + \frac{xsq^{2n+1}}{1 - sq^{2n+2}} \right) \mathcal{A}(1) \right).$$

Once s is set to 1, this reads, with the notation (14):

$$\mathcal{A}(1) = \mathcal{J}(-xq) - (\mathcal{J}(x) - 1)\mathcal{A}(1).$$

Solving for $\mathcal{A}(1)$ gives the first result of Theorem 3.3. ■

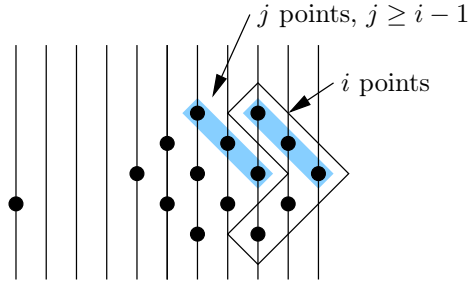


FIGURE 4. Peeling the rightmost diagonals in a self-dual alternating heap.

Remark: connection with staircase polyominoes and Dyck paths. We have proved these A -results to illustrate our recursive approach, but they are equivalent to previously published results involving staircase polyominoes and Dyck paths, respectively. More precisely, let us say that a non-empty fc element of A_n is *connected* (or: *has full support*) if every generator occurs in it. In this case, if we replace every point of the corresponding heap by a unit square, the resulting collection of squares is a *staircase polyomino* (Figure 5, left). The perimeter of this polyomino is $2n + 2$, and its area is the length of the fc element. Let $P(x, q)$ be the generating function of staircase polyominoes, counted by the half-perimeter (variable x) and the area (variable q). Then the generating function of connected fc elements of type A is $A^{(c)}(x, q) = P(x, q)/x$. By

discussing whether the first column of an fc heap of type A is empty or not, one can relate the series A and $A^{(c)}$ as follows:

$$A = 1 + xA + A^{(c)} + A^{(c)}xA, \tag{29}$$

so that

$$A = \frac{1 + A^{(c)}}{1 - x - xA^{(c)}} = \frac{1 + P/x}{1 - x - P}.$$

An expression of $P(x, q)$ can be found in [10, Thm. 3.2] or [12, Prop. 4.1]. With our notation, it reads

$$P(x, q) = \frac{x^2q}{1 - xq} \frac{H(xq^2)}{H(xq)},$$

where $H(x)$ is defined by (7). One then recovers the expression of A given in Theorem 3.1 using (8) and (9).

It follows from (12) and (11) that $J(xq)/J(x)$ and $H(xq)/H(x)$ admit the same limit as $q \rightarrow 1$, namely $(1 - x) \text{Cat}(x)/x$, where $\text{Cat}(x) = \sum_{n \geq 1} x^n \binom{2n}{n} / (n + 1)$ is the Catalan generating function. Hence the identities

$$A(x, q) = \frac{1}{1 - xq} \frac{J(xq)}{J(x)}, \quad A^{(c)}(x, q) = \frac{xq}{1 - xq} \frac{H(xq^2)}{H(xq)},$$

are q -analogues of $A^{(c)}(x, 1) = xA(x, 1) = \text{Cat}(x)$.

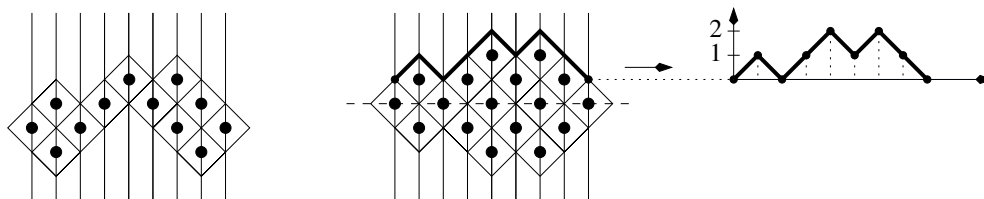


FIGURE 5. Left: A connected fc element of A_9 and the corresponding staircase polyomino, formed of square cells. Right: A connected fc involution of A_9 and the corresponding Dyck path, with the heights of the vertices shown.

Fully commutative involutions of type A are related in a similar fashion to staircase polyominoes that are invariant by reflection in a horizontal line, with half-perimeter/area generating function $\mathcal{P}(x, q)$:

$$\mathcal{A} = \frac{1 + \mathcal{P}/x}{1 - x - \mathcal{P}}.$$

These polyominoes are in bijection with the famous *Dyck paths* (Figure 5, right). More precisely, a symmetric staircase polyomino of half-perimeter n (necessarily even) and area a gives rise to a Dyck path of length $n - 2$ in which the *total height* (the sum of heights of the vertices) is $a - n + 1$. Hence, denoting by $D(x, q)$ the generating function of Dyck paths, counted by half-length and total height, we have

$$\mathcal{P}(x, q) = x^2q D(x^2q^2, q).$$

An expression for $D(x, q)$ can be found in [21, Ex. 5.2.12(c)]. With our notation, it gives

$$D(x^2q^2, q) = \frac{\mathcal{J}_e(xq^2)}{\mathcal{J}_e(xq)},$$

where $\mathcal{J}_e(x)$ is the even part of the series $\mathcal{J}(x)$ defined by (14). One then recovers the expression of \mathcal{A} given in Theorem 3.3 using (20) and (21).

4.2. TYPE B

As argued in [5, Sec. 4.4], fc elements of B_n come in two types.

- First, we have all alternating heaps over the n -point path having at most one point in the last column. We denote by $B^{(a)}(x)$ their generating function.
- The remaining fc elements are not alternating. With the notation of Figure 1, they are obtained as follows: one starts from an alternating heap over a path with vertices s_j, \dots, s_{n-1} (with $1 \leq j \leq n-1$), having exactly one point in the s_j -column and at most one point in the last column, and inflates the point in the s_j -column into a $<$ -shaped heap $s_j s_{j-1} \cdots s_1 t s_1 \cdots s_{j-1} s_j$ (see the third picture of [5, Fig. 7] for an illustration). We denote by $B^{(\text{na})}(x)$ the corresponding generating function.

The latter description gives $B^{(\text{na})}(x) = xq^2/(1-xq^2)A^{(1)}(x)$, where $A^{(1)}(x)$ counts alternating heaps over a path with one point in the first column and at most one point in the last one. Since the generating function of alternating heaps with an empty first column is xA , we have $A^{(1)}(x) = A - 1 - xA$. Thus:

$$B^{(\text{na})}(x) = \frac{xq^2}{1-xq^2} ((1-x)A - 1) \quad (30)$$

$$= \frac{xq^2}{1-xq^2} \left(\frac{1-x}{1-xq} \frac{J(xq)}{J(x)} - 1 \right). \quad (31)$$

We now focus on the generating function $B^{(a)}(x)$ of alternating fc elements of type B. We refine their enumeration by recording the size i of the rightmost diagonal: with the generators denoted as in Figure 1, this is the longest right factor of the heap of the form $s_{n-i} \cdots s_{n-1}$, with $s_0 = t$. We apply the recursive approach that led to (27) in Section 4.1. The only difference is that we can now have several points in the first column. The first three cases contribute as before (with $A(s)$ replaced by $B^{(a)}(s)$). However, the fourth case is now richer: when the point in the last column is higher than some point in the next-to-last column, removing the top point in each of the i rightmost columns may leave an alternating heap with a rightmost diagonal of size $i-1$ (instead of size $j \geq i$ in the A -case). This happens only if i is maximal, that is, equal to the number of generators. An example is shown in Figure 6, left. We shall see below that the contribution of these heaps is $xsqT(sq)$, where

$$T(s) = xsq + xsqT(s) + xsqT(sq), \quad (32)$$

which implies

$$T(s) = \sum_{m \geq 1} \frac{(xs)^m q^{\binom{m+1}{2}}}{(xsq)_m}. \quad (33)$$

Putting together all four cases gives:

$$B^{(a)}(s) = 1 + xB^{(a)}(1) + xsqB^{(a)}(s) + \frac{xsq}{1-sq} \left(B^{(a)}(1) - B^{(a)}(sq) \right) + xsqT(sq). \quad (34)$$

We now proceed as we did for the A -equation (27). Grouping the terms in $B^{(a)}(s)$ and in $B^{(a)}(1)$, and dividing by $(1-xsq)$, gives:

$$\begin{aligned} B^{(a)}(s) &= \frac{1}{1-xsq} + \frac{xsq}{1-xsq} T(sq) + \frac{x}{(1-sq)(1-xsq)} B^{(a)}(1) - \frac{xsq}{(1-sq)(1-xsq)} B^{(a)}(sq) \\ &= 1 + T(s) + \frac{x}{(1-sq)(1-xsq)} B^{(a)}(1) - \frac{xsq}{(1-sq)(1-xsq)} B^{(a)}(sq) \end{aligned}$$

by (32). Iterating the equation gives

$$B^{(a)}(s) = \sum_{n \geq 0} \frac{(-xs)^n q^{\binom{n+1}{2}}}{(sq)_n (xsq)_n} \left(1 + T(sq^n) + \frac{x}{(1-sq^{n+1})(1-xsq^{n+1})} B^{(a)}(1) \right). \quad (35)$$

Using the expression (33) of $T(s)$, we can rewrite

$$\begin{aligned} \sum_{n \geq 0} \frac{(-xs)^n q^{\binom{n+1}{2}}}{(sq)_n (xsq)_n} (1 + T(sq^n)) &= \sum_{n \geq 0} \frac{(-xs)^n q^{\binom{n+1}{2}}}{(sq)_n (xsq)_n} \sum_{m \geq 0} \frac{(xsq^n)^m q^{\binom{m+1}{2}}}{(xsq^{n+1})_m} \\ &= \sum_{m, n \geq 0} \frac{(xs)^{m+n} q^{\binom{m+n+1}{2}} (-1)^n}{(xsq)_{m+n} (sq)_n} \\ &= \sum_{N \geq 0} \frac{(xs)^N q^{\binom{N+1}{2}}}{(xsq)_N} \sum_{n=0}^N \frac{(-1)^n}{(sq)_n}. \end{aligned}$$

We now return to (35), where we set $s = 1$. This gives

$$B^{(a)}(1) = \sum_{n \geq 0} \frac{x^n q^{\binom{n+1}{2}}}{(xq)_n} \sum_{k=0}^n \frac{(-1)^k}{(q)_k} - B^{(a)}(1) \sum_{n \geq 0} \frac{(-x)^{n+1} q^{\binom{n+1}{2}}}{(q)_{n+1} (xq)_{n+1}}.$$

Solving for $B^{(a)}(1)$ gives

$$B^{(a)}(1) = \frac{K(x)}{J(x)}, \quad (36)$$

where J and K are defined by (5). Adding the contribution (31) of non-alternating fc elements gives the second result of Theorem 3.1.

We still have to explain the term $xsqT(sq)$ occurring in (34). Let $T(s) \equiv T(s; x, q)$ denote the generating function of alternating heaps such that i is maximal (and having, as always in this subsection, at most one point in the last column — and hence exactly one). Then the heaps that we need to count to complete the proof of (34) (i maximal, two points in the next-to-last column) are counted by $xsqT(sq)$, because they are obtained by adding one point in every column of a heap counted by $T(s)$, and then a final (rightmost) column containing one point (Figure 6, left). It remains to prove that $T(s)$ satisfies (32). This results again from a peeling procedure. The first term counts the heap reduced to one point, the second counts those such that the point in the last column is lower than any point in the next-to-last column, and the third one, as we have just explained, counts those in which the point in the last column is higher than one point in the next-to-last column. Another way to justify (33) is to observe that heaps counted by $T(s)$ are just integer partitions into distinct parts, where q counts the weight, and (xs) the size of the largest part. The integer m in (33) gives the number of parts.

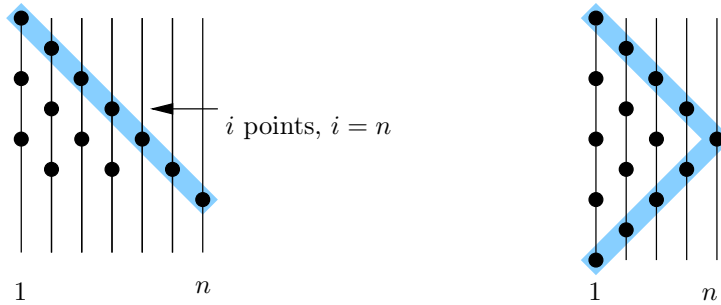


FIGURE 6. Left: An alternating heap where the parameter i is maximal. The corresponding integer partition is $(7, 4, 2)$. Right: The self-dual case.

■

We now restrict this argument to involutions, that is, to self-dual heaps. The generating function of non-alternating fc involutions is $\mathcal{B}^{(na)} = xq^2/(1-xq^2)\mathcal{A}^{(1)}(x)$, where $\mathcal{A}^{(1)}(x)$ counts

self-dual alternating heaps with one point in the first column and at most one point in the last column. With our notation, $\mathcal{A}^{(1)} = \mathcal{A} - 1 - x\mathcal{A}$, and by the first result of Theorem 3.3,

$$\mathcal{B}^{(\text{na})} = \frac{xq^2}{1-xq^2} (\mathcal{A}(1-x) - 1) \quad (37)$$

$$= \frac{xq^2}{1-xq^2} \left(\frac{(1-x)\mathcal{J}(-xq)}{\mathcal{J}(x)} - 1 \right). \quad (38)$$

We now consider the generating function $\mathcal{B}^{(\text{a})}$ of self-dual alternating heaps with at most one point in the last column. We refine it as before into $\mathcal{B}^{(\text{a})}(s)$, and adapt the argument that led to (28) in type *A*. The first two cases contribute as before (with \mathcal{A} replaced by $\mathcal{B}^{(\text{a})}$). The third one splits again into two sub-cases: either i is maximal (Figure 6, right), which gives the generating function

$$\mathcal{T}(s) = \sum_{m \geq 1} (xs)^m q^{\binom{m+1}{2}}, \quad (39)$$

or deleting the rightmost diagonals leaves a heap with a rightmost diagonal of size $j \geq i - 1$, as in Figure 4. This gives:

$$\mathcal{B}^{(\text{a})}(s) = 1 + x \left(1 + \frac{xsq}{1-sq^2} \right) \mathcal{B}^{(\text{a})}(1) - \frac{x^2 s^2 q^3}{1-sq^2} \mathcal{B}^{(\text{a})}(sq^2) + \mathcal{T}(s). \quad (40)$$

By iteration, we obtain

$$\mathcal{B}^{(\text{a})}(s) = \sum_{n \geq 0} \frac{(-x^2 s^2)^n q^{n(2n+1)}}{((sq^2))_n} \left(1 + \mathcal{T}(sq^{2n}) + x \left(1 + \frac{xsq^{2n+1}}{1-sq^{2n+2}} \right) \mathcal{B}^{(\text{a})}(1) \right).$$

Setting $s = 1$ and solving for $\mathcal{B}^{(\text{a})}(1)$ gives

$$\mathcal{J}(x)\mathcal{B}^{(\text{a})}(1) = \sum_{n \geq 0} \frac{(-x^2)^n q^{n(2n+1)}}{((q^2))_n} (1 + \mathcal{T}(q^{2n})),$$

where $\mathcal{J}(x)$ is defined by (14). By definition (39) of the series $\mathcal{T}(s)$, the right-hand side can be written as

$$\sum_{n, m \geq 0} \frac{(-x^2)^n q^{n(2n+1)}}{((q^2))_n} x^m q^{2nm + \binom{m+1}{2}} = \sum_{N \geq 0} x^N q^{\binom{N+1}{2}} \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{(-1)^k}{((q^2))_k} = \mathcal{K}(x)$$

where $\mathcal{K}(x)$ is defined in (15) (we have set $N := m + 2n$ and $k := n$ in the double sum). This finally gives

$$\mathcal{B}^{(\text{a})}(1) = \frac{\mathcal{K}(x)}{\mathcal{J}(x)}. \quad (41)$$

Adding the contribution (38) of non-alternating heaps gives the second result of Theorem 3.3. ■

Remark. In the next subsection, we need to count self-dual alternating heaps of type *B* in which the first column contains an odd number of points. Let us denote by $\mathcal{B}_{\text{odd}}^{(\text{a})}(s)$ the corresponding generating function. The peeling argument that led to (40) specializes into:

$$\mathcal{B}_{\text{odd}}^{(\text{a})}(s) = x \left(1 + \frac{xsq}{1-sq^2} \right) \mathcal{B}_{\text{odd}}^{(\text{a})}(1) - \frac{x^2 s^2 q^3}{1-sq^2} \mathcal{B}_{\text{odd}}^{(\text{a})}(sq^2) + \mathcal{T}_{\text{odd}}(s),$$

with

$$\mathcal{T}_{\text{odd}}(s) = \sum_{m \text{ odd}} (xs)^m q^{\binom{m+1}{2}}.$$

Comparing with (40) shows that the only difference is the replacement of $1 + \mathcal{T}(s)$ by $\mathcal{T}_{\text{odd}}(s)$. The iteration procedure yields:

$$\mathcal{B}_{\text{odd}}^{(\text{a})}(1) = \frac{xq\mathcal{K}_e(xq)}{\mathcal{J}(x)}, \quad (42)$$

where $\mathcal{K}_e(x)$ is the even part of $\mathcal{K}(x)$ as defined by (16).

4.3. TYPE D

As explained in [5], the structures of fc elements of type B and D are closely related, and this results in similar generating functions. In particular, we claim that the second equation of Proposition 4.6 in [5] translates, in our notation, as

$$D = 2B^{(a)} - 1 - xA + \frac{1}{xq}B^{(na)}. \quad (43)$$

Indeed, the definitions of the series M^* and Q^* given in [5, Sec. 1.3] in terms of certain paths, and the correspondence between these paths and heaps, show that, with our notation, $M^* = 1 + xA$ and $Q^* = B^{(a)}$. Proposition 4.6 of [5] thus reads

$$\begin{aligned} B &= B^{(a)} + \frac{x^2q^3}{1-xq^2}M^*(x)M(xq), \\ D &= 2B^{(a)} - (1 + xA) + \frac{xq^2}{1-xq^2}M^*(x)M(xq). \end{aligned}$$

The second term in the expression of B must thus be $B^{(na)}$, and reporting this in the expression of D gives (43). The third result of Theorem 3.1 now follows from (31), (36), and our expression (6) of A .

Following the analysis of [4, Prop. 3.1], we claim that the counterpart of (43) for fc *involutions* of type D is:

$$\mathcal{D} = 2\mathcal{B}_{\text{odd}}^{(a)} + 1 + x\mathcal{A} + \frac{1}{xq}\mathcal{B}^{(na)}, \quad (44)$$

where $\mathcal{B}_{\text{odd}}^{(a)}$ counts self-dual alternating heaps of type B with an odd number of points in the first column. Indeed, the definitions of the series M , Q and Q° in terms of paths given in [4, Sec. 1.4 and Prop. 3.1], and the correspondence between these paths and heaps, show that, with our notation,

$$\frac{M}{1-xM} = 1 + x\mathcal{A}, \quad \frac{Q}{1-xM} = \mathcal{B}^{(a)} \quad \text{and} \quad \frac{Q^\circ}{1-xM} = \mathcal{B}_{\text{odd}}^{(a)}.$$

With our notation, Proposition 3.1 of [4] thus reads

$$\begin{aligned} \mathcal{B} &= \mathcal{B}^{(a)} + \frac{x^2q^3}{1-xq^2} \frac{M(x)M(xq)}{1-xM(x)}, \\ \mathcal{D} &= 2\mathcal{B}_{\text{odd}}^{(a)} + 1 + x\mathcal{A} + \frac{xq^2}{1-xq^2} \frac{M(x)M(xq)}{1-xM(x)}. \end{aligned}$$

The second term in the expression of \mathcal{B} must thus be $\mathcal{B}^{(na)}$, and reporting this in the expression of \mathcal{D} gives (44). The third result of Theorem 3.3 now follows from (42), (38) and our expression (17) of \mathcal{A} . ■

4.4. NEW EXPRESSIONS FOR SOME PATH GENERATING FUNCTIONS

In the previous subsection, we have used two identities between generating functions that came from [5] and [4], namely (43) and (44), to express the series D and \mathcal{D} . In Section 5, we will use more of these identities when counting fc elements of type \tilde{A} . It is thus appropriate to give the explicit values of some generating functions for paths appearing in [5] and [4], and characterized therein by non-linear q -equations.

We first express the series involved in [5] in the enumeration of fc elements of types A , B , and D in terms of the series J and K of (5), and of the series H of (7).

Proposition 4.1. *The series denoted $M(x)$ and $M^*(x)$ in [5] admit the following explicit expressions:*

$$M(x) = \frac{H(xq)}{(1-x)H(x)}, \quad M^*(x) = \frac{H(xq)}{J(x)}.$$

Proof. The series denoted A^{FC} in [5] is, in our notation, $x\mathcal{A}$. Hence Proposition 2.7 from [5] gives $M^* = 1 + x\mathcal{A}$, and the above expression of M^* follows from Theorem 3.1 and (9).

Let us now consider the series M . Equation (4) in [5] gives

$$M(x) = \frac{M^*(x)}{1 - xM^*(x)},$$

and the above expression of $M(x)$ follows from the expression of $M^*(x)$ and (8). \blacksquare

Remark. Our proof of Theorem 3.1 only relies on the description of fc elements in terms of heaps. An alternative proof would have been to use the characterization of the series A , B and D in terms of non-linear q -equations given in [5], and to check that our expressions in terms of J and K satisfy these equations. One advantage of our recursive approach is that it *constructs* the series J and K , while one would not understand where they come from if we had simply checked the systems of [5]. In the next section however, we solve the \tilde{A} -case by checking the relevant q -equations of [5].

To conclude this section, we state the counterpart of Proposition 4.1 for involutions. Recall that the series \mathcal{J}_e is defined by combining (14) and (16).

Proposition 4.2. *Let us denote by $\mathcal{M}(x) \equiv \mathcal{M}(x, q)$ the series denoted by $M(x)$ in [4]. Then*

$$\mathcal{M}(x) = \frac{\mathcal{J}_e(xq)}{\mathcal{J}_e(x)}.$$

Proof. The series denoted \tilde{A} in [4] is our series $x\mathcal{A}$. Hence Proposition 3.1 in [4] reads

$$\frac{\mathcal{M}(x)}{1 - x\mathcal{M}(x)} = 1 + x\mathcal{A}.$$

Combined with the expression (17) of \mathcal{A} , this gives:

$$\mathcal{M}(x) = \frac{\mathcal{J}(x) + x\mathcal{J}(-xq)}{(1+x)\mathcal{J}(x) + x^2\mathcal{J}(-xq)}.$$

Using the explicit expression of \mathcal{J} , one checks that the numerator (resp. denominator) of this expression is $\mathcal{J}_e(xq)$ (resp. $\mathcal{J}_e(x)$). \blacksquare

5. AFFINE TYPE \tilde{A}

In this section, we first establish the simple expression of \tilde{A} announced in Theorem 1.1. As proved in [31, Prop. 3.3], fc elements in \tilde{A}_{n-1} are in one-to-one correspondence with alternating heaps over the n -point cycle (Figure 2, right). In [5], the series \tilde{A} was characterized in terms of two series \check{O} and \check{O}^* related to the series M and M^* of Proposition 4.1. This yields our nice expression of \tilde{A} in terms of J and its derivative. We then proceed similarly with the generating function $\tilde{\mathcal{A}}$ of fc involutions in type \tilde{A} (Theorem 1.2).

This computational approach does not explain the simplicity of \tilde{A} and $\tilde{\mathcal{A}}$. In the extended abstract [6], we provide bijective explanations of them.

5.1. ALL FULLY COMMUTATIVE ELEMENTS

Theorem 5.1. *With $J(x)$ and $H(x)$ defined by (5) and (7) above, the generating functions $\check{O}(x)$ and $\check{O}^*(x)$ defined in Corollary 2.4 of [5] are given by*

$$\check{O}(x) = \frac{1}{1-x} - x \frac{H'(x)}{H(x)} + xq \frac{H'(xq)}{H(xq)}, \quad (45)$$

$$\check{O}^*(x) = 1 - x \frac{J'(x)}{J(x)} + xq \frac{H'(xq)}{H(xq)}. \quad (46)$$

The generating function of fully commutative elements in type \tilde{A} is

$$\tilde{A} := \sum_{n \geq 1} \tilde{A}_{n-1}^{FC}(q) x^n = -x \frac{J'(x)}{J(x)} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n}.$$

Remarks

1. Above, we have taken $\tilde{A}_1 = A_2$ so that $\tilde{A}_1^{FC}(q) = (1+q)/(1-q)$. The coefficient $\tilde{A}_0^{FC}(q)$ of x^1 in $\tilde{A}(x, q)$ is irrelevant (and equal to 1).

2. Fully commutative elements of \tilde{A} having full support (that is, in which each generator occurs) can be seen as periodic staircase polyominoes. These objects have recently been studied for their own sake [2, 13], and in particular counted according to their perimeter. When proving the above theorem, we will obtain their length generating function as

$$-xq \frac{H'(xq)}{H(xq)} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n}. \quad (47)$$

This formula can be explained combinatorially by the methods of [6].

Proof. Both series \tilde{O} and \tilde{O}^* are expressed in terms of M and M^* in Eq. (2) of [5]. For instance,

$$\tilde{O}(x) = M(x) \left(1 + x^2 q \frac{\partial(xM)}{\partial x}(xq) \right). \quad (48)$$

The notation should be understood as follows: one first takes the derivative of xM , and then evaluates it at xq . Using the expression of M in terms of H (Proposition 4.1), the right-hand side can be written in terms of $H(x)$, $H(xq)$, $H(xq^2)$, $H'(xq)$ and $H'(xq^2)$. Now using the linear q -equation (11) satisfied by H , we can express $H(xq^2)$ and $H'(xq^2)$ in terms of $H(x)$, $H(xq)$, $H'(x)$ and $H'(xq)$. This yields the expression of \tilde{O} given in the theorem.

The expression of \tilde{O}^* can be proved similarly, starting from Eq. (2) in [5]. Alternatively, we can derive it from our expression of \tilde{O} since this equation implies that $\tilde{O}^* = \tilde{O}M^*/M = \tilde{O}(1-x)H/J$. One then checks that this coincides with (46) using the expression (8) of J in terms of H .

Consider now the series \tilde{A} . The second part of Eq. (1) from [5] tells us that

$$\tilde{A}(x) - \tilde{A}(xq) = \tilde{O}(xq) - 1 - 2 \frac{xq}{1 - xq} + \tilde{O}^*(x) - \tilde{O}^*(xq). \quad (49)$$

(We have used the fact that $\tilde{O}(x)$ has constant term 1.) We now use our expression of \tilde{O} :

$$\tilde{A}(x) - \tilde{A}(xq) = -\frac{xq}{1 - xq} - xq \frac{H'(xq)}{H(xq)} + xq^2 \frac{H'(xq^2)}{H(xq^2)} + \tilde{O}^*(x) - \tilde{O}^*(xq).$$

Since $\tilde{A}(x)$ has constant term 0, and $\tilde{O}^*(x)$ has constant term 1, iterating this equation gives:

$$\tilde{A}(x) = -\sum_{i \geq 1} \frac{xq^i}{1 - xq^i} - xq \frac{H'(xq)}{H(xq)} + \tilde{O}^*(x) - 1,$$

and the last statement of the theorem follows using our expression of $\tilde{O}^*(x)$. (We also expand the sum over i in powers of x .)

If we only want to count fc elements with full support, then the arguments of [5, Cor. 2.4] shows that we only have to drop the terms $\tilde{O}^*(x)$ and $\tilde{O}^*(xq)$ in (49). The expression (47) of their generating function follows. \blacksquare

5.2. FULLY COMMUTATIVE INVOLUTIONS

Theorem 5.2. *Let $\mathcal{M}(x)$ be given by Proposition 4.2. Let $\check{O}(x) = \sum_{n \geq 0} \check{O}_n(q) x^n$, where the polynomials $\check{O}_n(q)$ are defined in Proposition 3.3 of [4], and let*

$$\check{O}^*(x) = \frac{\mathcal{M}(x)}{1 - x\mathcal{M}(x)} \left(1 + x^2 q \frac{\partial(x\mathcal{M})}{\partial x}(xq) \right). \quad (50)$$

Then

$$\check{O}(x) = 1 - x \frac{\mathcal{J}'_e(x)}{\mathcal{J}_e(x)} + xq \frac{\mathcal{J}'_e(xq)}{\mathcal{J}_e(xq)}, \quad (51)$$

$$\check{O}^*(x) = 1 - x \frac{\mathcal{J}'(x)}{\mathcal{J}(x)} + xq \frac{\mathcal{J}'_e(xq)}{\mathcal{J}_e(xq)}, \quad (52)$$

where $\mathcal{J}(x)$ is defined by (15) and $\mathcal{J}_e(x)$ denotes its even part in x (see (16)).

The generating function of fully commutative involutions in type \tilde{A} is

$$\tilde{\mathcal{A}} = \sum_{n \geq 1} \tilde{\mathcal{A}}_{n-1}^{FC}(q) x^n = -x \frac{\mathcal{J}'(x)}{\mathcal{J}(x)}.$$

Remark. The coefficients of x^1 and x^2 in $\tilde{\mathcal{A}}$ are irrelevant.

Proof. The polynomials $\check{O}_n(q)$ and the series $\mathcal{M}(x)$ are defined in terms of Dyck-like paths in [4] (see Proposition 3.3 and Section 1.4, where the \check{F} notation is explained). These definitions lead to:

$$\check{O}(x) = \mathcal{M}(x) \left(1 + x^2 q \frac{\partial(x\mathcal{M})}{\partial x}(xq) \right).$$

We can now derive the expression (51) exactly as we proved (45) from (48), using Proposition 4.2 for the expression of \mathcal{M} and the linear q -equation (21) satisfied by \mathcal{J}_e .

The definition (50) of $\check{O}^*(x)$ now reads $\check{O}^*(x) = \check{O}(x)/(1 - x\mathcal{M}(x))$. Using the expression of \mathcal{M} given in Proposition 4.2, and the expression (20) of \mathcal{J} in terms of \mathcal{J}_e , one readily checks that this coincides with our expression of $\check{O}^*(x)$.

Consider now the series $\tilde{\mathcal{A}}$. Proposition 3.3 in [4] tells us that

$$\begin{aligned} \tilde{\mathcal{A}}(x) - \tilde{\mathcal{A}}(xq) &= \check{O}(xq) - 1 + \check{O}^*(x) - \check{O}^*(xq) \\ &= -xq \frac{\mathcal{J}'_e(xq)}{\mathcal{J}_e(xq)} + xq^2 \frac{\mathcal{J}'_e(xq^2)}{\mathcal{J}_e(xq^2)} + \check{O}^*(x) - \check{O}^*(xq) \end{aligned}$$

by (51). Iterating this formula gives

$$\tilde{\mathcal{A}}(x) = -xq \frac{\mathcal{J}'_e(xq)}{\mathcal{J}_e(xq)} + \check{O}^*(x) - 1 = -x \frac{\mathcal{J}'(x)}{\mathcal{J}(x)}$$

by (52). ■

Corollary 5.3. *The number of fully commutative involutions in \tilde{A}_{2m} is finite, equal to 4^m .*

Proof. We will prove this using the enumerative results of Theorem 5.2, though there are more direct combinatorial arguments based on the structure of these involutions, see [4].

By [4, Eq. (2)], the series $\mathcal{M}(x)$ satisfies

$$\mathcal{M}(x) = 1 + x^2 q \mathcal{M}(x) \mathcal{M}(xq).$$

Hence it is finite at $q = 1$, equal to $(1 - \sqrt{1 - 4x^2})/(2x^2)$. Then it follows from (50) that at $q = 1$,

$$\check{O}^*(x) = \frac{1}{2} \left(\frac{1}{\sqrt{1 - 4x^2}} + \frac{1}{1 - 2x} \right). \quad (53)$$

We want to determine the series

$$\lim_{q \rightarrow 1} \sum_{m \geq 0} \tilde{\mathcal{A}}_{2m}^{FC}(q) x^{2m+1},$$

which is the odd part of $\tilde{\mathcal{A}}(x, q)$, in the limit $q \rightarrow 1$. Let us compare the expressions of $\tilde{\mathcal{A}}$ and \check{O}^* in Theorem 5.2: the first and third terms in the expression of \check{O}^* are even in x (because of the derivative), and thus the series we want to compute is also the odd part of $\check{O}^*(x)$, that is (thanks to (53)), $x/(1 - 4x^2)$. The result follows. ■

6. ANOTHER RECURSIVE APPROACH: PEELING A COLUMN

In order to count fc elements in infinite types \tilde{B} , \tilde{C} and \tilde{D} , we need to count alternating heaps over a path, with no restriction on the number of points in the first and last columns. The recursive approach of the previous section applies, but the resulting expressions are complicated and we do not give them.

Instead, we describe in this section another recursive approach to count alternating heaps over a path. The principle simply consists in deleting the first column of the heap. This is essentially the idea behind the encoding of alternating heaps as paths in [5] but we will not use this encoding here. By forcing the first and/or last column to contain at most one point, we obtain new expressions for the series A and $B^{(a)}$. Remarkably, these expressions involve negative powers of the length variable q . This phenomenon has already been observed in polyomino enumeration (see eg. [18, 19] or [21, Ex. 5.5.2]).

6.1. ALTERNATING HEAPS OVER A PATH

We begin with a new expression for the series A .

Theorem 6.1. *The generating function $A(x, q) \equiv A$ of fully commutative elements of type A , defined by*

$$A = \sum_{n \geq 0} A_n^{FC}(q) x^n$$

and given by (6), can also be written as

$$A = \frac{\sum_{k \geq 0} x^k \sum_{i=0}^k \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix} q^{-i(k-i)}}{1 + \sum_{k \geq 1} x^k \sum_{i=0}^{k-1} \begin{bmatrix} k-1 \\ i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} q^{-i(k-i)}},$$

where $\begin{bmatrix} k \\ i \end{bmatrix}$ is the q -binomial coefficient:

$$\begin{bmatrix} k \\ i \end{bmatrix} = \frac{(q)_k}{(q)_i (q)_{k-i}}.$$

The next theorem gives a new expression for alternating heaps of type B (so far expressed by (36)) and an expression for alternating heaps over a path. By Theorem 3.4 of [5], these heaps form a subset of fc elements in type \tilde{C} . As before, the variable x records the number of vertices of the path (that is, the number of generators in the group), and q the number of points in the heap (the length of the corresponding fc element). By an *empty* path, we mean the n -point path when $n = 0$.

Theorem 6.2. *Let $N(x)$ and $S(x)$ be the following series:*

$$N(x) = \sum_{k \geq 1} x^k q^{-\binom{k}{2}} \frac{(-q)_{k-1}^2}{1 - q^k},$$

and

$$S(x) = N(x/q) - N(xq) = \sum_{k \geq 1} x^k q^{-\binom{k+1}{2}} (-q)_{k-1} (-q)_k. \quad (54)$$

The generating function of alternating heaps of type B , expressed in (36), can also be written as

$$B^{(a)} = 1 + S(xq) - xS(x)A, \quad (55)$$

where A is the generating function of fc elements of type A , expressed in Theorem 3.1 or alternatively in Theorem 6.1.

The generating function of alternating heaps over a non-empty path is

$$L(x) = N(x) - xS(x)B^{(a)},$$

where $B^{(a)}$ is the generating function of alternating heaps of type B , expressed by (36) or alternatively by (55).

Remark. As noted already, the expressions of A, B and D given in Theorem 3.1 define series in x with rational coefficients in q , with poles at roots of unity — even though we know from combinatorial reasons that these coefficients must be polynomials in q . In particular, it is not immediate to derive from Theorem 3.1 the value at $q = 1$ of these three series. Now with Theorem 6.1, we have an expression of A as a series in x whose coefficients are Laurent polynomials in q , and the specialization $q = 1$ is straightforward: the numerator of A becomes

$$\sum_{k \geq 0} x^k \binom{2k+1}{k} = \frac{1}{2x} \left(\frac{1}{\sqrt{1-4x}} - 1 \right),$$

and its denominator

$$1 + \sum_{k \geq 1} x^k \binom{2k-1}{k} = \frac{1}{2} \left(\frac{1}{\sqrt{1-4x}} + 1 \right),$$

so that we recover the Catalan generating function for $A(x, 1)$:

$$A(x, 1) = \sum_{n \geq 0} A_n^{FC}(1)x^n = \frac{1 - 2x - \sqrt{1-4x}}{2x^2} = \sum_{n \geq 0} \frac{1}{n+2} \binom{2n+2}{n+1} x^n.$$

Similarly, (55) expresses $B^{(a)}$ as a series in x whose coefficients are Laurent polynomials in q . From this and (30), one can derive a similar expression for $B = B^{(a)} + B^{(na)}$, and finally for D using (43). In these expressions, one can set $q = 1$ to recover Stembridge's enumeration of fc elements in B_n and D_{n+1} (see [33]). For the series $L(x)$ counting alternating heaps, we obtain the following periodicity result.

Corollary 6.3. *Write $L(x) = \sum_{n \geq 1} L_n x^n$, where $L_n \equiv L_n(q)$ counts alternating heaps over the n -point path. Then L_n is a rational fraction in q of denominator $(1 - q^n)$. Consequently, the sequence of its coefficients is ultimately periodic, with period dividing n .*

Proof. The expression of $L(x)$ given in Theorem 6.2 shows that L_n is a fraction in q with denominator $q^e(1 - q^n)$, for some integer e (indeed, this holds in $N(x)$, and the series S and $B^{(a)}$ have polynomial coefficients). But then the combinatorial description of L_n shows that $q = 0$ cannot be a pole. \blacksquare

Proof of Theorems 6.1 and 6.2. Let us begin with alternating heaps of type B , excluding the case B_0 , and denote their generating function by $\hat{B}^{(a)} := B^{(a)} - 1$. We enrich this series by taking into account (with a new variable u) the number of occurrences of the “leftmost” generator, denoted t in Figure 1. We denote by n the number of generators, and partition the set of such heaps into four classes.

- If the first column is empty, t does not occur. By Definition 2.3 of alternating heaps, the second column contains at most one point, so that deleting the first column leaves an alternating heap of type A_{n-1} ; the generating function for this class is thus xA .
- From now on t occurs. If $n = 1$ then the only element that contributes is t , with generating function xuq .
- From now on $n \geq 2$, and t occurs. If the second column is empty, that is, the generator s_1 does not occur, then there is only one occurrence of t , followed by an empty column and an alternating heap of type A_{n-2} . The generating function is thus x^2uqA .
- Otherwise, deleting every point of the first column leaves an alternating heap of type B_{n-1} with a non-empty first column. The generating function of such heaps is $\hat{B}^{(a)}(u) - xA$. Conversely, an alternating heap of type B_{n-1} with k points in the first column gives rise to four alternating heaps by adding a column: two with k points in the first column, one with $k-1$ points and one with $k+1$ points (Figure 7). However, if $k = 1$, the choice of $k-1$ occurrences of t gives a heap with no occurrence of t , which we have already counted in the first class. Hence the contribution of this last class is:

$$x \left(\frac{1}{uq} + 2 + uq \right) \left(\hat{B}^{(a)}(uq) - xA \right) - x(A - 1 - xA),$$

since $A - 1 - xA$ counts alternating heaps of type A with exactly one point in the first column.

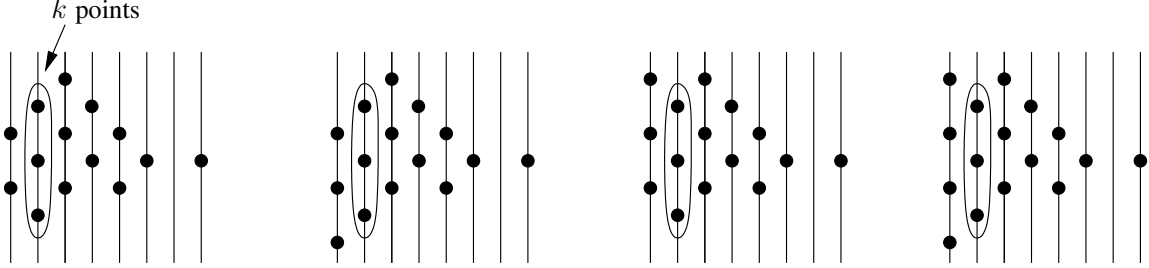


FIGURE 7. Adding a column to the left of an alternating heap of type B .

Putting together these four cases gives

$$\hat{B}^{(a)}(u) = x(1 + uq) - \frac{x^2(1 + uq)}{uq}A + \frac{x(1 + uq)^2}{uq}\hat{B}^{(a)}(uq). \quad (56)$$

This equation can be solved using the iteration method of Section 4.1:

$$\begin{aligned} \hat{B}^{(a)}(u) &= \sum_{k \geq 1} x^{k-1} u^{-k+1} q^{-\binom{k}{2}} (-uq)_{k-1}^2 \left(x(1 + uq^k) - \frac{x^2(1 + uq^k)}{uq^k} A \right) \\ &= \sum_{k \geq 1} x^k u^{-k+1} q^{-\binom{k}{2}} (-uq)_{k-1} (-uq)_k - A \sum_{k \geq 1} x^{k+1} u^{-k} q^{-\binom{k+1}{2}} (-uq)_{k-1} (-uq)_k. \end{aligned} \quad (57)$$

Setting $u = 1$ and adding the contribution of the empty heap gives (55).

We can now derive the expression of A given in Theorem 6.1. The series $\hat{B}^{(a)}(0)$ counts heaps with an empty first column, and thus coincides with the series xA . Let us generalize (54) by denoting

$$S(x, u) = \sum_{k \geq 1} x^k u^{-k} q^{-\binom{k+1}{2}} (-uq)_{k-1} (-uq)_k,$$

so that (57) reads

$$\hat{B}^{(a)}(u) = uS(xq, u) - xAS(x, u).$$

Extracting from this the coefficient of u^0 gives

$$xA = \hat{B}^{(a)}(0) = [u^{-1}]S(xq, u) - xA[u^0]S(x, u),$$

and hence

$$xA = \hat{B}^{(a)}(0) = \frac{[u^{-1}]S(xq, u)}{1 + [u^0]S(x, u)}.$$

Performing the coefficient extraction explicitly gives Theorem 6.1, using

$$(-uq)_k = \sum_{i=0}^k u^i q^{\binom{i+1}{2}} \begin{bmatrix} k \\ i \end{bmatrix}.$$

We now apply column peeling to general alternating heaps over the n -point path, counted by the series L . There are no more constraints on the first and last columns. As before, an additional variable u records the number of points in the first column. The four cases listed in the derivation of (56) are transformed as follows.

- If the first column is empty, then the second must contain at most one point. Thus what remains after deleting the first column is an alternating heap of type B_{n-1} , reflected in a vertical line. The generating function for this first case is therefore $xB^{(a)}$.

- From now on the first column is non-empty. If $n = 1$ then all heaps are alternating, with generating function $xuq/(1 - uq)$.
- Assume the second column is empty. Then there is just one point in the first column, followed by an empty column, and then a reflected alternating heap of type B_{n-2} . The generating function is $x^2uqB^{(a)}$.
- Finally, if there are k points in the second column, with $k \geq 1$, then we can have $k - 1$, k or $k + 1$ points in the first column. As before, the case where $k = 1$ and the first column is empty has already been counted. Hence the contribution of this final case is:

$$x \left(\frac{1}{uq} + 2 + uq \right) \left(L(uq) - xB^{(a)} \right) - x(B^{(a)} - 1 - xB^{(a)}),$$

since $B^{(a)} - 1 - xB^{(a)}$ counts alternating heaps of type B , reflected in a vertical line, and having exactly one point in the first column.

Putting together these four cases gives

$$L(u) = \frac{x}{1 - uq} - \frac{x^2(1 + uq)}{uq} B^{(a)} + \frac{x(1 + uq)^2}{uq} L(uq).$$

Iterating yields

$$L(u) = \sum_{k \geq 1} x^k u^{-k+1} q^{-\binom{k}{2}} \frac{(-uq)_{k-1}^2}{1 - uq^k} - B^{(a)} \sum_{k \geq 1} x^{k+1} u^{-k} q^{-\binom{k+1}{2}} (-uq)_{k-1} (-uq)_k.$$

Setting $u = 1$ gives the last result of Theorem 6.2. ■

6.2. SELF-DUAL ALTERNATING HEAPS OVER A PATH

We now restrict the enumeration to self-dual heaps. We begin with a new expression for the series \mathcal{A} , already expressed in Theorem 3.3.

Theorem 6.4. *The generating function $\mathcal{A}(x, q) \equiv \mathcal{A}$ of fully commutative involutions of type A , defined by*

$$\mathcal{A} = \sum_{n \geq 0} \mathcal{A}_n^{FC}(q) x^n$$

and given by (17), can also be written as:

$$\mathcal{A} = \frac{\sum_{j \geq 0} x^j \llbracket \begin{smallmatrix} j \\ \lfloor j/2 \rfloor \end{smallmatrix} \rrbracket q^{-\lfloor j/2 \rfloor^2 + \chi(j \text{ odd})}}{1 + \sum_{j \geq 1} (-x)^j \llbracket \begin{smallmatrix} j-1 \\ \lfloor j/2 \rfloor \end{smallmatrix} \rrbracket q^{-\lfloor j/2 \rfloor^2}}$$

where $\llbracket \begin{smallmatrix} k \\ i \end{smallmatrix} \rrbracket$ is the q^2 -binomial coefficient:

$$\llbracket \begin{smallmatrix} k \\ i \end{smallmatrix} \rrbracket = \frac{((q^2)_k)}{((q^2)_i ((q^2))_{k-i}} = \frac{(q^2; q^2)_k}{(q^2; q^2)_i (q^2; q^2)_{k-i}}.$$

The next theorem gives a new expression for self-dual alternating heaps of type B and an expression for self-dual alternating heaps over a path. These heaps form a subclass of fc involutions in type \tilde{C} .

Theorem 6.5. *Let $\mathcal{N}(x)$ and $\mathcal{S}(x)$ be the following series:*

$$\mathcal{N}(x) = \sum_{k \geq 1} x^k q^{-\binom{k}{2}} \frac{((-q^2)_{k-1}}{1 - q^k}, \quad (58)$$

$$\mathcal{S}(x) = \mathcal{N}(x/q) - \mathcal{N}(x) = \sum_{k \geq 1} x^k q^{-\binom{k+1}{2}} ((-q^2)_{k-1}). \quad (59)$$

The generating function of self-dual alternating heaps of type B , already given by (41), can also be written as

$$\mathcal{B}^{(a)} = 1 + \mathcal{S}(xq) + \mathcal{S}(xq^2) - x(\mathcal{S}(x) - \mathcal{S}(xq)) \mathcal{A}, \quad (60)$$

where \mathcal{A} is the generating function of fully commutative involutions of type A , given in Theorem 3.3 or alternatively in Theorem 6.4.

The generating function for self-dual alternating heaps over a non-empty path is

$$\mathcal{L} = \mathcal{N}(x) - x(\mathcal{S}(x) - \mathcal{S}(xq))\mathcal{B}^{(a)},$$

where $\mathcal{B}^{(a)}$ is the generating function of self-dual alternating heaps of type B , given by (41) or alternatively by (60).

Proof of Theorems 6.4 and 6.5. We begin with self-dual alternating heaps of type B , excluding again the case B_0 , and denote their generating function by $\hat{\mathcal{B}}^{(a)} := \mathcal{B}^{(a)} - 1$. We count them by specializing to self-dual heaps the argument that led to (56) for alternating heaps of type B . The first three cases described there now contribute $x\mathcal{A}$, xuq and $x^2uq\mathcal{A}$ respectively. In the fourth (and last) case, we observe that, in a self-dual heap, the first two columns cannot both contain k points. Hence the contribution of the fourth case is now

$$x \left(\frac{1}{uq} + uq \right) \left(\hat{\mathcal{B}}^{(a)}(uq) - x\mathcal{A} \right) - x(\mathcal{A} - 1 - x\mathcal{A}).$$

Putting together the four cases gives

$$\hat{\mathcal{B}}^{(a)}(u) = x(1 + uq) - \frac{x^2}{uq}(1 - uq)\mathcal{A} + \frac{x}{uq}(1 + u^2q^2)\hat{\mathcal{B}}^{(a)}(uq).$$

By iterating, we obtain:

$$\begin{aligned} \hat{\mathcal{B}}^{(a)}(u) &= \sum_{k \geq 1} x^k u^{1-k} q^{-\binom{k}{2}} (1 + uq^k) ((-u^2q^2))_{k-1} \\ &\quad - \mathcal{A} \sum_{k \geq 1} x^{k+1} u^{-k} q^{-\binom{k+1}{2}} (1 - uq^k) ((-u^2q^2))_{k-1}. \end{aligned} \quad (61)$$

Setting $u = 1$ and adding the contribution of the empty heap gives the expression (60) of $\mathcal{B}^{(a)}$.

We can now derive the alternative expression of \mathcal{A} given in Theorem 6.4. The series $\hat{\mathcal{B}}^{(a)}(0)$ counts heaps with an empty first column, and thus coincides with the series $x\mathcal{A}$. Let us generalize (59) by denoting

$$\mathcal{S}(x, u) = \sum_{k \geq 1} x^k u^{-k} q^{-\binom{k+1}{2}} ((-u^2q^2))_{k-1},$$

so that (61) reads

$$\hat{\mathcal{B}}^{(a)}(u) = u\mathcal{S}(xq, u) + u^2\mathcal{S}(xq^2, u) - x\mathcal{A}(\mathcal{S}(x, u) - u\mathcal{S}(xq, u)). \quad (62)$$

Extracting the coefficient of u^0 gives

$$x\mathcal{A} = \hat{\mathcal{B}}^{(a)}(0) = \frac{[u^{-1}]\mathcal{S}(xq, u) + [u^{-2}]\mathcal{S}(xq^2, u)}{1 + [u^0]\mathcal{S}(x, u) - [u^{-1}]\mathcal{S}(xq, u)}.$$

The coefficient extraction is performed explicitly thanks to

$$((-u^2q^2))_k = \sum_{i=0}^k u^{2i} q^{i(i+1)} \left[\begin{matrix} k \\ i \end{matrix} \right],$$

and one thus obtains Theorem 6.4.

Our final step is to count self-dual alternating heaps, by specializing to the self-dual case the argument that led to (56) for general alternating heaps. The first three cases described there now contribute $x\mathcal{B}^{(a)}$, $xqu/(1 - uq)$ and $x^2uq\mathcal{B}^{(a)}$, respectively. In the fourth case, we must exclude heaps in which the first two columns would have the same size. Hence the contribution of the fourth case is now:

$$x \left(\frac{1}{uq} + uq \right) \left(\mathcal{L}(uq) - x\mathcal{B}^{(a)} \right) - x \left(\mathcal{B}^{(a)} - 1 - x\mathcal{B}^{(a)} \right).$$

Putting together the four cases gives

$$\mathcal{L}(u) = \frac{x}{1-uv} - \frac{x^2}{uv}(1-uv)\mathcal{B}^{(a)} + \frac{x}{uv}(1+u^2q^2)\mathcal{L}(uq). \quad (63)$$

Iterating yields

$$\begin{aligned} \mathcal{L}(u) &= \sum_{k \geq 1} x^k u^{1-k} q^{-\binom{k}{2}} \frac{((-u^2q^2)_{k-1})}{1-uv^k} - \mathcal{B}^{(a)} \sum_{k \geq 1} x^{k+1} u^{-k} q^{-\binom{k+1}{2}} (1-uv^k) ((-u^2q^2)_{k-1}), \\ &= \sum_{k \geq 1} x^k u^{1-k} q^{-\binom{k}{2}} \frac{((-u^2q^2)_{k-1})}{1-uv^k} - x(\mathcal{S}(x, u) - u\mathcal{S}(xq, u)) \mathcal{B}^{(a)}. \end{aligned} \quad (64)$$

Setting $u = 1$ gives the second result of Theorem 6.5. \blacksquare

Later we will need the following result, which gives the generating functions of self-dual alternating heaps over a path, with parity constraints on the first and/or last columns. Recall the notation $F_e(x)$ and $F_o(x)$ for the even and odd parts (in x) of a series $F(x)$ (see (16)).

Proposition 6.6. *The generating function of self-dual alternating heaps of type B having an odd number of points in the first column, given by (42), admits the following alternative expression:*

$$\mathcal{B}_{\text{odd}}^{(a)} = \mathcal{S}_e(xq) + \mathcal{S}_o(xq^2) - x(\mathcal{S}_o(x) - \mathcal{S}_e(xq)) \mathcal{A}, \quad (65)$$

where \mathcal{S} is defined by (59) and \mathcal{A} counts fc involutions of type A (Theorems 3.1 and 6.4).

The generating function of self-dual alternating heaps over a path, having an odd number of points in the first column, is

$$\mathcal{L}_{\text{odd}} = \frac{1}{2} \left(\mathcal{N}(x) + \tilde{\mathcal{N}}(-x) \right) - x(\mathcal{S}_o(x) - \mathcal{S}_e(xq)) \mathcal{B}^{(a)},$$

where \mathcal{N} is defined by (58), $\mathcal{B}^{(a)}$ counts self-dual alternating heaps of type B (see (41) and (60)), and

$$\tilde{\mathcal{N}}(x) = \sum_{k \geq 1} x^k q^{-\binom{k}{2}} \frac{((-q^2)_{k-1})}{1+q^k}.$$

The generating function of self-dual alternating heaps over a path having an odd number of points in the first and last columns is

$$\mathcal{L}_{\text{odd}}^{\text{odd}} = \frac{1}{2} \left(\mathcal{N}_o(xq) + \tilde{\mathcal{N}}_o(xq) \right) - x(\mathcal{S}_o(x) - \mathcal{S}_e(xq)) \mathcal{B}_{\text{odd}}^{(a)}$$

where $\mathcal{B}_{\text{odd}}^{(a)}$ is given by (42) or (65).

Proof. Recall that the series $\hat{\mathcal{B}}^{(a)}(u)$ given by (62) counts alternating heaps of type B, and records, with the variable u , the number of points in the first column. Hence $\mathcal{B}_{\text{odd}}^{(a)}$ is simply the odd part (in u) of $\hat{\mathcal{B}}^{(a)}(u)$, specialized at $u = 1$:

$$\mathcal{B}_{\text{odd}}^{(a)} = \frac{1}{2} \left(\hat{\mathcal{B}}^{(a)}(1) - \hat{\mathcal{B}}^{(a)}(-1) \right).$$

Upon using (62), and noticing that x and u have the same parity in $\mathcal{S}(x, u)$, we obtain the first result of the proposition.

For the second one, use in the same spirit (64) and

$$\mathcal{L}_{\text{odd}} = \frac{1}{2} (\mathcal{L}(1) - \mathcal{L}(-1)).$$

Finally, in order to compute our last series $\mathcal{L}_{\text{odd}}^{\text{odd}}$, we will determine the generating function $\mathcal{L}^{\text{odd}}(u)$ of self-dual alternating heaps over a path having an odd number of points in the last column, where u records the number of points in the first column. In order to do this, we restrict to heaps with an odd number of points in the last column the argument that led to (63). The

first three cases contribute $x\mathcal{B}_{\text{odd}}^{(a)}$, $xuq/(1-u^2q^2)$ and $x^2uq\mathcal{B}_{\text{odd}}^{(a)}$ respectively. The fourth one contributes

$$x\left(\frac{1}{uq} + uq\right)\left(\mathcal{L}^{\text{odd}}(uq) - x\mathcal{B}_{\text{odd}}^{(a)}\right) - x(1-x)\mathcal{B}_{\text{odd}}^{(a)}.$$

Putting together the four contributions gives:

$$\mathcal{L}^{\text{odd}}(u) = \frac{xuq}{1-u^2q^2} - \frac{x^2}{uq}(1-uq)\mathcal{B}_{\text{odd}}^{(a)} + \frac{x}{uq}(1+u^2q^2)\mathcal{L}^{\text{odd}}(uq).$$

Iterating this gives the “odd” analogue of (64):

$$\mathcal{L}^{\text{odd}}(u) = \sum_{k \geq 1} (xq)^k u^{2-k} q^{-\binom{k}{2}} \frac{((-u^2q^2)_{k-1}}{1-u^2q^{2k}} - x(\mathcal{S}(x, u) - u\mathcal{S}(xq, u))\mathcal{B}_{\text{odd}}^{(a)}.$$

The expression of $\mathcal{L}_{\text{odd}}^{\text{odd}}$ follows by writing

$$\mathcal{L}_{\text{odd}}^{\text{odd}} = \frac{1}{2}(\mathcal{L}^{\text{odd}}(1) - \mathcal{L}^{\text{odd}}(-1)).$$

■

Remark. Theorem 6.4 gives \mathcal{A} as a series in x whose coefficients are Laurent polynomials in q (while the expression of Theorem 3.3 defines a series in x whose coefficients are rational functions of q , with poles at roots of unity). In particular, we can specialize Theorem 6.4 at $q = 1$ to recover the number of fc involutions in \mathcal{A}_n , already derived by Stembridge [33]. Similarly, the expression (60) of $\mathcal{B}^{(a)}$ can be specialized at $q = 1$. Combined with (37) and (44), this allows to recover the number of fc involutions in B_n and D_n . Finally, note that the expression of $\mathcal{B}_{\text{odd}}^{(a)}$ in Proposition 6.6 is also of the same type: a series in q whose coefficients are Laurent polynomials in q . The specialization $q = 1$ is again possible. For the series \mathcal{L} and its variants, we obtain the following periodicity results.

Corollary 6.7. Write $\mathcal{L}(x) = \sum_{n \geq 1} \mathcal{L}_n x^n$, where $\mathcal{L}_n \equiv \mathcal{L}_n(q)$ counts self-dual alternating heaps over the n -point path. Define similarly the series $\mathcal{L}_{\text{odd}, n}$ (resp. $\mathcal{L}_{\text{odd}, n}^{\text{odd}}$) for self-dual alternating heaps with an odd first (resp. first and last) column. Then \mathcal{L}_n (resp. $\mathcal{L}_{\text{odd}, n}$) is a rational fraction in q of denominator $(1 - q^n)$ (resp. $(1 - q^{2n})$). Consequently, the sequence of its coefficients is ultimately periodic, with period dividing n (resp. $2n$). The series $\mathcal{L}_{\text{odd}, n}^{\text{odd}}$ is a polynomial when n is even, and otherwise a rational fraction of denominator $(1 - q^{2n})$. In this case its coefficients form a periodic sequence, with period dividing $2n$.

Proof. This follows from the expressions of $\mathcal{L}(x)$, $\mathcal{L}_{\text{odd}}(x)$ and $\mathcal{L}_{\text{odd}}^{\text{odd}}(x)$ given in Theorem 6.5 and Proposition 6.6. ■

7. GENERATING FUNCTIONS FOR INFINITE TYPES

In this section, we complete our results for the infinite types. The \tilde{A} -case has been solved in Section 5, and we now derive the generating functions of fully commutative elements, and fully commutative involutions, in Coxeter groups of type \tilde{B} , \tilde{C} and \tilde{D} .

7.1. ALL FULLY COMMUTATIVE ELEMENTS

Recall that all series for finite types have a common form (Theorem 3.1). This is also true for the infinite types \tilde{B} , \tilde{C} and \tilde{D} . In particular, all involve the generating function $L(x)$ of alternating heaps over a path, determined in the previous section (Theorem 6.2).

Theorem 7.1. Let $\tilde{B} \equiv \tilde{B}(x, q)$, $\tilde{C} \equiv \tilde{C}(x, q)$, and $\tilde{D} \equiv \tilde{D}(x, q)$ be the generating functions of fully commutative elements of type \tilde{B} , \tilde{C} and \tilde{D} , defined respectively by

$$\tilde{B}(x, q) = \sum_{n \geq 0} \tilde{B}_n^{FC}(q)x^n, \quad \tilde{C}(x, q) = \sum_{n \geq 0} \tilde{C}_{n-1}^{FC}(q)x^n, \quad \tilde{D}(x, q) = \sum_{n \geq 0} \tilde{D}_{n+1}^{FC}(q)x^n.$$

Let $L(x)$ be the generating function of alternating heaps over a path, expressed in Theorem 6.2, and define the series $J(x)$ and $K(x)$ by (5) as before. Then each of the series \tilde{B} , \tilde{C} and \tilde{D} can be written as

$$R_0 L(x) + R_1 \frac{K(x)}{J(x)} + R_2 \frac{J(xq)}{J(x)} + R_3 + S, \quad (66)$$

where the series R_i are explicit rational series in x and q and S is a simple q -series. More precisely, the series R_0 , R_1 , R_2 and S are given by the following table:

	B	C	D
R_0	2	1	4
R_1	$-\frac{x + q(x-1) + xq^2(x-2)}{1-xq^2}$	$2 \frac{xq^2(1-x)}{1-xq^2}$	$4 \frac{q-x-xq+x^2q^2}{1-xq^2}$
R_2	$\frac{xq^2(1-x)(q-x(1+q)+x^2q^2)}{(1-xq)(1-xq^2)^2}$	$\frac{x^2q^4(1-x)^2}{(1-xq)(1-xq^2)^2}$	$\frac{(q-x-xq+x^2q^2)^2}{(1-xq)(1-xq^2)^2}$
S	$\sum_{n \geq 3} \frac{x^n q^{2(2n-1)}}{1-q^{2n-1}}$	0	$2 \sum_{n \geq 3} \frac{x^n q^{3n}}{1-q^n}$

and, with obvious notation,

$$R_3^B = \frac{x^3 q^4 (-3q^6 x - 2xq^5 + q^4 x + 4q^4 + 2q^3 - xq^2 - q^2 + q + 1)}{(1-q)(1-xq^2)^2},$$

$$R_3^C = \frac{x^3 q^4 (1+q)(1-3q+4q^2+2xq^3-3xq^4)}{(1-q)(1-xq^2)^2},$$

$$R_3^D = \frac{x^3 q^5 (-3q^6 x - 3q^5 x + 4q^4 + 3q^3 - 2xq^2 + q + 2)}{(1-q)(1-xq^2)^2}.$$

Remarks

1. The coefficients of x^0 , x^1 and x^2 in our three series are irrelevant, since the corresponding groups are not well-defined. We have simply adjusted these coefficients so as to make the rational part R_3 a multiple of x^3 .
2. Again, we can feed a computer algebra system with our expressions and expand then in x to obtain the series $\tilde{W}_n^{FC}(q)$ for small values of n . The first few are reported in Table 3.

n	3	4
\tilde{A}_{n-1}^{FC}	$\frac{[1,2,3]}{1-q}$	$\frac{[1,4,9,12,8]}{1-q^2}$
\tilde{B}_n^{FC}	$\frac{(1+q)^2 [1,3,7,11,14,15,12,7,2,0,3,1,1]}{(1+q+q^2)(1-q^5)}$	$\frac{(1+q)[1,4,10,18,24,29,29,27,19,12,3,3,9,8,7,3,1,1]}{1-q^7}$
\tilde{C}_{n-1}^{FC}	$\frac{[1,3,5,7,6,4,2]}{1-q^3}$	$\frac{[1,3,5,7,6,4,2,2,2,2]}{1-q}$
\tilde{D}_{n+1}^{FC}	$\frac{(1+q)[1,4,10,17,17,13,4,9,5,4]}{1-q^3}$	$\frac{[1,6,20,46,78,100,100,78,35,14,36,44,36,16,8,4]}{1-q^4}$

TABLE 3. Length generating functions of fc elements in affine Coxeter groups. The list $[a_0, \dots, a_k]$ stands for the polynomial $a_0 + a_1 q + \dots + a_k q^k$, and the notation \bar{a} means $-a$. The \tilde{A} -results come from Theorem 5.1.

Before we prove Theorem 7.1, let us show that it implies the periodicity results of [5, Thm. 4.1]. Note that the *exact* values of the periods were found in [25].

Corollary 7.2. *The coefficients of the series $\tilde{B}_n^{FC}(q)$ (resp. $\tilde{C}_{n-1}^{FC}(q)$, $\tilde{D}_{n+1}^{FC}(q)$) form a periodic sequence of period dividing $n(2n-1)$ (resp. n , n).*

Proof. We start from the expressions of Theorem 7.1. The series R_1 and R_2 , once expanded in x , have polynomial coefficients in q . The same holds for $K(x)/J(x)$ (by (36)) and $J(xq)/J(x)$ (by (6)). The coefficient of x^n in R_3 is a fraction with denominator $(1 - q)$. Then the results follow from the properties of $L_n(q)$ (Corollary 6.3) and the values of the series S . ■

Proof of Theorem 7.1. We begin with the simplest case, which is \tilde{C}_n . Recall that the Coxeter graph of \tilde{C}_n is the $(n + 1)$ -point path (Figure 1), which explains why the generating function we consider is $\sum_n \tilde{C}_n^{FC} x^{n+1} = \sum_n \tilde{C}_{n-1}^{FC} x^n$.

For $n \geq 2$, Theorem 3.4 in [5] describes the set of heaps encoding fc elements of \tilde{C}_n as the disjoint union of five sets, which we will enumerate separately.

- First come alternating heaps over the $(n + 1)$ -point path. The associated generating function is

$$L(x) - xL_1 - x^2L_2,$$

where L_i counts alternating heaps over the i -point path. Since the coefficients of x and x^2 in our final series are irrelevant, we do not need the expressions of L_1 or L_2 .

- Then come *zigzags*. Their generating function, derived in the proof of [5, Prop. 4.2], is

$$\sum_{n \geq 2} x^{n+1} \left(\frac{2nq^{2n+2}}{1 - q} + (2n - 2)q^{2n+1} \right) = 2 \frac{x^3 q^5 (1 + q - xq^3)}{(1 - q)(1 - xq^2)^2}.$$

- Then come *left peaks*. They are obtained as follows: starting from an alternating heap over the $(n - j + 1)$ -point path (with $1 \leq j \leq n - 1$, and vertices indexed by s_j, \dots, s_{n-1}, u), having exactly one point in the s_j -column, one inflates this point into a $<$ -shaped heap $s_j s_{j-1} \cdots s_1 t s_1 \cdots s_{j-1} s_j$ (this is closely related to the description of non-alternating fc elements of type B in Section 4.2). The corresponding generating function is

$$LP(x) = \frac{xq^2}{1 - xq^2} \left(B^{(a)}(x) - 1 - xq - xB^{(a)}(x) \right), \quad (67)$$

where the term between parentheses counts alternating heaps with at least two columns and exactly one point in the first column.

- The generating function $RP(x)$ of *right peaks* coincides with $LP(x)$.
- The fifth class consists of *left-right peaks*. To obtain them, one starts from an alternating heap H over a path with vertices s_j, \dots, s_k , having one point in its first and last columns (with $1 \leq j < k \leq n - 1$), inflates the point in the s_j -column into a $<$ -shaped heap $s_j s_{j-1} \cdots s_1 t s_1 \cdots s_{j-1} s_j$, and symmetrically inflates the point in the s_k -column into a $>$ -shaped heap $s_k s_{k+1} \cdots s_{n-1} u s_{n-1} \cdots s_{k+1} s_k$. Accordingly, the generating function is

$$LRP(x) = \left(\frac{xq^2}{1 - xq^2} \right)^2 \left(A(x) - 1 - 2xA(x) + x^2A(x) + x - xq \right), \quad (68)$$

where the term between parentheses counts alternating heaps of type A with at least two columns and one point in the first and last columns.

We now add all contributions, and inject the expressions (6) and (36) of A and $B^{(a)}$. This gives the form (66) for the series

$$\sum_{n \geq 2} \tilde{C}_n^{FC} x^{n+1} = \sum_{n \geq 3} \tilde{C}_{n-1}^{FC} x^n,$$

for the values of S, R_0, R_1, R_2 listed above, but with a different value of R_3 . Removing from R_3 its terms in x^1 and x^2 gives the announced value of R_3^C .

We now move to the \tilde{B} -case. For $n \geq 2$, Section 3.2 in [5] describes the heaps encoding fc elements of \tilde{B}_{n+1} in terms of a transformation Δ_t acting on fc heaps of type \tilde{C}_n (which we have described above). This transformation maps an fc heap H of type \tilde{C}_n on a set of heaps of type \tilde{B}_{n+1} , obtained by replacing each occurrence of the generator t in H by one or several copies of t_1 and/or t_2 (with the notation of Figure 1). The union of these sets is disjoint, and consists of

all fc heaps of type \tilde{B}_{n+1} . Therefore we now count heaps of $\Delta_t(X)$, for each of the five families X of fc heaps of type \tilde{C}_n listed above.

- Given an alternating heap H of \tilde{C}_n , the length generating function of the set $\Delta_t(H)$ is $q^{|H|}\alpha_t(H)$, where

$$\alpha_t(H) = 2 + \begin{cases} 0 & \text{if } H \text{ contains at least two occurrences of } t, \\ -1 & \text{if } H \text{ contains no occurrence of } t, \\ q & \text{if } H \text{ contains exactly one occurrence of } t. \end{cases} \quad (69)$$

In the second case, erasing the first (empty) column of H leaves an alternating heap of type B on $n \geq 2$ points, reflected in a vertical line. In the third case, H is an alternating heap of type B on $n+1 \geq 3$ points with a non-empty last column, reflected in a vertical line. Hence, the generating function of fc heaps of type \tilde{B}_{n+1} arising from alternating heaps of type \tilde{C}_n is

$$2(L(x) - xL_1 - x^2L_2) - x(B^{(a)}(x) - 1 - xB_1^{(a)}) + q(B^{(a)}(x) - 1 - xB_1^{(a)} - x^2B_2^{(a)} - xB^{(a)}(x) + x + x^2B_1^{(a)}), \quad (70)$$

where $B_i^{(a)}$ counts alternating heaps of type B over the i -point path. Again, we do not need the expressions of L_1 , L_2 , $B_1^{(a)}$ or $B_2^{(a)}$.

- The generating function of heaps arising from zigzags is given in [5, Prop. 4.3]. It reads

$$\begin{aligned} \sum_{n \geq 2} x^{n+1} \left(\frac{(2n+3)q^{2n+4}}{1-q} + \frac{q^{2(2n+1)}}{1-q^{2n+1}} + (2n+2)q^{2n+3} + (2n-2)q^{2n+2} \right) \\ = \sum_{n \geq 2} x^{n+1} \frac{q^{2(2n+1)}}{1-q^{2n+1}} + \frac{x^3q^6(-q^4x - 4q^3x + q^2 + 4q + 2)}{(1-q)(1-xq^2)^2}. \end{aligned}$$

- Each left peak H of \tilde{C}_n gives rise to one element of \tilde{B}_{n+1} , of size $|H|+1$. The associated generating function is thus $qLP(x)$, with $LP(x)$ given by (67).
- Right peaks behave under the map Δ_t as alternating elements do, with one special case: the right peak H corresponding to the element $ts_1 \cdots s_{n-1}us_{n-1} \cdots s_1t$ does not give rise to two heaps of size $|H|$, but to nine heaps of \tilde{B}_{n+1} : four of size $|H|$, four of size $|H|+1$ and one of size $|H|+2$. By adapting the argument that led us to (70), we obtain the generating function of heaps arising from right peaks as

$$\begin{aligned} RP_{\Delta_t} = 2RP(x) - \frac{x^2q^2}{1-xq^2}(A(x) - 1 - xA(x)) + (2+4q+q^2) \sum_{n \geq 2} x^{n+1}q^{2n+1} \\ + \frac{xq^3}{1-xq^2}(A(x) - 1 - x(1+q) - 2x(A(x) - 1) + x^2A(x)), \quad (71) \end{aligned}$$

where $RP(x) = LP(x)$ is given by (67). In the above expression, the second term counts right peaks with an empty first column, and the fourth one is q times the generating function of right peaks having exactly one point in the first column.

- Each left-right peak H of \tilde{C}_n gives rise to one element of \tilde{B}_{n+1} , of size $|H|+1$. The associated generating function is thus $qLRP(x)$, with $LRP(x)$ given by (68).

Adding all contributions, and injecting the expressions (6) and (36) of A and $B^{(a)}$, gives the form (66) for the series

$$\sum_{n \geq 2} \tilde{B}_{n+1}^{FC}(q)x^{n+1} = \sum_{n \geq 3} \tilde{B}_n^{FC}(q)x^n,$$

for the values of S, R_0, R_1 and R_2 listed above. The value of R_3 that we obtain only differs from the one given in the theorem by the coefficients of x^1 and x^2 .

Let us finally address the \tilde{D} -case. Again, Section 3.2 in [5] describes the heaps encoding fc elements of \tilde{D}_{n+2} (for $n \geq 2$) in terms of a transformation $\Delta_{t,u}$ acting on fc heaps of type \tilde{C}_n . The transformation Δ_t that we used to generate elements of \tilde{B}_{n+1} was acting on the first column of a heap. We can roughly describe the new transformation $\Delta_{t,u}$ by saying that it also modifies the last column, in a symmetric fashion. We refer to [5] for details. We will now count heaps of $\Delta_{t,u}(X)$, for each of the five families X of fc heaps of type \tilde{C}_n listed above.

- Given an alternating heap H of \tilde{C}_n , the length generating function of the set $\Delta_{t,u}(H)$ is $q^{|H|}\alpha_t(H)\alpha_u(H)$, where $\alpha_t(H)$ is defined by (69), and $\alpha_u(H)$ is defined similarly in terms of the generator u . A careful case by case study gives the generating function of fc elements of type \tilde{D}_{n+2} arising from alternating heaps of \tilde{C}_n as

$$4L(x) - 3x^2A(x) + 2(q-2)x(A(x) - xA(x)) - 4x(B^{(a)}(x) - A(x)) \\ + (4q + q^2)(A(x) - 2xA(x) + x^2A(x)) + 4q(B^{(a)}(x) - xB^{(a)}(x) - A(x) + xA(x)) + \text{Pol}_2(x),$$

where $\text{Pol}_2(x)$ is a polynomial in x of degree at most 2 (recall that the first three coefficients of our final series $\tilde{D}(x, q)$ are irrelevant). The first term corresponds to the generic case where a heap gives rise to 4 heaps of the same size. The second (resp. third, fourth, fifth, sixth) term corrects this contribution for heaps H such that $\{|H|_t, |H|_u\} = \{0\}$ (resp. $\{0, 1\}$, $\{0, i\}$, $\{1\}$, $\{1, i\}$, with $i \geq 2$).

- The generating function of heaps arising from zigzags is given in [5, Prop. 4.4]. It reads

$$\sum_{n \geq 2} x^{n+1} \left(\frac{(2n+6)q^{2n+5}}{1-q} + \frac{2q^{3(n+1)}}{1-q^{n+1}} + (2n+4)q^{2n+4} + (2n-2)q^{2n+3} \right) \\ = 2 \sum_{n \geq 3} x^n \frac{q^{3n}}{1-q^n} + 2 \frac{x^3 q^7 (-q^4 x - 3q^3 x + q^2 + 3q + 1)}{(1-xq^2)^2 (1-q)}.$$

- Applying $\Delta_{t,u}$ to a right peak boils down to applying Δ_t , and then replacing the only occurrence of u by $u_1 u_2$. Hence the generating function of heaps of \tilde{D}_{n+2} arising from a right peak of \tilde{C}_n is qRP_{Δ_t} , where RP_{Δ_t} is given by (71).
- The case of left peaks is symmetric, and gives another term qRP_{Δ_t} .
- Finally, each left-right peak H of \tilde{C}_n gives rise to one element of \tilde{D}_{n+2} , of size $|H| + 2$. The associated generating function is thus $q^2LRP(x)$, with $LRP(x)$ given by (68).

Adding all contributions gives an expression of the form (66) for the series

$$\sum_{n \geq 2} \tilde{D}_{n+2}^{FC}(q)x^{n+1} = \sum_{n \geq 3} \tilde{D}_{n+1}^{FC}(q)x^n,$$

for the values of S, R_1, R_2 listed in the theorem. The series R_3 only differs from R_3^D by a polynomial in x of degree 2. \blacksquare

7.2. FULLY COMMUTATIVE INVOLUTIONS

We now establish the corresponding results for fc involutions. Again, the three series that we obtain have a similar form. In particular, each of them involves the generating function $\mathcal{L}(x)$ of self-dual alternating heaps over a path, determined in the previous section (Theorem 6.5), or one of its variants obtained by imposing parity conditions on the first and/or last column (Proposition 6.6). Recall that the \tilde{A} -case was solved in Section 5.

Theorem 7.3. *Let $\tilde{\mathcal{B}} \equiv \tilde{\mathcal{B}}(x, q)$, $\tilde{\mathcal{C}} \equiv \tilde{\mathcal{C}}(x, q)$, and $\tilde{\mathcal{D}} \equiv \tilde{\mathcal{D}}(x, q)$ be the generating functions of fully commutative involutions of type \tilde{B} , \tilde{C} and \tilde{D} , defined respectively by*

$$\tilde{\mathcal{B}}(x, q) = \sum_{n \geq 0} \tilde{\mathcal{B}}_n^{FC}(q)x^n, \quad \tilde{\mathcal{C}}(x, q) = \sum_{n \geq 0} \tilde{\mathcal{C}}_{n-1}^{FC}(q)x^n, \quad \tilde{\mathcal{D}}(x, q) = \sum_{n \geq 0} \tilde{\mathcal{D}}_{n+1}^{FC}(q)x^n.$$

Define the series $\mathcal{J}(x)$ and $\mathcal{K}(x)$ by (14) and (15) as before, and let $\mathcal{K}_e(x)$ denote the even part of $\mathcal{K}(x)$ in x (see (16)). Then each of the series $\tilde{\mathcal{B}}, \tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ can be written as

$$\mathcal{R}_0 \mathcal{L}_0(x) + \mathcal{R} \frac{\mathcal{K}_e(xq)}{\mathcal{J}(x)} + \mathcal{R}_1 \frac{\mathcal{K}(x)}{\mathcal{J}(x)} + \mathcal{R}_2 \frac{\mathcal{J}(-xq)}{\mathcal{J}(x)} + \mathcal{R}_3 + \mathcal{S}, \quad (72)$$

where $\mathcal{L}_0(x)$ is the generating function of self-dual alternating heaps (Theorem 6.5) or one of its variants (Proposition 6.6), \mathcal{R} and the \mathcal{R}_i are explicit rational series in x and q , and \mathcal{S} is a simple q -series. The series $\mathcal{R}_0, \mathcal{L}_0, \mathcal{R}, \mathcal{R}_1, \mathcal{R}_2$ and \mathcal{S} are given by the following table:

	\mathcal{B}	\mathcal{C}	\mathcal{D}
\mathcal{R}_0	2	1	4
\mathcal{L}_0	\mathcal{L}_{odd}	\mathcal{L}	$\mathcal{L}_{\text{odd}}^{\text{odd}}$
\mathcal{R}	$2x^2q^3 \frac{1-x}{1-xq^2}$	0	$4xq \frac{x+q(1-x)-x^2q^2}{1-xq^2}$
\mathcal{R}_1	$\frac{x+q(1-x)-x^2q^2}{1-xq^2}$	$2 \frac{xq^2(1-x)}{1-xq^2}$	0
\mathcal{R}_2	$\frac{xq^2(1-x)(x+q(1-x)-x^2q^2)}{(1-xq^2)^2}$	$\frac{x^2q^4(1-x)^2}{(1-xq^2)^2}$	$\frac{(x+q(1-x)-x^2q^2)^2}{(1-xq^2)^2}$
\mathcal{S}	$\sum_{n \geq 3} \frac{x^n q^{2(2n-1)}}{1-q^{2n-1}}$	0	$2 \sum_{n \geq 3} \frac{x^n q^{4n}}{1-q^{2n}}$

and, with obvious notation,

$$\begin{aligned} \mathcal{R}_3^{\mathcal{B}} &= x^3 q^4 \frac{x(q^2 - 2q^3 + 3q^4 - 3q^6) - 1 + 3q - 5q^2 + 4q^4}{(1-q)(1-xq^2)^2}, \\ \mathcal{R}_3^{\mathcal{C}} &= \frac{x^3 q^4 (1 - 3q - 5q^2 + (2x+5)q^3 + (3x+4)q^4 - 4xq^5 - 3xq^6)}{(1-q^2)(1-xq^2)^2}, \\ \mathcal{R}_3^{\mathcal{D}} &= x^3 q^5 \frac{xq^2(2 - 2q + 3q^3 - 2q^4 - 3q^5) - 2 + 3q - q^2 - 5q^3 + 3q^4 + 4q^5}{(1-q^2)(1-xq^2)^2}. \end{aligned}$$

Remark. As before, the coefficients of x^0, x^1 and x^2 in our series are irrelevant, and we have adjusted them so that the fractions \mathcal{R}_3 are multiples of x^3 . Expanding our series in x gives the first few values of the series $\tilde{\mathcal{W}}_n^{FC}(q)$, reported in Table 4.

n	3	4
$\tilde{\mathcal{A}}_{n-1}^{FC}$	[1,3]	$\frac{[1,4,2,0,3,4]}{1-q^4}$
$\tilde{\mathcal{B}}_n^{FC}$	$\frac{[1,5,8,11,14,16,16,11,8,3,1,5,3,4,2,2,1,1]}{(1-q^5)(1-q^6)/(1-q)}$	$\frac{[1,6,11,10,12,13,13,12,6,1,2,5,6,7,6,5,4,2,1,1]}{(1+q)(1-q^7)}$
$\tilde{\mathcal{C}}_{n-1}^{FC}$	$\frac{[1,4,4,4,1,2,2]}{(1+q)(1-q^3)}$	$\frac{[1,4,2,0,3,2,0,4,0,4,2]}{1-q^2}$
$\tilde{\mathcal{D}}_{n+1}^{FC}$	$\frac{[1,5,6,4,7,4,12,1,0,4,1,4,1,4]}{1-q^6}$	$\frac{[1,6,10,6,7,6,8,4,5,2,8,6,5,6,6,4,2,4]}{1-q^8}$

TABLE 4. Length generating functions of fc involutions in affine Coxeter groups. The list $[a_0, \dots, a_k]$ stands for the polynomial $a_0 + a_1q + \dots + a_kq^k$, and the notation \bar{a} means $-a$. The $\tilde{\mathcal{A}}$ -results come from Theorem 5.2. Recall that $\tilde{\mathcal{A}}_{2m}(q)$ is a polynomial.

Before we prove Theorem 7.3, let us show that it implies the periodicity results of [4, Prop. 3.4]. Again, we refer to [25] for the *exact* values of the periods.

Corollary 7.4. *The coefficients of the series $\tilde{\mathcal{B}}_n^{FC}(q)$ (resp. $\tilde{\mathcal{C}}_{n-1}^{FC}(q)$, $\tilde{\mathcal{D}}_{n+1}^{FC}(q)$) form a periodic sequence of period dividing $2n(2n-1)$ (resp. $2n, 2n$). Moreover, if n is even, the period of $\tilde{\mathcal{C}}_{n-1}^{FC}(q)$ divides n .*

Proof. We start from the expressions of Theorem 7.3. The series \mathcal{R} , \mathcal{R}_1 and \mathcal{R}_2 , once expanded in x , have polynomial coefficients in q . The same holds for $\mathcal{K}_e(xq)/\mathcal{J}(x)$ (by (42)), $\mathcal{K}(x)/\mathcal{J}(x)$ (by (41)) and $\mathcal{J}(-xq)/\mathcal{J}(x)$ (by (17)). The coefficient of x^n in \mathcal{R}_3 is a fraction with denominator $(1-q)$ (in the B -case) or $(1-q^2)$ (in the C - and D -cases). Then the results follow from the properties of $\mathcal{L}_n(q)$ and its variants (Corollary 6.7) and the values of the series \mathcal{S} . ■

Proof of Theorem 7.3. By [4, Lemma 1.4], a heap encodes an involution if and only if it is self-dual. Hence we revisit the proof of Theorem 7.1, by restricting the enumeration to self-dual heaps.

Again we begin with the simplest case, which is \tilde{C}_n , and we enumerate self-dual heaps in the five disjoint sets listed in [5, Theorem 3.4]. The expressions that we obtain are perfect analogues of those determined in the proof of Theorem 7.1.

- For self-dual alternating heaps over the $(n+1)$ -point path, the generating function is

$$\mathcal{L}(x) - x\mathcal{L}_1 - x^2\mathcal{L}_2,$$

where \mathcal{L}_i counts self-dual alternating heaps over the i -point path.

- The generating function for self-dual zigzags, derived in the proof of [4, Prop. 3.4], is

$$\sum_{n \geq 2} \frac{2q^{2n+3}}{1-q^2} x^{n+1} = \frac{2x^3q^7}{(1-q^2)(1-xq^2)}.$$

- For self-dual left peaks, the generating function is the analogue of (67):

$$\mathcal{LP}(x) = \frac{xq^2}{1-xq^2} \left(\mathcal{B}^{(a)}(x) - 1 - xq - x\mathcal{B}^{(a)}(x) \right). \quad (73)$$

- The generating function $\mathcal{RP}(x)$ of self-dual right peaks coincides with $\mathcal{LP}(x)$.
- Finally, for self-dual left-right peaks we obtain the self-dual analogue of (68):

$$\mathcal{LRP}(x) = \left(\frac{xq^2}{1-xq^2} \right)^2 \left(\mathcal{A}(x) - 1 - 2x\mathcal{A}(x) + x^2\mathcal{A}(x) + x - xq \right). \quad (74)$$

Adding all contributions, and injecting the expressions (17) and (41) of \mathcal{A} and $\mathcal{B}^{(a)}$ gives the form (72) for $\tilde{C}(x, q)$, for the values of \mathcal{R}_0 , \mathcal{L}_0 , \mathcal{R} , \mathcal{R}_1 , \mathcal{R}_2 and \mathcal{S} listed above. The value of \mathcal{R}_3 is different, but one obtains \mathcal{R}_3^C by subtracting from \mathcal{R}_3 its terms in x^1 and x^2 .

We now move to the \tilde{B} -case. In the proof of Theorem 7.1, we have described the set of fc elements of \tilde{B}_{n+1} as the disjoint union of sets $\Delta_t(H)$, for H an fc element of \tilde{C}_n . Now we have to determine, for each H , which elements in $\Delta_t(H)$ are self-dual. Fortunately, the answer is simple: if H itself is not self-dual, then no element in $\Delta_t(H)$ is self-dual. If H is self-dual, then all elements of $\Delta_t(H)$ are self-dual, except in the two following cases:

- if H is the special right peak $ts_1 \cdots s_{n-1}us_{n-1} \cdots s_1t$, then exactly three of the nine elements of $\Delta_t(H)$ are self-dual: two of size $|H|$ and one of size $|H| + 2$;
- if H is either alternating or a (non-special) right peak, and has a positive even number of points in its first column, then $\Delta_t(H)$ contains no self-dual heap.

In particular, if H is either alternating or a non-special right peak, the length generating function of self-dual heaps in $\Delta_t(H)$ is $q^{|H|}a_t(H)$ with

$$a_t(H) = \begin{cases} 2 & \text{if } H \text{ contains an odd number of occurrences of } t, \text{ at least equal to } 3, \\ 1 & \text{if } H \text{ contains no occurrence of } t, \\ 2+q & \text{if } H \text{ contains exactly one occurrence of } t, \\ 0 & \text{otherwise.} \end{cases} \quad (75)$$

With these changes in mind, we revisit the derivation of the series $\tilde{B}(x, q)$ performed in the proof of Theorem 7.1, and adapt it to self-dual heaps.

- The contribution of involutions of \tilde{B}_{n+1} obtained from self-dual alternating heaps of \tilde{C}_n is found to be:

$$2(\mathcal{L}_{\text{odd}}(x) - x\mathcal{L}_{\text{odd},1} - x^2\mathcal{L}_{\text{odd},2}) + x\left(\mathcal{B}^{(a)}(x) - 1 - x\mathcal{B}_1^{(a)}\right) \\ + q\left(\mathcal{B}^{(a)}(x) - 1 - x\mathcal{B}_1^{(a)} - x^2\mathcal{B}_2^{(a)} - x\mathcal{B}^{(a)}(x) + x + x^2\mathcal{B}_1^{(a)}\right),$$

where $\mathcal{L}_{\text{odd},i}$ (resp. $\mathcal{B}_i^{(a)}$) counts self-dual alternating heaps (resp. of type B) over the i -point path. The main difference with (70) is, in the first term, the restriction to heaps with an odd number of points in the first column. This also results in a sign change in the second term.

- The generating function of self-dual heaps arising from zigzags is given in the proof of [4, Prop. 3.4] and reads

$$\sum_{n \geq 2} x^{n+1} \left(\frac{q^{2n+4}}{1-q} + \frac{q^{2(2n+1)}}{1-q^{2n+1}} \right) = \sum_{n \geq 2} x^{n+1} \frac{q^{2(2n+1)}}{1-q^{2n+1}} + \frac{x^3 q^8}{(1-q)(1-xq^2)}.$$

- Each self-dual left peak H of \tilde{C}_n gives rise to one fc involution of \tilde{B}_{n+1} , of size $|H| + 1$. The associated generating function is thus $q\mathcal{LP}(x)$, with $\mathcal{LP}(x)$ given by (73).
- For right peaks, we adapt the derivation of (71) to the self-dual case, keeping in mind the above restrictions (i) and (ii). We obtain the following generating function:

$$\mathcal{RP}_{\Delta_t} = 2\mathcal{RP}_{\text{odd}}(x) + \frac{x^2 q^2}{1-xq^2} (\mathcal{A}(x) - 1 - x\mathcal{A}(x)) \\ + \frac{xq^3}{1-xq^2} (\mathcal{A}(x) - 1 - x(1+q) - 2x(\mathcal{A}(x) - 1) + x^2\mathcal{A}(x)) + (2+q^2) \sum_{n \geq 2} x^{n+1} q^{2n+1}, \quad (76)$$

where $\mathcal{RP}_{\text{odd}}(x)$ counts self-dual right peaks with an odd number of points in the first column (and at least three columns). The main difference with (71) is, as before, the restriction on the parity of the first column in the first term. This results in a sign change in the second term.

Since self-dual right peaks with an odd first column are obtained by inserting a $>$ -shaped heap to the right of a self-dual alternating heap of type B with an odd first column, which gives:

$$\mathcal{RP}_{\text{odd}}(x) = \frac{xq^2}{1-xq^2} \left(\mathcal{B}_{\text{odd}}^{(a)} - xq - x\mathcal{B}_{\text{odd}}^{(a)} \right),$$

which should be compared to (67).

- Finally, for self-dual heaps arising from left-right peaks, we obtain the series $q\mathcal{LRP}(x)$, where $\mathcal{LRP}(x)$ is given by (74).

Adding all contributions, and injecting the expressions (17), (41) and (42) of \mathcal{A} , $\mathcal{B}^{(a)}$ and $\mathcal{B}_{\text{odd}}^{(a)}$, gives an expression of the form (72) for the series

$$\sum_{n \geq 2} \tilde{\mathcal{B}}_{n+1}^{FC}(q)x^{n+1} = \sum_{n \geq 3} \tilde{\mathcal{B}}_n^{FC}(q)x^n,$$

for the values of $\mathcal{R}_0, \mathcal{L}_0, \mathcal{R}, \mathcal{R}_1, \mathcal{R}_2$ and \mathcal{S} listed in the theorem. The value of \mathcal{R}_3 only differs from \mathcal{R}_3^B by the coefficients of x^1 and x^2 .

Let us finally address the \tilde{D} -case. As in the proof of Theorem 7.1, we use the transformation $\Delta_{t,u}$ acting on fc heaps H of type \tilde{C}_n . Again, the description of self-dual heaps found in $\Delta_{t,u}(H)$ is simple. If H is not self-dual, then no heap of $\Delta_{t,u}(H)$ will be. If H is self-dual, then all heaps of $\Delta_{t,u}(H)$ are self-dual, with the following exceptions:

- (i) if H is the special left peak $us_{n-1} \cdots s_1 t s_1 \cdots s_{n-1} u$ or the special right peak $t s_1 \cdots s_{n-1} u s_{n-1} \cdots s_1 t$, then exactly three elements of $\Delta_{t,u}(H)$ are self-dual: two of size $|H| + 1$, one of size $|H| + 3$;
- (ii) if H is either alternating, or a non-special left peak, or a non-special right peak *with a positive even number of points in its first or last column*, then no heap of $\Delta_{t,u}(H)$ is self-dual.

In particular, if H is alternating, the generating function of self-dual elements in $\Delta_{t,u}(H)$ is $q^{|H|} a_t(H) a_u(H)$, where a_t is defined by (75), and a_u is defined symmetrically.

We now count self-dual heaps of $\Delta_{t,u}(X)$, for each of the five families X of self-dual fc heaps of type \tilde{C}_n listed above.

- For heaps arising from self-dual alternating heaps of \tilde{C}_n , a careful case by case study involving the parity of the first and last columns gives the generating function as:

$$4\mathcal{L}_{\text{odd}}^{\text{odd}}(x) + x^2 \mathcal{A}(x) + 2(2+q)x(1-x)\mathcal{A}(x) + 4x(\mathcal{B}_{\text{odd}}^{(a)}(x) - (1-x)\mathcal{A}(x)) \\ + (4q+q^2)(1-x)^2 \mathcal{A}(x) + 4q((1-x)\mathcal{B}_{\text{odd}}^{(a)}(x) - (1-x)^2 \mathcal{A}(x)) + \text{Pol}_2(x),$$

where $\text{Pol}_2(x)$ is a polynomial in x of degree at most 2. The first term corresponds to the generic case of a heap with odd first and right columns. The second (resp. third, fourth, fifth, sixth) term corrects or completes this contribution for heaps H such that $\{|H|_t, |H|_u\} = \{0\}$ (resp. $\{0, 1\}$, $\{0, i\}$, $\{1\}$, $\{1, i\}$), with $i \geq 3$ odd.

- The generating function of self-dual heaps arising from zigzags is given at the end of the proof of [4, Prop. 3.4] and reads

$$\sum_{n \geq 2} x^{n+1} \left(\frac{2q^{2n+6}}{1-q^2} + \frac{2q^{4n+4}}{1-q^{2n+2}} \right) = 2 \sum_{n \geq 3} x^n \frac{q^{4n}}{1-q^{2n}} + 2 \frac{x^3 q^{10}}{(1-xq^2)(1-q^2)}.$$

- If H is a self-dual right peak, the self-dual elements of $\Delta_{t,u}(H)$ are obtained from self-dual elements of $\Delta_t(H)$ by replacing their only occurrence of u by $u_1 u_2$. Hence the generating function of heaps of \tilde{D}_{n+2} arising from a self-dual right peak of \tilde{C}_n is $q\mathcal{R}\mathcal{P}_{\Delta_t}$, where $\mathcal{R}\mathcal{P}_{\Delta_t}$ is given by (76).
- The case of self-dual left peaks is symmetric, and gives another term $q\mathcal{R}\mathcal{P}_{\Delta_t}$.
- Finally, each self-dual left-right peak H of \tilde{C}_n gives rise to one fc involution of \tilde{D}_{n+2} , of length $|H| + 2$. The associated generating function is thus $q^2 \mathcal{L}\mathcal{R}\mathcal{P}(x)$, with $\mathcal{L}\mathcal{R}\mathcal{P}(x)$ given by (74).

Adding all contributions, and injecting the expressions (17) and (42) of \mathcal{A} and $\mathcal{B}_{\text{odd}}^{(a)}$, gives an expression of the form (72) for the series

$$\sum_{n \geq 2} \tilde{\mathcal{D}}_{n+2}^{FC}(q) x^{n+1} = \sum_{n \geq 3} \tilde{\mathcal{D}}_{n+1}^{FC}(q) x^n,$$

with the values of $\mathcal{L}_0, \mathcal{R}_0, \mathcal{R}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{S}$ given in the theorem. As before, the value of \mathcal{R}_3 only differs from \mathcal{R}_3^D by a polynomial in x of degree 2. ■

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R. BIAGIOLI, F. JOUHET, AND P. NADEAU: UNIV LYON, UNIVERSITÉ CLAUDE BERNARD LYON 1, CNRS
UMR 5208, INSTITUT CAMILLE JORDAN, F-69622 VILLEURBANNE CEDEX, FRANCE
E-mail address: `biagioli, jouhet, nadeau@math.univ-lyon1.fr`

M. BOUSQUET-MÉLOU: CNRS, LABRI, UNIVERSITÉ DE BORDEAUX, 351 COURS DE LA LIBÉRATION, 33405
TALENCE, FRANCE
E-mail address: `mireille.bousquet@labri.fr`