New Finite Rogers-Ramanujan Identities

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Abstract. We present two general finite extensions for each of the two Rogers-Ramanujan identities. Of these one can be derived directly from Watson's transformation formula by specialization or through Bailey's method, the second similar formula can be proved either by using the first formula and the q-Gosper algorithm, or through the so-called Bailey lattice.

Keywords: Rogers-Ramanujan identities, Watson's transformation, Bailey chain, Bailey lattice

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1 Introduction

The famous Rogers-Ramanujan identities (see [4]) may be stated as follows:

$$1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \qquad (1.1)$$

$$1 + \sum_{k=1}^{\infty} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \qquad (1.2)$$

Throughout this paper we suppose that q is a complex number such that $0 < |q| < 1$. For any integer n define the *q-shifted factorial* $(a)_n$ by $(a)_0 = 1$ and

$$(a)_n = (a;q)_n := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1, 2, \ldots, \\ \left[(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^n)\right]^{-1}, & n = -1, -2, \ldots. \end{cases}$$

Note that $\frac{1}{(q)_n} = 0$ if $n < 0$. For $m \geq 1$ we will also use the compact notations:

$$(a_1, \ldots, a_m)_n := (a_1)_n \cdots (a_m)_n, \qquad (a_1, \ldots, a_m)_\infty := \lim_{n \to \infty} (a_1, \ldots, a_m)_n.$$

The following finite forms of the Rogers-Ramanujan identities were proposed by Andrews [2] (see also [3, 9]) as a shortcut to the Rogers-Ramanujan identities:

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k(q)_{n-k}} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-k)/2}}{(q)_{n-k}(q)_{n+k}}, \tag{1.3}$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q)_k(q)_{n-k}} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-3k)/2}}{(q)_{n-k}(q)_{n+k}}, \tag{1.4}$$

where n is a nonnegative integer. These identities were first proved by Watson [22] from his q-analogue of Whipple's transformation formula and were first stated in this form by Andrews. Later Ekhad and Tre [11] and Paule [16] also used (1.3) as the starting point of the computer proofs of the first Rogers-Ramanujan identity (1.1).

In the literature there are many other finite forms of the Rogers-Ramanujan identities due to Schur [17], Watson [22], Andrews [2], Bressoud [10], and Warnaar [20]. We refer the reader to Sills [19] and Berkovich and Warnaar [8] for recent developments of this subject.

The Bailey lemma [7] provides one of the most effective methods for proving q-series identities of Rogers-Ramanujan type. The main purpose of this paper is to study further generalizations of Andrews' identities (1.3) and (1.4) by using Bailey's lemma, Bailey's chain and lattice, though we provide also alternative proofs.

First of all it is easy to derive the following general identities from Watson's classical q-Whipple transformation [12, (III.18)].

Theorem 1.1. *If at least one of a, b and c is of the form q^n, $n = 1, 2, \ldots$, then the following identities hold*

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d, e)_k} (abcdeq^{-3})^k$$
$$= \frac{(q, ab/q, bc/q, ac/q)_\infty}{(a, b, c, abc/q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de/q)_k}{(q, q^3/abc, d, e)_k} q^k, \tag{1.5}$$

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(aq, bq, cq, dq, eq)_k} (abcdeq^{-1})^k$$
$$= \frac{(q, ab, bc, ac)_\infty}{(aq, bq, cq, abc/q)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de)_k}{(q, q^2/abc, dq, eq)_k} q^k. \tag{1.6}$$

Indeed, by collecting the terms indexed by k and $-k$ (resp. $-k-1$) together, we see that (1.5) (resp. (1.6)) is in fact the limiting case $a \to 1$ (resp. $a \to q$) of Watson's classical q-Whipple transformation [12, (III.18)] up to some obvious parameter replacements.

Note that, letting $d, e \to 0$ in (1.5) and (1.6), we obtain two simpler identities.

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Corollary 1.2. *We have*

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c)_k}{(a, b, c)_k}(abc)^k q^{k^2-2k} = \frac{(q, ab/q, bc/q, ac/q)_\infty}{(a, b, c, abc/q^2)_\infty}\sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c)_k}{(q, q^3/abc)_k}q^k, \quad (1.7)$$

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c)_k}{(aq, bq, cq)_k}(abc)^k q^{k^2} = \frac{(q, ab, bc, ac)_\infty}{(aq, bq, cq, abc/q)_\infty}\sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c)_k}{(q, q^2/abc)_k}q^k, \quad (1.8)$$

provided that at least one of a, b and c is of the form q^n, $n = 1, 2, \ldots$.

In a recent paper [14, (7.16) and (7.24)] Liu stated (1.7) and (1.8) *incorrectly* as non-terminating series. For example, if $bc = q$, the left-hand side of (1.7) becomes

$$\sum_{k=-\infty}^{\infty} \frac{(q/a)_k}{(a)_k}a^k q^{k^2-k} = \frac{(q)_\infty}{(a)_\infty},$$

while the right-hand side of (1.7) is equal to 0 (since $ab/q = 1$). Similarly, if $bc = 1$, the left-hand side of (1.8) becomes

$$\sum_{k=-\infty}^{\infty} \frac{(q/a)_k}{(aq)_k}a^k q^{k^2} = \frac{(q)_\infty}{(aq)_\infty},$$

while the right-hand side of (1.8) is equal to 0. Thus (1.7) and (1.8) do not hold for non-terminating series. Therefore Theorems 13 and 14 in [14] are invalid too. A valid version of the latter theorems will be given in this paper within the framework of Bailey chain.

Since the identities (1.5) and (1.6) are between two terminating series, setting $a = q^{n+1}$, $b = q^{l+1}$, $c = q^{m+1}$, $d = q^{u+1}$ and $e = q^{v+1}$, we can rewrite them, respectively, as follows:

$$\sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{l+m+n-k}(q)_{u+v+k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}(q)_{v+k}}$$

$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-k)/2}(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_u(q)_v(q)_{u+v}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{v-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k}(q)_{v+k}}, \quad (1.9)$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}(q)_{l+m+n-k+1}(q)_{u+v+k+1}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k+1}(q)_{v+k+1}}$$

$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2+3k)/2}(q)_{l+m+1}(q)_{m+n+1}(q)_{l+n+1}(q)_u(q)_v(q)_{u+v+1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{v-k}(q)_{l+k+1}(q)_{m+k+1}(q)_{n+k+1}(q)_{u+k+1}(q)_{v+k+1}},$$
$$(1.10)$$

where l, m, n, u, v are nonnegative integers.

Starting from the identities (1.5) and (1.6), we can derive the following new identities, which were originally found through computer experiments.

Theorem 1.3. *If at least one of a, b and c is of the form q^n, $n = 1, 2, \ldots$, then the following identities hold*

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d/q, e/q)_k} (abcdeq^{-3})^k$$
$$= \frac{(q, ab/q, bc/q, ac/q)_\infty}{(a, b, c, abc/q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de/q^2)_k}{(q, q^3/abc, d, e)_k} q^k, \tag{1.11}$$

and

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d/q, e/q)_k} (abcdeq^{-4})^k$$
$$= \frac{(q, ab/q, bc/q, ac/q)_\infty}{(a, b, c, abc/q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de/q^2)_k}{(q, q^3/abc, d, e)_k} q^{2k}. \tag{1.12}$$

Setting $a = q^{n+1}$, $b = q^{l+1}$, $c = q^{m+1}$, $d = q^{u+1}$ and $e = q^{v+1}$, where $l, m, n, u, v \in \mathbb{N}$, the above two identities can be written, respectively, as follows:

$$\sum_{k=0}^{\infty} \frac{q^{k^2} (q)_{l+m+n-k} (q)_{u+v+k-1}}{(q)_k (q)_{l-k} (q)_{m-k} (q)_{n-k} (q)_{u+k} (q)_{v+k}}$$
$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-k)/2} (q)_{l+m} (q)_{l+n} (q)_{m+n} (q)_{u-1} (q)_{v-1} (q)_{u+v-1}}{(q)_{l-k} (q)_{m-k} (q)_{n-k} (q)_{u-k} (q)_{v-k} (q)_{l+k} (q)_{m+k} (q)_{n+k} (q)_{u+k-1} (q)_{v+k-1}}, \tag{1.13}$$

and

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k} (q)_{l+m+n-k} (q)_{u+v+k-1}}{(q)_k (q)_{l-k} (q)_{m-k} (q)_{n-k} (q)_{u+k} (q)_{v+k}}$$
$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-3k)/2} (q)_{l+m} (q)_{l+n} (q)_{m+n} (q)_{u-1} (q)_{v-1} (q)_{u+v-1}}{(q)_{l-k} (q)_{m-k} (q)_{n-k} (q)_{u-k} (q)_{v-k} (q)_{l+k} (q)_{m+k} (q)_{n+k} (q)_{u+k-1} (q)_{v+k-1}}. \tag{1.14}$$

It is clear that letting $l, m, u, v \to \infty$ in (1.9) and (1.13) (resp. (1.10) and (1.14)) we obtain (1.3) (resp. (1.4)).

In Section 2 we will show connections between Theorems 1.1 and 1.3 and the Bailey chain concept. In particular, we shall give a valid version (cf. Corollaries 2.2 and 2.3 below) of Liu's Theorems 13 and 14 by iterating the Bailey lemma. Since the left-hand sides of (1.11) and (1.12) are *not well-poised*, a modified version of Bailey's lemma, which is called the Bailey lattice, is needed to prove (1.11) while (1.12) is simply a combination of the previous ones. We will give two proofs of (1.11) in Section 3 and the proof of (1.12) in Section 4. Finally we derive some interesting special cases of Theorems 1.1 and 1.3 in Section 5.

2   Bailey's lemma revisited

The following theorem can be found in [6, Chapter 12] (see also [5, 15]). Its original form is due to Bailey [7]. The full power of the Bailey lemma was discovered by Andrews when he observed the iterative nature of the lemma, leading to the Bailey chain.

Theorem 2.1 (Bailey's lemma). *If (α_n, β_n) is a Bailey pair with parameter x, i.e., they are related by*

$$\beta_n = \sum_{r \geq 0} \frac{\alpha_r}{(q)_{n-r}(xq)_{n+r}}, \qquad \forall\, n \geq 0, \tag{2.1}$$

then (α_n', β_n') is also a Bailey pair with parameter x, where

$$\alpha_n' = \frac{(\rho_1, \rho_2)_n (xq/\rho_1\rho_2)^n}{(xq/\rho_1)_k (xq/\rho_2)_n} \alpha_n,$$

$$\beta_n' = \sum_{r \geq 0} \frac{(\rho_1, \rho_2)_r (xq/\rho_1\rho_2)_{n-r} (xq/\rho_1\rho_2)^r}{(q)_{n-r}(xq/\rho_1)_n (xq/\rho_2)_n} \beta_r.$$

Thus we can iterate the above procedure to produce the so-called Bailey chain:

$$(\alpha_n, \beta_n) \longrightarrow (\alpha_n', \beta_n') \longrightarrow (\alpha_n'', \beta_n'') \longrightarrow \cdots$$

When $x = 1$ we can symmetrize Bailey's lemma as follows.

Corollary 2.2. *If two sequences (α_n) and (β_n) are related by*

$$\beta_n = \sum_{r=-\infty}^{\infty} \frac{\alpha_r}{(q)_{n-r}(q)_{n+r}}, \qquad \forall\, n \geq 0,$$

then

$$\sum_{n=-\infty}^{\infty} \frac{(\rho_1, \rho_2, q^{-N})_n}{(q/\rho_1, q/\rho_2, q^{N+1})_n} \left(\frac{q^{1+N}}{\rho_1\rho_2} \right)^n (-1)^n q^{-\binom{n}{2}} \alpha_n$$

$$= \frac{(q, q/\rho_1\rho_2)_N}{(q/\rho_1, q/\rho_2)_N} \sum_{n \geq 0} \frac{(\rho_1, \rho_2, q^{-N})_n q^n \beta_n}{(\rho_1\rho_2 q^{-N})_n}. \tag{2.2}$$

Proof. Note that the identity (2.1) for (α_n', β_n') can be written as

$$\sum_{n \geq 0} \frac{(\rho_1, \rho_2, q^{-N})_n}{(xq/\rho_1, xq/\rho_2, xq^{N+1})_n} \left(\frac{xq^{1+N}}{\rho_1\rho_2} \right)^n (-1)^n q^{-\binom{n}{2}} \alpha_n$$

$$= \frac{(xq, xq/\rho_1\rho_2)_N}{(xq/\rho_1, xq/\rho_2)_N} \sum_{n \geq 0} \frac{(\rho_1, \rho_2, q^{-N})_n q^n \beta_n}{(\rho_1\rho_2 q^{-N}/x)_n}. \tag{2.3}$$

So, setting $x = 1$ and replacing α_n by $\alpha_n + \alpha_{-n}$ for $n \geq 1$ in (2.3) yields (2.2). ∎

Remark. Set $A_n = q^n(q)_{2n}\beta_n$, $B_n = (-1)^n q^{-\binom{n}{2}}\alpha_n$, $\rho_1 = q/b$ and $\rho_2 = q/c$ in the above corollary we obtain a valid version of Theorem 13 in [14] with $a = q^{1+N}$.

Let $\alpha_0 = \beta_0 = 1$ and

$$\alpha_n = (-1)^n (q^{\binom{n}{2}} + q^{\binom{-n}{2}}), \qquad \beta_n = 0, \qquad n \geq 1.$$

Then the q-binomial formula implies that (α_n, β_n) is a Bailey pair with parameter $x = 1$. Namely,

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}}}{(q)_{n-r}(q)_{n+r}} = \delta_{n,0}.$$

Moving to the right in the Bailey chain we get

$$\alpha'_n = \frac{(\rho_1, \rho_2)_n (q/\rho_1\rho_2)^n}{(q/\rho_1, q/\rho_2)_n}(-1)^n \left(q^{\binom{n}{2}} + q^{\binom{-n}{2}}\right),$$

$$\beta'_N = \frac{(q/\rho_1\rho_2)_N}{(q, q/\rho_1, q/\rho_2)_N}.$$

The corresponding identity (2.1) reads

$$\sum_{n=-\infty}^{\infty} \frac{(\rho_1, \rho_2)_n (q/\rho_1\rho_2)^n (-1)^n q^{\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}(q/\rho_1, q/\rho_2)_n} = \frac{(q/\rho_1\rho_2)_N}{(q, q/\rho_1, q/\rho_2)_N}.$$

Now the next Bailey pair is

$$\alpha''_n = \frac{(\rho_1, \rho_2, \rho_3, \rho_4)_n}{(q/\rho_1, q/\rho_2, q/\rho_3, q/\rho_4)_n}\left(\frac{q^2}{\rho_1\rho_2\rho_3\rho_4}\right)^n (-1)^n \left(q^{\binom{n}{2}} + q^{\binom{-n}{2}}\right),$$

$$\beta''_N = \sum_{n \geq 0} \frac{(q/\rho_3\rho_4)_{N-n}(\rho_3, \rho_4, q/\rho_1\rho_2)_n}{(q)_{N-n}(q/\rho_3, q/\rho_4)_N(q, q/\rho_1, q/\rho_2)_n}\left(\frac{q}{\rho_3\rho_4}\right)^n.$$

The corresponding identity (2.1) is nothing else than (1.5) with $a = q^{1+N}$ by the following substitution:

$$\rho_1 = q/b, \ \rho_2 = q/c, \ \rho_3 = q/d, \ \rho_4 = q/e.$$

Similarly, when $x = q$, we can symmetrize Bailey's lemma as follows.

Corollary 2.3. *If two sequences (α_n) and (β_n) are related by*

$$\beta_n = \sum_{r=-\infty}^{\infty} \frac{\alpha_r}{(q)_{n-r}(q)_{n+r+1}}, \qquad \forall\, n \geq 0,$$

then

$$\sum_{n=-\infty}^{\infty} \frac{(\rho_1, \rho_2, q^{-N})_n}{(q^2/\rho_1, q^2/\rho_2, q^{N+2})_n}\left(\frac{q^{2+N}}{\rho_1\rho_2}\right)^n (-1)^n q^{-\binom{n}{2}}\frac{\alpha_n}{1-q}$$

$$= \frac{(q^2, q^2/\rho_1\rho_2)_N}{(q^2/\rho_1, q^2/\rho_2)_N}\sum_{n \geq 0} \frac{(\rho_1, \rho_2, q^{-N})_n q^n \beta_n}{(\rho_1\rho_2 q^{-N-1})_n}.$$

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Proof. In Bailey's lemma set $x = q$ and replace α_n by $\frac{1}{1-q}(\alpha_n + \alpha_{-n-1})$. ∎

Remark. Set $A_n = q^n(q)_{2n+1}\beta_n$, $B_n = (-1)^n q^{-\binom{n}{2}}\alpha_n$, $\rho_1 = q/b$ and $\rho_2 = q/c$ in the above corollary we obtain a valid version of Theorem 14 in [14] with $a = q^{1+N}$.

Similarly, let

$$\alpha_n = (-1)^n(q^{\binom{n}{2}} + q^{\binom{-n-1}{2}}), \qquad \beta_n = \delta_{n,0}.$$

Then the q-binomial formula implies that (α_n, β_n) is a Bailey pair with parameter $x = q$, i.e.,

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}}}{(q)_{n-r}(q)_{n+r+1}} = \delta_{n,0}.$$

Moving to the right in the Bailey chain we find

$$\alpha'_n = \frac{(\rho_1, \rho_2)_n (q^2/\rho_1\rho_2)^n}{(q^2/\rho_1, q^2/\rho_2)_n} \frac{(-1)^n}{1-q} \left(q^{\binom{n}{2}} + q^{\binom{-n-1}{2}}\right),$$

$$\beta'_N = \frac{(q^2/\rho_1\rho_2)_N}{(q, q^2/\rho_1, q^2/\rho_2)_N}.$$

The corresponding identity (2.1) reads

$$\sum_{n=-\infty}^{\infty} \frac{(\rho_1, \rho_2)_n (q^2/\rho_1\rho_2)^n (-1)^n q^{\binom{n}{2}}}{(q)_{N-n}(q)_{N+n+1}(q^2/\rho_1, q^2/\rho_2)_n} = \frac{(q^2/\rho_1\rho_2)_N}{(q, q^2/\rho_1, q^2/\rho_2)_N}.$$

Finally the next Bailey pair is

$$\alpha''_n = \frac{(\rho_1, \rho_2, \rho_3, \rho_4)_n}{(q^2/\rho_1, q^2/\rho_2, q^2/\rho_3, q^2/\rho_4)_n} \left(\frac{q^2}{\rho_1\rho_2\rho_3\rho_4}\right)^n \frac{(-1)^n}{1-q} \left(q^{\binom{n}{2}} + q^{\binom{-n-1}{2}}\right),$$

$$\beta''_N = \sum_{n \geq 0} \frac{(q^2/\rho_3\rho_4)_{N-n}(\rho_3, \rho_4, q^2/\rho_1\rho_2)_n}{(q)_{N-n}(q^2/\rho_3, q^2/\rho_4)_N(q, q^2/\rho_1, q^2/\rho_2)_n} \left(\frac{q^2}{\rho_3\rho_4}\right)^n.$$

The corresponding identity (2.1) is nothing else than (1.6) with $a = q^{1+N}$ by the following substitution:

$$\rho_1 = q/b, \ \rho_2 = q/c, \ \rho_3 = q/d, \ \rho_4 = q/e.$$

3   Two proofs of (1.11)

3.1   First Proof of (1.11)

We shall prove the equivalent form (1.13). Writing

$$(1 - q^{M+N+2})(q)_M(q)_N = (q)_M(q)_{N+1} + q^{N+1}(q)_{M+1}(q)_N,$$

with $M = l + m + n - k$ and $N = u + v + k - 1$ we see that the left-hand side of (1.13) multiplied by $(1 - q^{M+N+2})$ is equal to

$$L = \sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{l+m+n-k}(q)_{u+v+k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}(q)_{v+k}} + \sum_{k=0}^{\infty} \frac{q^{k^2+k+u+v}(q)_{l+m+n-k+1}(q)_{u+v+k-1}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}(q)_{v+k}}.$$

Note that the first sum is exactly the left-hand side of (1.9) while the second sum corresponds to the left-hand side of (1.10) with $u \to u - 1$ and $v \to v - 1$. Combining any two terms indexed by k and $-k$ (respectively, $-k - 1$) in the right-hand side of (1.9) (respectively, (1.10)) then yields

$$L = -\frac{(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_{u+v}}{(q)_l^2(q)_m^2(q)_n^2(q)_u(q)_v} + \sum_{k=0}^{\infty} f_k,$$

where

$$f_k = \frac{(-1)^k q^{(5k^2-k)/2}(1+q^k)(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_u(q)_v(q)_{u+v}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{v-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k}(q)_{v+k}}$$
$$+ \frac{(-1)^k q^{(5k^2+3k)/2+u+v}(1-q^{2k+1})(q)_{l+m+1}(q)_{m+n+1}(q)_{l+n+1}(q)_{u-1}(q)_{v-1}(q)_{u+v-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k-1}(q)_{v-k-1}(q)_{l+k+1}(q)_{m+k+1}(q)_{n+k+1}(q)_{u+k}(q)_{v+k}}.$$

On the other hand, combining the terms indexed with k and $-k$ in the right-hand side of (1.13) multiplied by $(1 - q^{M+N+2})$ yields

$$R = -\frac{(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_{u+v-1}(1-q^{l+m+n+u+v+1})}{(q)_l^2(q)_m^2(q)_n^2(q)_{u-1}(q)_{v-1}} + \sum_{k=0}^{\infty} g_k,$$

where

$$g_k = \frac{(-1)^k q^{(5k^2-k)/2}(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_{u-1}(q)_{v-1}(q)_{u+v-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{v-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k-1}(q)_{v+k-1}}$$
$$\times (1-q^{l+m+n+u+v+1})\left(1 + \frac{q^k(q)_{u-k}(q)_{v-k}}{(q)_{u+k}(q)_{v+k}}\right).$$

Now applying the q-Gosper's algorithm we see that

$$f_k - g_k = F(k+1) - F(k), \tag{3.1}$$

where

$$F(k) = \frac{(-1)^k q^{(5k^2-3k)/2+u+v}(1-q^{l+m+n+k+1})(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_{u-1}(q)_{v-1}(q)_{u+v-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{v-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k-1}(q)_{v+k-1}}.$$

Summing (3.1) over $k \geq 0$ yields $L = R$. ∎

Remark. Since

$$\frac{q^{(5k^2-k)/2}}{(q)_{v+k-1}} = \frac{q^{(5k^2-k)/2}}{(q)_{v+k}} - \frac{q^{(5k^2+k)/2}}{(q)_{v+k}} q^v,$$

by (1.9) we have

$$\sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{l+m+n-k}(q)_{u+v+k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}(q)_{v+k}}$$

$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-k)/2}(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_u(q)_{v-1}(q)_{u+v}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{v-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k}(q)_{v+k-1}}.$$

3.2 Proof of (1.11) through the Bailey lattice

Now it is possible to prove (1.11) in the context of Bailey pairs, but one needs a modified version of Bailey's lemma, which allows to change the parameter x between two iterations of Bailey's lemma. This was defined as the Bailey lattice by Agarwal, Andrews and Bressoud in [1] (see also [18, 21]), where they proved with this tool the full Andrews-Gordon identity:

Theorem 3.1 (Bailey lattice). *If (α_n, β_n) is a Bailey pair with parameter x then (α'_n, β'_n) is a Bailey pair with parameter xq^{-1}, where $\alpha'_0 = 1$ and for $n \geq 1$*

$$\alpha'_n = (1-x)(x/\rho_1\rho_2)^n \frac{(\rho_1, \rho_2)_n}{(x/\rho_1, x/\rho_2)_n} \left(\frac{\alpha_n}{1 - xq^{2n}} - xq^{2n-2} \frac{\alpha_{n-1}}{1 - xq^{2n-2}} \right),$$

and for $n \geq 0$

$$\beta'_n = \sum_{r \geq 0} \frac{(\rho_1, \rho_2)_r (x/\rho_1\rho_2)_{n-r} (x/\rho_1\rho_2)^r}{(q)_{n-r}(x/\rho_1)_n(x/\rho_2)_n} \beta_r.$$

Consider for $n \geq 0$

$$\alpha_n = (-1)^n q^{\binom{n}{2}} \frac{1 - q^{2n+1}}{1 - q}, \qquad \beta_n = \delta_{n,0}. \tag{3.2}$$

Then by the q-binomial formula, (α_n, β_n) is a Bailey pair with parameter $x = q$, and we can move to the right in the Bailey chain to get the following Bailey pair with parameter $x = q$:

$$\alpha'_n = \frac{(\rho_1, \rho_2)_n (q^2/\rho_1\rho_2)^n}{(q^2/\rho_1, q^2/\rho_2)_n} (-1)^n q^{\binom{n}{2}} \frac{1 - q^{2n+1}}{1 - q},$$

$$\beta'_N = \frac{(q^2/\rho_1\rho_2)_N}{(q, q^2/\rho_1, q^2/\rho_2)_N}.$$

In the next step, apply Theorem 3.1 to assert that (α''_n, β''_n) is a Bailey pair with parameter $x = 1$, where $\alpha''_0 = 1$ and for $n \geq 1$

$$\alpha''_n = (1-q)(q/\rho_3\rho_4)^n \frac{(\rho_3, \rho_4)_n}{(q/\rho_3, q/\rho_4)_n} \left(\frac{\alpha'_n}{1 - q^{2n+1}} - q^{2n-1} \frac{\alpha'_{n-1}}{1 - q^{2n-1}} \right),$$

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and for $N \geq 0$

$$\beta_N'' = \sum_{n \geq 0} \frac{(\rho_3, \rho_4)_n (q/\rho_3\rho_4)_{N-n}(q/\rho_3\rho_4)^n}{(q)_{N-n}(q/\rho_3)_N(q/\rho_4)_N} \beta_n'.$$

Setting $\rho_1 = q^2/d$, $\rho_2 = q^2/e$, $\rho_3 = q/b$ and $\rho_4 = q/c$ yields

$$\begin{aligned}
\beta_N'' &= \sum_{n \geq 0} \frac{(q/b, q/c)_n (bc/q)_{N-n}(bc/q)^n}{(q)_{N-n}(b,c)_N} \frac{(de/q^2)_n}{(q, d, e)_n} \\
&= \frac{(bc/q)_N}{(q, b, c)_N} \sum_{n \geq 0} \frac{(q^{-N}, q/b, q/c, de/q^2)_n}{(q, d, e, q^{2-N}/bc)_n} q^n.
\end{aligned} \tag{3.3}$$

On the other hand, writing $\beta_N'' = \sum_{n \geq 0} \dfrac{\alpha_n''}{(q)_{N-n}(q)_{N+n}}$, we get

$$\begin{aligned}
\beta_N'' &= \frac{1}{(q,q)_N} \left[1 + \sum_{n \geq 1} \frac{(q^{-N}, q/b, q/c)_n}{(q^{1+N}, b, c)_n}(bcq^{N-1})^n \frac{(q^2/d, q^2/e)_n}{(d, e)_n}(deq^{-2})^n \right] \\
&\quad + \frac{1}{(q,q)_N} \sum_{n \geq 1} \frac{(q^{-N}, q/b, q/c)_n}{(q^{1+N}, b, c)_n}(bcq^{N-1})^n \frac{(q^2/d, q^2/e)_{n-1}}{(d, e)_{n-1}}q(deq^{-1})^{n-1} \\
&= \frac{1}{(q,q)_N} \sum_{n \geq 0} \frac{(q^{-N}, q/b, q/c, q^2/d, q^2/e)_n}{(q^{1+N}, b, c, d, e)_n}(bcdeq^{N-3})^n \\
&\quad + \frac{1}{(q,q)_N} \sum_{n \geq 1} \frac{(q^{-N}, q/b, q/c, q/d, q/e)_n}{(q^{1+N}, b, c, d/q, e/q)_n}(bcdeq^{N-2})^n \\
&= \frac{1}{(q,q)_N} \sum_{n \in \mathbb{Z}} \frac{(q^{-N}, q/b, q/c, q/d, q/e)_n}{(q^{1+N}, b, c, d/q, e/q)_n}(bcdeq^{N-2})^n,
\end{aligned} \tag{3.4}$$

where the last equality follows by replacing n by $-n$ in the first sum. Finally, equating (3.3) and (3.4) yields (1.11), where a is replaced by q^{1+N}.

3.3 Remarks

In [18, Lemma 4.3] Schilling and Warnaar proved another transformation, different but closely related to Theorem 3.1. If we apply the latter formula to the same pair (3.2), then we can derive the following extension of (1.4), different from the previous ones:

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, bq, cq, dq, eq)_k}(abcdeq^{-1})^k \\
= \frac{(q, ab, bc, ac)_\infty}{(a, bq, cq, abc/q)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de)_k}{(q, q^2/abc, dq, eq)_k} q^k,
\end{aligned} \tag{3.5}$$

where at least one of a, b, c is of the form q^n, $n = 1, 2, \ldots$.

However, Eq. (3.5) can be easily deduced from (1.6). This is because the left-hand side of (3.5) can be written as

$$\sum_{k=-\infty}^{\infty} \frac{(1-aq^k)(q/a, q/b, q/c, q/d, q/e)_k}{(1-a)(aq, bq, cq, dq, eq)_k}(abcdeq^{-1})^k,$$

and we have

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(aq, bq, cq, dq, eq)_k}(abcde)^k = 0 \tag{3.6}$$

since

$$\frac{(q/a)_{-k-1}}{(aq)_{-k-1}}a^{-k-1} = \frac{(1/a)_{k+1}}{(a)_{k+1}}a^{k+1} = -\frac{(q/a)_k}{(aq)_k}a^k. \tag{3.7}$$

An identity similar to (3.5) is as follows:

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, bq, cq, dq, eq)_k}(abcde)^k$$

$$= \frac{a(q, ab, bc, ac)_\infty}{q(a, bq, cq, abc/q)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de)_k}{(q, q^2/abc, dq, eq)_k}q^k, \tag{3.8}$$

where at least one of a, b, c is of the form q^n, $n = 1, 2, \ldots$.

Eq.(3.8) is also a consequence of (1.6), because the left-hand side of (3.8) may be written as

$$\sum_{k=-\infty}^{\infty} \frac{(1-aq^k)(q/a, q/b, q/c, q/d, q/e)_k}{(1-a)(aq, bq, cq, dq, eq)_k}(abcde)^k.$$

But we have (3.6) and, by (3.7), Eq. (1.6) may be rewritten as

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(aq, bq, cq, dq, eq)_k}(abcdeq)^k$$

$$= -\frac{(q, ab, bc, ac)_\infty}{q(aq, bq, cq, abc/q)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de)_k}{(q, q^2/abc, dq, eq)_k}q^k.$$

Note that (3.8) is an extension of

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q)_k(q)_{n-k}} = q^{-n} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-5k)/2}}{(q)_{n-k}(q)_{n+k}}.$$

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Similarly to (3.5) and (3.8), we can prove

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(aq, bq, cq, dq, e)_k} (abcdeq^{-1})^k$$

$$= \frac{(q, ab, bc, ac)_\infty}{(1-e)(aq, bq, cq, abc/q)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de)_k}{(q, q^2/abc, dq, eq)_k} q^k, \qquad (3.9)$$

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(aq, bq, cq, dq, e)_k} (abcde)^k$$

$$= \frac{e(q, ab, bc, ac)_\infty}{q(1-e)(aq, bq, cq, abc/q)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de)_k}{(q, q^2/abc, dq, eq)_k} q^k, \qquad (3.10)$$

where at least one of a, b, c is of the form q^n, $n = 1, 2, \ldots$.

Furthermore, writing

$$\frac{1}{(e/q)_k} = \frac{1}{(1-e/q)} \left(\frac{1}{(e)_k} - \frac{1}{(e)_k} eq^{k-1} \right),$$

we have

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d, e/q)_k} (abcdeq^{-3})^k$$

$$= \frac{1}{1-e/q} \sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d, e/q)_k} (abcdeq^{-3})^k$$

$$- \frac{e/q}{1-e/q} \sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d, e/q)_k} (abcdeq^{-2})^k$$

$$= \sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d, e/q)_k} (abcdeq^{-3})^k.$$

The last equality holds because of the following relation

$$\frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d, e/q)_k} (abcdeq^{-3})^k = \frac{(q/a, q/b, q/c, q/d, q/e)_{-k}}{(a, b, c, d, e/q)_{-k}} (abcdeq^{-2})^{-k}.$$

Thus, by (1.5), we have

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a/q, b, c, d, e)_k} (abcdeq^{-3})^k$$

$$= \sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d, e/q)_k} (abcdeq^{-3})^k$$

$$= \frac{(q, ab/q, bc/q, ac/q)_\infty}{(a, b, c, abc/q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q/a, q/b, q/c, de/q)_k}{(q, q^3/abc, d, e)_k} q^k,$$

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where at least one of a, b, c is of the form q^n, $n = 1, 2, \ldots$.

Finally, setting $a = q^{n+1}$, $b = q^{l+1}$, $c = q^{m+1}$, $d = q^{u+1}$ and $e = q^{v+1}$, we can rewrite the last identity as follows:

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-k)/2} (q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_u(q)_{v-1}(q)_{u+v}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{v-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k}(q)_{v+k-1}}$$
$$= \sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{l+m+n-k}(q)_{u+v+k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}(q)_{v+k}}.$$

4 Proof of (1.12)

We prove the equivalent form (1.14). Writing $(q)_{l+m+n-k} = (q)_{l+m+n-k+1} + (q)_{l+m+n-k}q^{l+m+n-k+1}$, the left-hand side of (1.14) is equal to

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}(q)_{l+m+n-k+1}(q)_{u+v+k-1}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}(q)_{v+k}} + \sum_{k=0}^{\infty} \frac{q^{k^2+l+m+n+1}(q)_{l+m+n-k}(q)_{u+v+k-1}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}(q)_{v+k}}.$$

Note that the first sum is equal to the left-hand side of (1.10) with $u \to u-1$ and $v \to v-1$, while the second sum corresponds to the left-hand side of (1.13). Now substituting the above sums by the corresponding right-hand sides of (1.10) and (1.13) with $k \to -k$ in the second sum, and then putting any two terms indexed by k and $-k-1$ together, the left-hand side of (1.14) may be written as $\sum_{k=0}^{\infty} S_k$, where

$$S_k = \frac{(-1)^k q^{(5k^2+3k)/2}(1-q^{2k+1})(q)_{l+m+1}(q)_{m+n+1}(q)_{l+n+1}(q)_{u-1}(q)_{v-1}(q)_{u+v-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k-1}(q)_{v-k-1}(q)_{l+k+1}(q)_{m+k+1}(q)_{n+k+1}(q)_{u+k}(q)_{v+k}}$$
$$+ \frac{(-1)^k q^{(5k^2+k)/2+l+m+n+1}(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_{u-1}(q)_{v-1}(q)_{u+v-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k-1}(q)_{v-k-1}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k}(q)_{v+k}}$$
$$\times \left(1 - \frac{q^{4k+2}(1-q^{l-k})(1-q^{m-k})(1-q^{n-k})}{(1-q^{l+k+1})(1-q^{m+k+1})(1-q^{n+k+1})} \right), \tag{4.1}$$

while the right-hand side of (1.14) may be written as $\sum_{k=0}^{\infty} T_k$, where

$$T_k = \frac{(-1)^k q^{(5k^2+3k)/2}(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_{u-1}(q)_{v-1}(q)_{u+v-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k-1}(q)_{v-k-1}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k}(q)_{v+k}}$$
$$\times \left(1 - \frac{q^{2k+1}(1-q^{l-k})(1-q^{m-k})(1-q^{n-k})}{(1-q^{l+k+1})(1-q^{m+k+1})(1-q^{n+k+1})} \right). \tag{4.2}$$

It is easy to see that $S_k = T_k$ $(k \geq 0)$, which is equivalent to

$$(1-ab)(1-bc)(1-ac)(1-d^2) + d^2(1-a/d)(1-b/d)(1-c/d)(1-abcd)$$
$$= (1-ad)(1-bd)(1-cd)(1-abc/d),$$

where $a = q^{l+\frac{1}{2}}$, $b = q^{m+\frac{1}{2}}$, $c = q^{n+\frac{1}{2}}$ and $d = q^{k+\frac{1}{2}}$. Therefore $\sum_{k \geq 0} S_k = \sum_{k \geq 0} T_k$. This completes the proof. ∎

5   Some special cases

In this section we give some interesting special cases and limiting cases of Theorems 1.1 and 1.3. Letting $e \to 0$ in (1.5) and (1.6), we obtain

Corollary 5.1. *For $|bc| < |q|$, the following identity holds*

$$\sum_{k=-\infty}^{\infty} \frac{(q/a, q/b, q/c, q/d)_k}{(a, b, c, d)_k} (-abcd)^k q^{(k^2-5k)/2} = \frac{(q, bc/q)_\infty}{(b, c)_\infty} \sum_{k=0}^{\infty} \frac{(q/b, q/c, ad/q)_k}{(q, d, a)_k} \left(\frac{bc}{q}\right)^k,$$
(5.1)

and for $|bc| < 1$ there holds

$$\sum_{k=-\infty}^{\infty} \frac{(q/b, q/c, q/d, q/e)_k}{(bq, cq, dq, eq)_k} (-bcde)^k q^{(k^2-k)/2} = \frac{(q, bc)_\infty}{(bq, cq)_\infty} \sum_{k=0}^{\infty} \frac{(q/b, q/c, de)_k}{(q, dq, eq)_k} (bc)^k, \quad (5.2)$$

provided that at least one of a, b and c is of the form q^n, $n = 1, 2, \ldots$.

Letting $a = q^l$ and $l \to \infty$ in (1.11) and (1.12), we obtain

Corollary 5.2. *For $|bc| < |q|$, there holds*

$$\sum_{k=-\infty}^{\infty} \frac{(q/b, q/c, q/d, q/e)_k}{(b, c, d/q, e/q)_k} (-bcde)^k q^{(k^2-5k)/2} = \frac{(q, bc/q)_\infty}{(b, c)_\infty} \sum_{n=0}^{\infty} \frac{(q/b, q/c, de/q^2)_k}{(q, d, e)_k} \left(\frac{bc}{q}\right)^k,$$

and for $|bc| < 1$ there holds

$$\sum_{k=-\infty}^{\infty} \frac{(q/b, q/c, q/d, q/e)_k}{(b, c, d/q, e/q)_k} (-bcde)^k q^{(k^2-7k)/2} = \frac{(q, bc/q)_\infty}{(b, c)_\infty} \sum_{n=0}^{\infty} \frac{(q/b, q/c, de/q^2)_k}{(q, d, e)_k} (bc)^k.$$

Letting $v \to \infty$ in (1.9)–(1.14) we obtain the following two corollaries.

Corollary 5.3. *For $l, m, n, u \in \mathbb{N}$, there holds*

$$\sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{l+m+n-k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}}$$
$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-k)/2}(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_u}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k}},$$
(5.3)

and

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}(q)_{l+m+n-k+1}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}}$$
$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-3k)/2}(q)_{l+m+1}(q)_{l+n+1}(q)_{m+n+1}(q)_u}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{l+k+1}(q)_{m+k+1}(q)_{n+k+1}(q)_{u+k+1}}.$$
(5.4)

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Corollary 5.4. *For $l, m, n, u \in \mathbb{N}$, there holds*

$$\sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{l+m+n-k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}}$$
$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-k)/2}(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_{u-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k-1}}, \tag{5.5}$$

and

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}(q)_{l+m+n-k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}}$$
$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(5k^2-3k)/2}(q)_{l+m}(q)_{m+n}(q)_{l+n}(q)_{u-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k-1}}. \tag{5.6}$$

Replacing q by q^{-1} in (5.3)–(5.6) and noticing that $(q^{-1}; q^{-1})_m = (-1)^m q^{-\binom{m+1}{2}}(q; q)_m$, we get the following results:

Corollary 5.5. *For $l, m, n, u \in \mathbb{N}$, there holds*

$$\sum_{k=0}^{n} \frac{q^{k^2+uk}(q)_{l+m+n-k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}}$$
$$= \sum_{k=-n}^{n} \frac{(-1)^k q^{(3k^2-k)/2}(q)_{l+m}(q)_{m+n}(q)_{l+n}(q)_u}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k}},$$

and

$$\sum_{k=0}^{n} \frac{q^{k^2+(u+1)k}(q)_{l+m+n-k+1}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k+1}}$$
$$= \sum_{k=-n-1}^{n} \frac{(-1)^k q^{(3k^2+k)/2}(q)_{l+m+1}(q)_{m+n+1}(q)_{l+n+1}(q)_u}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{l+k+1}(q)_{m+k+1}(q)_{n+k+1}(q)_{u+k+1}}.$$

Corollary 5.6. *For $l, m, n, u \in \mathbb{N}$, there holds*

$$\sum_{k=0}^{\infty} \frac{q^{k^2+uk}(q)_{l+m+n-k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}}$$
$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(3k^2-k)/2}(q)_{l+m}(q)_{l+n}(q)_{m+n}(q)_{u-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k-1}},$$

and

$$\sum_{k=0}^{\infty} \frac{q^{k^2+(u-1)k}(q)_{l+m+n-k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}}$$
$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(3k^2+k)/2}(q)_{l+m}(q)_{m+n}(q)_{l+n}(q)_{u-1}}{(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u-k}(q)_{l+k}(q)_{m+k}(q)_{n+k}(q)_{u+k-1}}.$$

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Observing the symmetry of the denominators in the right-hand sides of (1.9) and (1.10), we obtain

Corollary 5.7. *For $l, m, n, u, v \in \mathbb{N}$, there holds*

$$\frac{1}{(q)_{l+m}(q)_{l+n}(q)_u(q)_v} \sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{l+m+n-k}(q)_{u+v+k}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k}(q)_{v+k}}$$

$$= \frac{1}{(q)_{l+u}(q)_{l+v}(q)_m(q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{l+u+v-k}(q)_{m+n+k}}{(q)_k(q)_{l-k}(q)_{u-k}(q)_{v-k}(q)_{m+k}(q)_{n+k}}, \qquad (5.7)$$

and

$$\frac{1}{(q)_{l+m+1}(q)_{l+n+1}(q)_u(q)_v} \sum_{k=0}^{\infty} \frac{q^{k^2+k}(q)_{l+m+n-k+1}(q)_{u+v+k+1}}{(q)_k(q)_{l-k}(q)_{m-k}(q)_{n-k}(q)_{u+k+1}(q)_{v+k+1}}$$

$$= \frac{1}{(q)_{l+u+1}(q)_{l+v+1}(q)_m(q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{l+u+v-k+1}(q)_{m+n+k+1}}{(q)_k(q)_{l-k}(q)_{u-k}(q)_{v-k}(q)_{m+k+1}(q)_{n+k+1}}. \qquad (5.8)$$

Letting $l, u, v \to \infty$ in (5.7) and (5.8), we get

$$\frac{1}{(q)_{\infty}} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k(q)_{n-k}(q)_{m-k}} = \frac{1}{(q)_m(q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2}(q)_{m+n+k}}{(q)_k(q)_{m+k}(q)_{n+k}}, \qquad (5.9)$$

$$\frac{1}{(q)_{\infty}} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q)_k(q)_{n-k}(q)_{m-k}} = \frac{1}{(q)_m(q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2+k}(q)_{m+n+k+1}}{(q)_k(q)_{m+k+1}(q)_{n+k+1}}, \qquad (5.10)$$

Furthermore, letting $m \to \infty$ in (5.9) and (5.10), we obtain the following remarkable identities:

$$\frac{1}{(q)_{\infty}} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k(q)_{n-k}} = \frac{1}{(q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k(q)_{n+k}}, \qquad (5.11)$$

$$\frac{1}{(q)_{\infty}} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q)_k(q)_{n-k}} = \frac{1}{(q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q)_k(q)_{n+k+1}}.$$

In what follows we assume that $\frac{1}{n!} = 0$ if $n < 0$. Letting q tend to 1 in (1.9), we obtain

Corollary 5.8. *For $l, m, n, u, v \in \mathbb{N}$, there holds*

$$\sum_{k=-\infty}^{\infty} (-1)^k \binom{l+m}{l+k} \binom{m+n}{m+k} \binom{n+l}{n+k} \binom{u+v}{u+k} \binom{u+v}{v+k}$$

$$= \binom{u+v}{u} \sum_{k=0}^{\infty} \frac{(l+m+n-k)!(u+v+k)!}{k!(l-k)!(m-k)!(n-k)!(u+k)!(v+k)!}.$$

In particular, when $l = m = n = u = v$, we get

$$\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^5 = \binom{2n}{n} \sum_{k=0}^{n} \binom{3n-k}{n-k} \binom{2n+k}{k} \binom{2n}{n+k}^2. \qquad (5.12)$$

Replacing k by $n - k$ in the right-hand side of (5.12) yields

$$\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^5 = \binom{2n}{n} \sum_{k=0}^{n} \binom{3n-k}{n-k} \binom{2n+k}{k}^2 \binom{2n}{k}^2.$$

Similarly, letting q tend to 1 in (5.3), or first letting l tend to infinity in (1.9) and then letting q tend to 1, we obtain the following formulae:

Corollary 5.9. *For $l, m, n, u, v \in \mathbb{N}$, there holds*

$$\sum_{k=-\infty}^{\infty} (-1)^k \binom{l+m}{l+k} \binom{m+n}{m+k} \binom{n+l}{n+k} \binom{2u}{u+k}$$

$$= \frac{(2u)!}{u!} \sum_{k=0}^{\infty} \frac{(l+m+n-k)!}{k!(l-k)!(m-k)!(n-k)!(u+k)!},$$

and

$$\sum_{k=-\infty}^{\infty} (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} \binom{u+v}{u+k} \binom{u+v}{v+k}$$

$$= \binom{u+v}{u} \sum_{k=0}^{\infty} \frac{(m+n)!(u+v+k)!}{k!(m-k)!(n-k)!(u+k)!(v+k)!}.$$

In particular, when $l = m = n = u = v$, we get

$$\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^4 = \binom{2n}{n} \sum_{k=0}^{n} \binom{3n-k}{n-k} \binom{2n}{n+k} \binom{n}{k}$$

$$= \binom{2n}{n} \sum_{k=0}^{n} \binom{2n+k}{k} \binom{2n}{n+k}^2. \qquad (5.13)$$

From (5.12) and (5.13) one sees that $\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^m$ is divisible by $\binom{2n}{n}$ for $m = 4, 5$. In a separate paper [13], we have generalized this result by showing that the alternating sum

$$\sum_{k=-n}^{n} (-1)^k \prod_{i=1}^{m} \binom{n_i + n_{i+1}}{n_i + k} \qquad (n_{m+1} = n_1)$$

is a nonnegative integer divisible by all the binomial coefficients $\binom{n_j + n_{j+1}}{n_j}$ for $1 \leq j \leq m$.

Acknowledgments

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