SOME MORE SEMI-FINITE FORMS OF BILATERAL BASIC HYPERGEOMETRIC SERIES

FRÉDÉRIC JOUHET*

Abstract. We prove some new semi-finite forms of bilateral basic hypergeometric series. One of them yields in a direct limit Bailey's celebrated ${}_6\psi_6$ summation formula, answering a question recently raised by Chen and Fu (*Semi-Finite Forms of Bilateral Basic Hypergeometric Series*, Proc. Amer. Math. Soc., to appear).

1. Introduction

There is a standard method for obtaining a bilateral identity from a *unilateral terminating identity*, which was already utilized by Cauchy [7] in his second proof of Jacobi's [10] famous triple product identity. The idea of this method is to start from a finite unilateral summation and to shift the index of summation, say k $(0 \leq k \leq 2n)$, by n :

$$\sum_{k=0}^{2n} a(k) = \sum_{k=-n}^{n} a(k+n), \tag{1.1}$$

and then let $n \to \infty$ whenever it is possible after some manipulations. The same method has also been exploited by Bailey [4, Secs. 3 and 6], [5], Slater [14, Sec. 6.2], Schlosser [13] and Schlosser and the author [11].

Recently, Chen and Fu [8] used a method different from the previous one, as they started from *unilateral infinite summations* to derive *semi-finite forms of bilateral basic hypergeometric series*. The process can be summarized as follows :

$$\sum_{k\geq 0} a(k) = \sum_{k\geq -n} a(k+n), \tag{1.2}$$

and then let $n \to \infty$ whenever it is possible after some manipulations. The right-hand side of (1.1) (resp. (1.2)) can be seen as a finite (resp. semi-finite) form of a bilateral series. Chen and Fu have found in [8] semi-finite forms of Ramanujan's ${}_1\psi_1$ summation formula (cf. [9, Appendix (II.29)]), of a ${}_2\psi_2$ formula due to Bailey [9, Ex. 5.20(i)], and of Bailey's [4, Eq. (4.7)] ${}_6\psi_6$ summation formula ([9, Appendix (II.33)]), which can be written as follows :

$$\begin{aligned} &{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, \frac{qa^2}{bcde}\right] \\ &\qquad = \frac{(q, aq, q/a, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de)_\infty}{(q/b, q/c, q/d, q/e, aq/b, aq/c, aq/d, aq/e, qa^2/bcde)_\infty} \end{aligned} \tag{1.3}$$

2000 *Mathematics Subject Classification.* 33D15.

Key words and phrases. bilateral basic hypergeometric series, q-series, Bailey's ${}_6\psi_6$ summation.

*Partially supported by EC's IHRP Programme, grant HPRN-CT-2001-00272, "Algebraic Combinatorics in Europe".

(see the end of this introduction for the notations), where $|q| < 1$ and $|qa^2/bcde| < 1$. At the end of [8], Chen and Fu mention the problem of finding a proof of (1.3) using a semi-finite (or even finite) form which would yield (1.3) in a direct limit. Indeed, after letting $n \to \infty$ in their semi-finite form of (1.3), one needs to use Ramanujan's ${}_1\psi_1$ summation formula to derive (1.3).
In this paper, we use the method developped in [8] to find, among other results, a new semi-finite form of (1.3) which answers the question raised by Chen and Fu. After explaining some notations in the end of this introduction, we show in section 2 how the method in [8] can be applied to yield in a direct limit (1.3), starting from a nonterminating extension of Jackson's formula due to Bailey [9, Appendix (II.25)]. We give two other applications of this method in section 3, which yield in a direct limit a transformation formula for a ${}_6\psi_6$ series proved in [11] and a transformation formula for a ${}_8\psi_8$ series in terms of two ${}_8\phi_7$ series and a ${}_8\psi_8$ series.

Other proofs of Bailey's very-well-poised ${}_6\psi_6$ summation had been given by Bailey [4], Slater and Lakin [15], Andrews [1], Askey and Ismail [3], Askey [2], Schlosser [12] and Schlosser and the author [11]. It is worth noting that the elegant proof of Askey and Ismail in [3] uses an argument of *analytic continuation* together with the shift (1.2), but used from right to left.

Notation: It is appropriate to recall some standard notations for *q-series* and *basic hypergeometric series.*

Let q be a fixed complex parameter (the "base") with $0 < |q| < 1$. The *q-shifted factorial* is defined for any complex parameter a by

$$(a)_\infty \equiv (a;q)_\infty := \prod_{j \geq 0} (1 - aq^j)$$

and

$$(a)_k \equiv (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty},$$

where k is any integer. Since the same base q is used throughout this paper, it may be readily omitted (in notation, writing $(a)_k$ instead of $(a;q)_k$, etc) which will not lead to any confusion. For brevity, write

$$(a_1, \ldots, a_m)_k := (a_1)_k \cdots (a_m)_k,$$

where k is an integer or infinity. Further, recall the definition of *basic hypergeometric series*,

$${}_s\phi_{s-1}\left[\begin{matrix} a_1, \ldots, a_s \\ b_1, \ldots, b_{s-1} \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_s)_k}{(q, b_1, \ldots, b_{s-1})_k} z^k,$$

and of *bilateral basic hypergeometric series*,

$${}_s\psi_s\left[\begin{matrix} a_1, \ldots, a_s \\ b_1, \ldots, b_s \end{matrix}; q, z\right] := \sum_{k=-\infty}^{\infty} \frac{(a_1, \ldots, a_s)_k}{(b_1, \ldots, b_s)_k} z^k.$$

See Gasper and Rahman's text [9] for a comprehensive study of the theory of basic hypergeometric series. In particular, the computations in this paper rely on some elementary identities for q-shifted factorials, listed in [9, Appendix I].

2. A new semi-finite form of Bailey's ${}_6\psi_6$ summation formula

Consider Bailey's nonterminating extension of Jackson's ${}_8\phi_7$ summation [9, Appendix (II.25)]

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a,\, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f\end{matrix}; q, q\right]\\
&\quad=\frac{b}{a}\frac{(aq, c, d, e, f, bq/a, bq/c, bq/d, bq/e, bq/f)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a, b^2q/a)_\infty}\\
&\qquad\times {}_8\phi_7\left[\begin{matrix}b^2/a,\, qb/\sqrt{a}, -qb/\sqrt{a}, b, bc/a, bd/a, be/a, bf/a\\ b/\sqrt{a}, -b/\sqrt{a}, bq/a, bq/c, bq/d, bq/e, bq/f\end{matrix}; q, q\right]\\
&\qquad\qquad+\frac{(aq, b/a, aq/cd, aq/ce, aq/cf, aq/de, aq/df, aq/ef)_\infty}{(aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a)_\infty},
\end{aligned}
\tag{2.1}
$$

where $qa^2 = bcdef$.
Note that (2.1) can be proved by specializing $qa^2 = bcdef$ in Bailey's 3-term transformation formula for a nonterminating very-well-poised ${}_8\phi_7$ [9, Appendix (III.37)], which was the starting point in [8] to prove (1.3), and then using the sum of a very-well-poised ${}_6\phi_5$ [9, Appendix (II.20)].
Now, using (2.1), we can derive the following semi-finite form of (1.3).

Proposition 2.1.

$$
\begin{aligned}
&\sum_{k\geq -n}\frac{(aq^{-n}, q\sqrt{a}, -q\sqrt{a}, bq^n, c, d, e, f)_k}{(q^{1+n}, \sqrt{a}, -\sqrt{a}, aq^{1-n}/b, aq/c, aq/d, aq/e, aq/f)_k}q^k\\
&=\frac{(aq, c, d, e, f, bq^{1+2n}/a, bq^{1+n}/c, bq^{1+n}/d, bq^{1+n}/e, bq^{1+n}/f)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, bcq^n/a, bdq^n/a, beq^n/a, bfq^n/a, b^2q^{1+2n}/a)_\infty}\\
&\qquad\times\frac{b^{n+1}}{a}\frac{(q, q/a)_n}{(b, b/a)_n}\\
&\times {}_8\phi_7\left[\begin{matrix}b^2q^{2n}/a,\, bq^{1+n}/\sqrt{a}, -bq^{1+n}/\sqrt{a}, b, bcq^n/a, bdq^n/a, beq^n/a, bfq^n/a\\ bq^n/\sqrt{a}, -bq^n/\sqrt{a}, bq^{1+2n}/a, bq^{1+n}/c, bq^{1+n}/d, bq^{1+n}/e, bq^{1+n}/f\end{matrix}; q, q\right]\\
&\qquad+\frac{(aq, aq/cd, aq/ce, aq/cf, aq/de, aq/df, aq/ef, bq^n/a)_\infty}{(aq/c, aq/d, aq/e, aq/f, bcq^n/a, bdq^n/a, beq^n/a, bfq^n/a)_\infty}\\
&\qquad\qquad\times\frac{(q, q/a)_n}{(b, q/c, q/d, q/e, q/f)_n},
\end{aligned}
\tag{2.2}
$$

where $b = qa^2/cdef$.

Proof. By shifting the index of summation by n, the left-hand side of (2.1) is equal to

$$
\begin{aligned}
&\frac{1-aq^{2n}}{1-a}\frac{(a, b, c, d, e, f)_n}{(q, aq/b, aq/c, aq/d, aq/e, aq/f)_n}q^n\\
&\quad\times\sum_{k\geq -n}\frac{(aq^n, \sqrt{a}q^{1+n}, -\sqrt{a}q^{1+n}, bq^n, cq^n, dq^n, eq^n, fq^n)_k}{(q^{1+n}, \sqrt{a}q^n, -\sqrt{a}q^n, aq/b, aq^{1+n}/c, aq^{1+n}/d, aq^{1+n}/e, aq^{1+n}/f)_k}q^k.
\end{aligned}
$$

Next, on both sides of (2.1), replace a, c, d, e and f by aq^{-2n}, cq^{-n}, dq^{-n}, eq^{-n} and fq^{-n} respectively. Note that the condition $qa^2 = bcdef$ is equivalent to $b = qa^2/cdef$, thus b remains unchanged. We get

$$\frac{1-a}{1-aq^{-2n}}\frac{(aq^{-2n},b,cq^{-n},dq^{-n},eq^{-n},fq^{-n})_n}{(q,aq^{1-2n}/b,aq^{1-n}/c,aq^{1-n}/d,aq^{1-n}/e,aq^{1-n}/f)_n}q^n$$
$$\times\sum_{k\geq -n}\frac{(aq^{-n},q\sqrt{a},-q\sqrt{a},bq^n,c,d,e,f)_k}{(q^{1+n},\sqrt{a},-\sqrt{a},aq^{1-n}/b,aq/c,aq/d,aq/e,aq/f)_k}q^k$$
$$=\frac{b}{aq^{-2n}}\frac{(aq^{1-2n},cq^{-n},dq^{-n},eq^{-n},fq^{-n})_\infty}{(aq^{1-2n}/b,aq^{1-n}/c,aq^{1-n}/d,aq^{1-n}/e,aq^{1-n}/f)_\infty}$$
$$\times\frac{(bq^{1+2n}/a,bq^{1+n}/c,bq^{1+n}/d,bq^{1+n}/e,bq^{1+n}/f)_\infty}{(bcq^n/a,bdq^n/a,beq^n/a,bfq^n/a,b^2q^{1+2n}/a)_\infty}$$
$$\times\,{}_8\phi_7\left[\begin{matrix}b^2q^{2n}/a,\ bq^{1+n}/\sqrt{a},-bq^{1+n}/\sqrt{a},b,bcq^n/a,bdq^n/a,beq^n/a,bfq^n/a\\ bq^n/\sqrt{a},-bq^n/\sqrt{a},bq^{1+2n}/a,bq^{1+n}/c,bq^{1+n}/d,bq^{1+n}/e,bq^{1+n}/f\end{matrix};q,q\right]$$
$$+\frac{(aq^{1-2n},bq^{2n}/a,aq/cd,aq/ce,aq/cf,aq/de,aq/df,aq/ef)_\infty}{(aq^{1-n}/c,aq^{1-n}/d,aq^{1-n}/e,aq^{1-n}/f,bcq^n/a,bdq^n/a,beq^n/a,bfq^n/a)_\infty}.$$

This can be rewritten as

$$\sum_{k\geq -n}\frac{(aq^{-n},q\sqrt{a},-q\sqrt{a},bq^n,c,d,e,f)_k}{(q^{1+n},\sqrt{a},-\sqrt{a},aq^{1-n}/b,aq/c,aq/d,aq/e,aq/f)_k}q^k$$
$$=\frac{1-aq^{-2n}}{1-a}\frac{bq^n}{a}\frac{(q,aq^{1-2n}/b)_n}{(b,aq^{-2n})_n}\frac{(aq^{1-2n})_\infty}{(aq^{1-2n}/b)_\infty}$$
$$\times\frac{(c,d,e,f,bq^{1+2n}/a,bq^{1+n}/c,bq^{1+n}/d,bq^{1+n}/e,bq^{1+n}/f)_\infty}{(aq/c,aq/d,aq/e,aq/f,bcq^n/a,bdq^n/a,beq^n/a,bfq^n/a,b^2q^{1+2n}/a)_\infty}$$
$$\times\,{}_8\phi_7\left[\begin{matrix}b^2q^{2n}/a,\ bq^{1+n}/\sqrt{a},-bq^{1+n}/\sqrt{a},b,bcq^n/a,bdq^n/a,beq^n/a,bfq^n/a\\ bq^n/\sqrt{a},-bq^n/\sqrt{a},bq^{1+2n}/a,bq^{1+n}/c,bq^{1+n}/d,bq^{1+n}/e,bq^{1+n}/f\end{matrix};q,q\right]$$
$$+\frac{1-aq^{-2n}}{1-a}q^{-n}\frac{(q,aq^{1-2n}/b)_n(aq^{1-2n})_\infty}{(aq^{-2n},b,cq^{-n},dq^{-n},eq^{-n},fq^{-n})_n}$$
$$\times\frac{(bq^{2n}/a,aq/cd,aq/ce,aq/cf,aq/de,aq/df,aq/ef)_\infty}{(aq/c,aq/d,aq/e,aq/f,bcq^n/a,bdq^n/a,beq^n/a,bfq^n/a)_\infty}. \quad (2.3)$$

Now we use the three following elementary identities to simplify the right-hand side of (2.3) :

$$\frac{(xq^{-2n})_\infty}{(xq^{-2n})_n} = (-1)^n x^n q^{-(n^2+n)/2}(q/x)_n(x)_\infty, \quad (2.4)$$
$$(xq^{-2n})_n = (-1)^n x^n q^{-(3n^2+n)/2}(q^{n+1}/x)_n, \quad (2.5)$$
$$(xq^{-n})_n = (-1)^n x^n q^{-(n^2+n)/2}(q/x)_n, \quad (2.6)$$

and we obtain (2.2) after simplifications. □

Now, one may let $n\to\infty$ in (2.2), assuming $|qa^2/cdef|<1$ (i.e. $|b|<1$), while appealing to Tannery's theorem [6] for being able to interchange limit and summation. As the first term on the right-hand side of (2.2) tends to 0, this gives immediately Bailey's ${}_6\psi_6$ summation formula (1.3) with b replaced by f.

3. Other consequences

We give in this section two other applications of the previous process. Consider first the following transformation formula for a non terminating very-well-poised

${}_8\phi_7$ series [9, Appendix (III.23)]

$$
\begin{aligned}
{}_8\phi_7&\left[\begin{matrix} a,\, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f\end{matrix}; q, \frac{q^2a^2}{bcdef}\right] \\
&= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f)_\infty}{(aq/e, aq/f, \lambda q/ef, \lambda q)_\infty} \\
&\quad\times {}_8\phi_7\left[\begin{matrix} \lambda,\, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b/a, \lambda c/a, \lambda d/a, e, f \\ \sqrt{\lambda}, -\sqrt{\lambda}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f\end{matrix}; q, \frac{aq}{ef}\right], \qquad (3.1)
\end{aligned}
$$

where $\lambda = qa^2/bcd$, $|q^2a^2/bcdef| < 1$ and $|aq/ef| < 1$.
Note that (3.1) is nothing else but the $n \to \infty$ case of Bailey's [4] transformation formula for a very-well-poised ${}_{10}\phi_9$ series [9, Appendix (III.28)], which was the starting point in [11] for the derivation of (1.3). Now, using (3.1), we can prove the following semi-finite identity.

Proposition 3.1.

$$
\begin{aligned}
&\sum_{k\geq -n} \frac{(aq^{-n}, q\sqrt{a}, -q\sqrt{a}, bq^n, c, d, e, f)_k}{(q^{1+n}, \sqrt{a}, -\sqrt{a}, aq^{1-n}/b, aq/c, aq/d, aq/e, aq/f)_k}\left(\frac{q^2a^2}{bcdef}\right)^k \\
&\quad= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f)_\infty}{(aq/e, aq/f, \lambda q/ef, \lambda q)_\infty}\frac{(\lambda b/a, q/a, aq/\lambda c, aq/\lambda d)_n}{(b, q/\lambda, q/c, q/d)_n} \\
&\quad\times \sum_{k\geq -n} \frac{(\lambda q^{-n}, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda bq^n/a, \lambda c/a, \lambda d/a, e, f)_k}{(q^{1+n}, \sqrt{\lambda}, -\sqrt{\lambda}, aq^{1-n}/b, aq/c, aq/d, \lambda q/e, \lambda q/f)_k}\left(\frac{aq}{ef}\right)^k, \qquad (3.2)
\end{aligned}
$$

where $\lambda = qa^2/bcd$ *and* $|q^2a^2/bcdef| < 1$.

Proof. By shifting the index of summation by n on both sides of (3.1), we get

$$
\begin{aligned}
&\frac{1-aq^{2n}}{1-a}\frac{(a, b, c, d, e, f)_n}{(q, aq/b, aq/c, aq/d, aq/e, aq/f)_n}\left(\frac{q^2a^2}{bcdef}\right)^n \\
&\quad\times \sum_{k\geq -n} \frac{(aq^n, \sqrt{a}q^{1+n}, -\sqrt{a}q^{1+n}, bq^n, cq^n, dq^n)_k}{(q^{1+n}, \sqrt{a}q^n, -\sqrt{a}q^n, aq^{1+n}/b, aq^{1+n}/c, aq^{1+n}/d)_k} \\
&\qquad\times \frac{(eq^n, fq^n)_k}{(aq^{1+n}/e, aq^{1+n}/f)_k}\left(\frac{q^2a^2}{bcdef}\right)^k \\
&= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f)_\infty}{(aq/e, aq/f, \lambda q/ef, \lambda q)_\infty}\frac{1-\lambda q^{2n}}{1-\lambda}\frac{(\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f)_n}{(q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f)_n}\left(\frac{aq}{ef}\right)^n \\
&\quad\times \sum_{k\geq -n} \frac{(\lambda q^n, \sqrt{\lambda}q^{1+n}, -\sqrt{\lambda}q^{1+n}, \lambda bq^n/a, \lambda cq^n/a, \lambda dq^n/a)_k}{(q^{1+n}, \sqrt{\lambda}q^n, -\sqrt{\lambda}q^n, aq^{1+n}/b, aq^{1+n}/c, aq^{1+n}/d)_k} \\
&\qquad\times \frac{(eq^n, fq^n)_k}{(\lambda q^{1+n}/e, \lambda q^{1+n}/f)_k}\left(\frac{aq}{ef}\right)^k. \qquad (3.3)
\end{aligned}
$$

Next, on both sides of (3.3), replace a, c, d, e and f by aq^{-2n}, cq^{-n}, dq^{-n}, eq^{-n} and fq^{-n} respectively. Note that the condition $\lambda = qa^2/bcd$ implies that λ is replaced by λq^{-2n}. This yields

$$
\sum_{k\geq -n} \frac{(aq^{-n}, q\sqrt{a}, -q\sqrt{a}, bq^n, c, d, e, f)_k}{(q^{1+n}, \sqrt{a}, -\sqrt{a}, aq^{1-n}/b, aq/c, aq/d, aq/e, aq/f)_k}\left(\frac{q^2a^2}{bcdef}\right)^k
$$

$$= \frac{1-aq^{-2n}}{1-a}\frac{1-\lambda}{1-\lambda q^{-2n}}\frac{(\lambda q^{-2n},\lambda b/a,\lambda cq^{-n}/a,\lambda dq^{-n}/a)_n}{(aq^{-2n},b,cq^{-n},dq^{-n})_n}\left(\frac{a}{\lambda}\right)^n$$
$$\times\frac{(aq^{1-2n},aq/ef,\lambda q/e,\lambda q/f)_\infty}{(\lambda q^{1-2n},aq/e,aq/f,\lambda q/ef)_\infty}$$
$$\times\sum_{k\geq -n}\frac{(\lambda q^{-n},q\sqrt{\lambda},-q\sqrt{\lambda},\lambda bq^n/a,\lambda c/a,\lambda d/a,e,f)_k}{(q^{1+n},\sqrt{\lambda},-\sqrt{\lambda},aq^{1-n}/b,aq/c,aq/d,\lambda q/e,\lambda q/f)_k}\left(\frac{aq}{ef}\right)^k,$$

which is (3.2) after using the simplifications (2.4) and (2.6) on the right-hand side. □

By letting $n\to\infty$ in (3.2), assuming $|qa^2/cdef|<1$ while appealing to Tannery's theorem [6] for being able to interchange limit and summation, one gets the following transformation formula, which was derived in [11]

$$\begin{aligned}&{}_6\psi_6\left[\begin{matrix}q\sqrt{a},-q\sqrt{a},c,d,e,f\\ \sqrt{a},-\sqrt{a},aq/c,aq/d,aq/e,aq/f\end{matrix};q,\frac{qa^2}{cdef}\right]\\&=\frac{(aq,q/a,aq/ef,aq/cd,\lambda q/e,\lambda q/f,aq/\lambda c,aq/\lambda d)_\infty}{(aq/e,aq/f,q/c,q/d,\lambda q,q/\lambda,\lambda q/ef,b)_\infty}\\&\quad\times{}_6\psi_6\left[\begin{matrix}q\sqrt{\lambda},-q\sqrt{\lambda},\lambda c/a,\lambda d/a,e,f\\ \sqrt{\lambda},-\sqrt{\lambda},aq/c,aq/d,\lambda q/e,\lambda q/f\end{matrix};q,\frac{qa^2}{cdef}\right],\end{aligned}\tag{3.4}$$

where $\lambda=qa^2/bcd$, and b is now an extra parameter on the right-hand side.
As explained in [11], an iteration of (3.4) and an appropriate specialization of both extra parameters appearing on the right-hand side immediately establishes Bailey's formula (1.3).

Now, we consider the next level in the hierarchy of identities for very-well-poised nonterminating basic hypergeometric series, which is Bailey's four-term ${}_{10}\phi_9$ transformation [9, Appendix (III.39)]

$$\begin{aligned}&{}_{10}\phi_9\left[\begin{matrix}a,\ q\sqrt{a},-q\sqrt{a},b,c,d,e,f,g,h\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h\end{matrix};q,q\right]\\&+\frac{(aq,b/a,c,d,e,f,g,h,bq/c,bq/d,bq/e,bq/f,bq/g,bq/h)_\infty}{(b^2q/a,a/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h,bc/a,bd/a,be/a,bf/a,bg/a,bh/a)_\infty}\\&\quad\times{}_{10}\phi_9\left[\begin{matrix}b^2/a,\ qb/\sqrt{a},-qb/\sqrt{a},b,bc/a,bd/a,be/a,bf/a,bg/a,bh/a\\ b/\sqrt{a},-b/\sqrt{a},bq/a,bq/c,bq/d,bq/e,bq/f,bq/g,bq/h\end{matrix};q,q\right]\\&=\frac{(aq,b/a,\lambda q/f,\lambda q/g,\lambda q/h,bf/\lambda,bg/\lambda,bh/\lambda)_\infty}{(\lambda q,b/\lambda,aq/f,aq/g,aq/h,bf/a,bg/a,bh/a)_\infty}\\&\quad\times{}_{10}\phi_9\left[\begin{matrix}\lambda,\ q\sqrt{\lambda},-q\sqrt{\lambda},b,\lambda c/a,\lambda d/a,\lambda e/a,f,g,h\\ \sqrt{\lambda},-\sqrt{\lambda},\lambda q/b,aq/c,aq/d,aq/e,\lambda q/f,\lambda q/g,\lambda q/h\end{matrix};q,q\right]\\&+\frac{(aq,b/a,f,g,h,bq/f,bq/g,bq/h,\lambda c/a,\lambda d/a,\lambda e/a,abq/\lambda c,abq/\lambda d,abq/\lambda e)_\infty}{(b^2q/\lambda,\lambda/b,aq/c,aq/d,aq/e,aq/f,aq/g,aq/h,bc/a,bd/a,be/a,bf/a,bg/a,bh/a)_\infty}\\&\times{}_{10}\phi_9\left[\begin{matrix}b^2/\lambda,\ qb/\sqrt{\lambda},-qb/\sqrt{\lambda},b,bc/a,bd/a,be/a,bf/\lambda,bg/\lambda,bh/\lambda\\ b/\sqrt{\lambda},-b/\sqrt{\lambda},bq/\lambda,abq/\lambda c,abq/\lambda d,abq/\lambda e,bq/f,bq/g,bq/h\end{matrix};q,q\right],\end{aligned}\tag{3.5}$$

where $\lambda=qa^2/cde$ and $q^2a^3=bcdefgh$.
We can deduce from (3.5) the following semi-finite identity.

Proposition 3.2.

$$\sum_{k\geq -n}\frac{(aq^{-n},q\sqrt{a},-q\sqrt{a},b,cq^n,d,e,f,g,h)_k}{(q^{1+n},\sqrt{a},-\sqrt{a},aq/b,aq^{1-n}/c,aq/d,aq/e,aq/f,aq/g,aq/h)_k}q^k$$
$$+\alpha_n\ {}_{10}\phi_9\left[\begin{matrix}b^2/a,\ bq/\sqrt{a},-bq/\sqrt{a},bq^{-n},bcq^n/a,bd/a,be/a,bf/a,bg/a,bh/a\\ b/\sqrt{a},-b/\sqrt{a},bq^{1+n}/a,bq^{1-n}/c,bq/d,bq/e,bq/f,bq/g,bq/h\end{matrix};q,q\right]$$
$$=\beta_n\sum_{k\geq -n}\frac{(\lambda q^{-n},q\sqrt{\lambda},-q\sqrt{\lambda},b,\lambda cq^n/a,\lambda d/a,\lambda e/a,f,g,h)_k}{(q^{1+n},\sqrt{\lambda},-\sqrt{\lambda},\lambda q/b,aq^{1-n}/c,aq/d,aq/e,\lambda q/f,\lambda q/g,\lambda q/h)_k}q^k$$
$$+\gamma_n\ {}_{10}\phi_9\left[\begin{matrix}b^2/\lambda,\ bq/\sqrt{\lambda},-bq/\sqrt{\lambda},bq^{-n},bcq^n/a,bd/a,be/a,bf/\lambda,bg/\lambda,bh/\lambda\\ b/\sqrt{\lambda},-b/\sqrt{\lambda},bq^{1+n}/\lambda,abq^{1-n}/\lambda c,abq/\lambda d,abq/\lambda e,bq/f,bq/g,bq/h\end{matrix};q,q\right], \tag{3.6}$$

where $\lambda=qa^2/cde$, $c=q^2a^3/bdefgh$, *and*

$$\alpha_n=-\frac{b}{a}\frac{(q,q/a,c/b)_n}{(q/b,c/a)_n}\frac{(bq^{1+n}/a,cq^n,aq,bq/c)_\infty}{(bcq^n/a,b^2q/a,aq/b,aq/c)_\infty}$$
$$\times\frac{(bq/d,bq/e,bq/f,bq/g,bq/h,d,e,f,g,h)_\infty}{(bd/a,be/a,bf/a,bg/a,bh/a,aq/d,aq/e,aq/f,aq/g,aq/h)_\infty},$$

$$\beta_n=\frac{(q/a,\lambda c/a,aq/\lambda d,aq/\lambda e,b/a)_n}{(q/\lambda,c,q/d,q/e,b/\lambda)_n}$$
$$\times\frac{(aq,bf/\lambda,bg/\lambda,bh/\lambda,\lambda q/f,\lambda q/g,\lambda q/h,bq^n/a)_\infty}{(\lambda q,bf/a,bg/a,bh/a,aq/f,aq/g,aq/h,bq^n/\lambda)_\infty},$$

$$\gamma_n=\frac{(q,q/a,b/a,aq/\lambda d,aq/\lambda e,\lambda c/ab)_n}{(c,c/a,q/b,q/d,q/e,qb/\lambda)_n}\frac{(bq^n/a,aq,f,g,h)_\infty}{(bcq^n/a,b^2q/\lambda,aq/f,aq/g,aq/h)_\infty}$$
$$\times\frac{(\lambda c/a,\lambda d/a,\lambda e/a,bq/f,bq/g,bq/h,abq/\lambda c,abq/\lambda d,abq/\lambda e)_\infty}{(bd/a,be/a,bf/a,bg/a,bh/a,\lambda/b,aq/c,aq/d,aq/e)_\infty}.$$

Proof. In the first and the third summations of (3.5), shift the index of summation k by n, and replace a, b, d, e, f, g and h by aq^{-2n}, bq^{-n}, dq^{-n}, eq^{-n}, fq^{-n}, gq^{-n} and hq^{-n} respectively. Note that the condition $\lambda=qa^2/cde$ implies that λ is replaced by λq^{-2n}, and the condition $c=q^2a^3/bdefgh$ leaves c unchanged. The first ${}_{10}\phi_9$ is then equal to

$$\delta_n\sum_{k\geq -n}\frac{(aq^{-n},q\sqrt{a},-q\sqrt{a},b,cq^n,d,e,f,g,h)_k}{(q^{1+n},\sqrt{a},-\sqrt{a},aq/b,aq^{1-n}/c,aq/d,aq/e,aq/f,aq/g,aq/h)_k}q^k,$$

where

$$\delta_n=\frac{(aq^{-2n},bq^{-n},c,dq^{-n},eq^{-n},fq^{-n},gq^{-n},hq^{-n})_n}{(q,aq^{1-n}/b,aq^{1-2n}/c,aq^{1-n}/d,aq^{1-n}/e,aq^{1-n}/f,aq^{1-n}/g,aq^{1-n}/h)_n}$$
$$\times\frac{1-a}{1-aq^{-2n}}q^n,$$

and (3.5) is then equivalent to

$$\sum_{k\geq -n}\frac{(aq^{-n},q\sqrt{a},-q\sqrt{a},b,cq^n,d,e,f,g,h)_k}{(q^{1+n},\sqrt{a},-\sqrt{a},aq/b,aq^{1-n}/c,aq/d,aq/e,aq/f,aq/g,aq/h)_k}q^k$$

$$+\frac{a_n}{\delta_n}\,{}_{10}\phi_9\left[\begin{matrix}b^2/a,\ bq/\sqrt{a}, -bq/\sqrt{a}, bq^{-n}, bcq^n/a, bd/a, be/a, bf/a, bg/a, bh/a\\ b/\sqrt{a}, -b/\sqrt{a}, bq^{1+n}/a, bq^{1-n}/c, bq/d, bq/e, bq/f, bq/g, bq/h\end{matrix};q,q\right]$$

$$=\frac{b_n}{\delta_n}\sum_{k\geq -n}\frac{(\lambda q^{-n}, q\sqrt{\lambda}, -q\sqrt{\lambda}, b, \lambda cq^n/a, \lambda d/a, \lambda e/a, f, g, h)_k}{(q^{1+n}, \sqrt{\lambda}, -\sqrt{\lambda}, \lambda q/b, aq^{1-n}/c, aq/d, aq/e, \lambda q/f, \lambda q/g, \lambda q/h)_k}q^k$$

$$+\frac{c_n}{\delta_n}\,{}_{10}\phi_9\left[\begin{matrix}b^2/\lambda,\ bq/\sqrt{\lambda}, -bq/\sqrt{\lambda}, bq^{-n}, bcq^n/a, bd/a, be/a, bf/\lambda, bg/\lambda, bh/\lambda\\ b/\sqrt{\lambda}, -b/\sqrt{\lambda}, bq^{1+n}/\lambda, abq^{1-n}/\lambda c, abq/\lambda d, abq/\lambda e, bq/f, bq/g, bq/h\end{matrix};q,q\right],$$

where

$$a_n=\frac{(aq^{1-2n}, bq^n/a, c, dq^{-n}, eq^{-n}, fq^{-n}, gq^{-n}, hq^{-n})_\infty}{(b^2q/a, aq^{-n}/b, aq^{1-2n}/c, aq^{1-n}/d, aq^{1-n}/e, aq^{1-n}/f, aq^{1-n}/g, aq^{1-n}/h)_\infty}\times\frac{(bq^{1-n}/c, bq/d, bq/e, bq/f, bq/g, bq/h)_\infty}{(bcq^n/a, bd/a, be/a, bf/a, bg/a, bh/a)_\infty},$$

$$b_n=\frac{(aq^{1-2n}, bq^n/a, \lambda q^{1-n}/f, \lambda q^{1-n}/g, \lambda q^{1-n}/h, bf/\lambda, bg/\lambda, bh/\lambda)_\infty}{(\lambda q^{1-2n}, bq^n/\lambda, aq^{1-n}/f, aq^{1-n}/g, aq^{1-n}/h, bf/a, bg/a, bh/a)_\infty}\frac{1-\lambda}{1-\lambda q^{-2n}}q^n\times\frac{(\lambda q^{-2n}, bq^{-n}, \lambda c/a, \lambda dq^{-n}/a, \lambda eq^{-n}/a, fq^{-n}, gq^{-n}, hq^{-n})_n}{(q, \lambda q^{1-n}/b, aq^{1-2n}/c, aq^{1-n}/d, aq^{1-n}/e, \lambda q^{1-n}/f, \lambda q^{1-n}/g, \lambda q^{1-n}/h)_n},$$

$$c_n=\frac{(aq^{1-2n}, bq^n/a, fq^{-n}, gq^{-n}, hq^{-n}, bq/f, bq/g, bq/h)_\infty}{(b^2q/\lambda, \lambda q^{-n}/b, aq^{1-2n}/c, aq^{1-n}/d, aq^{1-n}/e, aq^{1-n}/f, aq^{1-n}/g, aq^{1-n}/h)_\infty}\times\frac{(\lambda c/a, \lambda dq^{-n}/a, \lambda eq^{-n}/a, abq^{1-n}/\lambda c, abq/\lambda d, abq/\lambda e)_\infty}{(bcq^n/a, bd/a, be/a, bf/a, bg/a, bh/a)_\infty}.$$

Using the simplifications (2.4) and (2.6), we get $a_n/\delta_n = \alpha_n$, $b_n/\delta_n = \beta_n$ and $c_n/\delta_n = \gamma_n$, which yields (3.6). □

Let $n \to \infty$ in (3.6), assuming $|q^2a^3/bdefgh| = |c| < 1$ and $|\lambda c/a| = |aq/de| < 1$ while appealing to Tannery's theorem [6] for being able to interchange limit and summation. One gets the following transformation formula

$$\begin{aligned}&{}_8\psi_8\left[\begin{matrix}q\sqrt{a}, -q\sqrt{a}, b, d, e, f, g, h\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/d, aq/e, aq/f, aq/g, aq/h\end{matrix};q,c\right]\\
&=\frac{(aq, q/a, \lambda c/a, aq/\lambda d, aq/\lambda e, b/a, bf/\lambda, bg/\lambda, bh/\lambda, \lambda q/f, \lambda q/g, \lambda q/h)_\infty}{(\lambda q, q/\lambda, c, q/d, q/e, b/\lambda, bf/a, bg/a, bh/a, aq/f, aq/g, aq/h)_\infty}\\
&\quad\times {}_8\psi_8\left[\begin{matrix}q\sqrt{\lambda}, -q\sqrt{\lambda}, b, \lambda d/a, \lambda e/a, f, g, h\\ \sqrt{\lambda}, -\sqrt{\lambda}, \lambda q/b, aq/d, aq/e, \lambda q/f, \lambda q/g, \lambda q/h\end{matrix};q,\frac{\lambda c}{a}\right]\\
&+\frac{b}{a}\frac{(q, q/a, c/b, aq, bq/c, bq/d, bq/e, bq/f, bq/g, bq/h, d, e, f)_\infty}{(q/b, c/a, b^2q/a, aq/b, aq/c, bd/a, be/a, bf/a, bg/a, bh/a, aq/d, aq/e, aq/f)_\infty}\\
&\times\frac{(g,h)_\infty}{(aq/g, aq/h)_\infty}\,{}_8\phi_7\left[\begin{matrix}b^2/a,\ bq/\sqrt{a}, -bq/\sqrt{a}, bd/a, be/a, bf/a, bg/a, bh/a\\ b/\sqrt{a}, -b/\sqrt{a}, bq/d, bq/e, bq/f, bq/g, bq/h\end{matrix};q,c\right]\\
&\quad+\frac{(q, q/a, b/a, aq/\lambda d, aq/\lambda e, \lambda c/ab, aq, f, g, h)_\infty}{(c, c/a, q/b, q/d, q/e, qb/\lambda, b^2q/\lambda, aq/f, aq/g, aq/h)_\infty}\\
&\quad\times\frac{(\lambda c/a, \lambda d/a, \lambda e/a, bq/f, bq/g, bq/h, abq/\lambda c, abq/\lambda d, abq/\lambda e)_\infty}{(bd/a, be/a, bf/a, bg/a, bh/a, \lambda/b, aq/c, aq/d, aq/e)_\infty}\end{aligned}$$

$$\times\, {}_8\phi_7\left[\begin{matrix} b^2/\lambda,\, bq/\sqrt{\lambda}, -bq/\sqrt{\lambda}, bd/a, be/a, bf/\lambda, bg/\lambda, bh/\lambda \\ b/\sqrt{\lambda}, -b/\sqrt{\lambda}, abq/\lambda d, abq/\lambda e, bq/f, bq/g, bq/h \end{matrix}; q, \frac{\lambda c}{a}\right], \qquad (3.7)$$

where $c = q^2a^3/bdefgh$, $\lambda = qa^2/cde$, $|c| < 1$ and $|\lambda c/a| < 1$.
Note that when $\lambda = a$ or when $b = a$, this identity is trivial.
On the other hand, identity (3.7) can be derived from classical known identities as follows : first, apply [9, (5.6.1)] to both ${}_8\psi_8$ series of (3.7), then the use of the transformation [9, (2.10.1)] gives some cancellations, and the remaining identity is finally a special case of the theta function identity [9, Ex. 5.22].

Aknowledgments. We thank Michael Schlosser for his valuable comments and pointing out a mistake in a previous version of this paper.

References

[1] G. E. Andrews, *Applications of basic hypergeometric functions*, SIAM Rev. **16** (1974), 441–484.
[2] R. Askey, *The very well poised ${}_6\psi_6$. II*, Proc. Amer. Math. Soc. **90** (1984), 575–579.
[3] R. Askey and M. E. H. Ismail, *The very well poised ${}_6\psi_6$*, Proc. Amer. Math. Soc. **77** (1979), 218–222.
[4] W. N. Bailey, *Series of hypergeometric type which are infinite in both directions*, Quart. J. Math. (Oxford) **7** (1936), 105–115.
[5] W. N. Bailey, *On the basic bilateral hypergeometric series ${}_2\psi_2$*, Quart. J. Math. (Oxford) (2) **1** (1950), 194–198.
[6] T. J. l'A. Bromwich, *An introduction to the theory of infinite series*, 2nd ed., Macmillan, London, 1949.
[7] A.-L. Cauchy, *Mémoire sur les fonctions dont plusieurs valeurs sont liées entre elles par une équation linéaire, et sur diverses transformations de produits composés d'un nombre indéfini de facteurs*, C. R. Acad. Sci. Paris **17** (1843), 523; reprinted in *Oeuvres de Cauchy*, Ser. 1 **8**, Gauthier-Villars, Paris (1893), 42–50.
[8] W. Y. C. Chen and A. M. Fu, *Semi-Finite Forms of Bilateral Basic Hypergeometric Series*, Proc. Amer. Math. Soc., to appear.
[9] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Second Edition, Encyclopedia of Mathematics And Its Applications 96, Cambridge University Press, Cambridge, 2004.
[10] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Regiomonti. Sumptibus fratrum Bornträger, 1829; reprinted in Jacobi's Gesammelte Werke, vol. 1, (Reimer, Berlin, 1881–1891), pp. 49–239; reprinted by Chelsea (New York, 1969); now distributed by the Amer. Math. Soc., Providence, RI.
[11] F. Jouhet and M. Schlosser, *Another proof of Bailey's ${}_6\psi_6$ summation*, Aequationes Math. **70** (1-2)(2005), 43–50.
[12] M. Schlosser, *A simple proof of Bailey's very-well-poised ${}_6\psi_6$ summation*, Proc. Amer. Math. Soc. **130** (2001), 1113–1123.
[13] M. Schlosser, *Abel–Rothe type generalizations of Jacobi's triple product identity*, in "Theory and Applications of Special Functions. A Volume Dedicated to Mizan Rahman" (M. E. H. Ismail and E. Koelink, eds.), Dev. Math. **13** (2005), 383–400.
[14] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, London/New York, 1966.
[15] L. J. Slater and A. Lakin, *Two proofs of the ${}_6\psi_6$ summation theorem*, Proc. Edinburgh Math. Soc. (2) **9** (1953–57), 116–121.

Institut Girard Desargues, Université Claude Bernard (Lyon 1), 69622 Villeurbanne Cedex, France
E-mail address: jouhet@igd.univ-lyon1.fr
URL: http://igd.univ-lyon1.fr/home/jouhet