The N-dimensional matching polynomial

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To P. Cartier and A. K. Zvonkin

Heilmann et Lieb ont introduit le polynôme de couplage $\mu(G,x)$ d'un graphe $G{=}(V,E)$. Nous prolongeons leur définition en munissant chaque sommet de G d'une forme linéaire N-dimensionnelle (ou bien d'un vecteur) et chaque arête d'une forme symétrique bilinéaire. On attache donc à tout r-couplage de G le produit des formes linéaires des sommets qui ne sont pas saturés par le couplage, multiplié par le produit des poids des r arêtes du couplage, où le poids d'une arête est la valeur de sa forme évaluée sur les deux vecteurs de ses extrémités. En multipliant par $(-1)^r$ et en sommant sur tous les couplages, nous obtenons notre polynôme de couplage N-dimensionnel. Si $N{=}1$, le théorème principal de l'article de Heilmann et Lieb affirme que tous les zéros de $\mu(G,x)$ sont réels. Si $N{=}2$, cependant, nous avons trouvé des graphes exceptionnels où il n'y a aucun zéro réel, même si chaque arête est munie du produit scalaire canonique. Toutefois, la théorie de la dualité développée dans [12] reste valable en N dimensions. Elle donne notamment une nouvelle interprétation à la transformation de Bargmann-Segal, aux diagrammes de Feynman et aux produits de Wick.

Heilmann and Lieb have introduced the matching polynomial $\mu(G,x)$ of a graph $G{=}(V,E)$. We extend their definition by associating to every vertex of G an N-dimensional linear form (or a vector) and to every edge a symmetric bilinear form. For every r-matching of G we define its weight as the product of the linear forms of the vertices not covered by the matching, multiplied by the product of the weights of the r edges of the matching, where the weight of an edge is the value of its form evaluated at the two vectors of its end points. Multiplying by $(-1)^r$ and summing over all matchings, we get our N-dimensional matching polynomial. If $N{=}1$, the Heilmann-Lieb theorem affirms that all zeroes of $\mu(G,x)$ are real. If $N{=}2$, however, there are exceptional graphs whithout any real zero at all, even if the canonical scalar product is associated to every edge. Nevertheless, the duality theory developed in [12] remains valid in N dimensions. In particular, it brings new light to the Bargmann-Segal transform, to the Feynman diagrams, and to the Wick products.

1. Introduction

Let V be a finite set of cardinality n. A graph $G = (V, E)$ is called simple if and only if the set of its edges E is a subset of $\binom{V}{2}$, the family of all 2-element subsets of V. For such graphs we define the *complement* by $\overline{G} := (V, \overline{E})$ with $\overline{E} := \binom{V}{2} \setminus E$.

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For every $r \geq 0$, an *r-matching* in G is a set of r edges of G, no two of which have a vertex in common. If $r = n/2$, then the matching is called *perfect*.

Moreover, let us consider a Euclidean space $\mathbf{R}$ of dimension N, $\mathbf{R} = \mathbb{R}^N$, with its canonical scalar product $\langle x, y\rangle = \sum_{i=1}^N x_i y_i$ (remember that $n = |V|$ is the number of vertices of G). Let $e_1, \ldots, e_N$ be the canonical (orthonormal) basis of $\mathbf{R}$. We use it to interpret polynomials and partial derivatives (or ∇) in the usual way. Moreover, let $\langle x, y\rangle_P := \langle x, Py\rangle$, $P = P^T$ and $P > 0$, be an additional scalar product.

Let us attach to each vertex $v \in V$ of our graph $G = (V, E)$ a vector $a_v \in \mathbf{R}$, and to each edge $\{u, v\} \in E$ the scalar product $\langle a_u, a_v\rangle_P$. We use the notation $a_P G = (aV, a_P E)$ for this "decorated" graph. Let us look at a matching of $a_P G$. We multiply the product of the scalar products of the edges of our matching and the product of the scalar products $\langle x, a_v\rangle_P$, $x \in \mathbf{R}$, of the vertices of $a_P G$ which are not covered by any edge of the matching. This gives a homogeneous polynomial of degree $n - 2r$ ($n = |V|$), if the matching contains r edges. If we multiply this polynomial by $(-1)^r$ and sum these expressions over all matchings of $a_P G$, we get the *signed N-dimensional matching polynomial* $\mu(a_P G, x)$. If we do not multiply by $(-1)^r$ before summing, we get the *unsigned N-dimensional matching polynomial* $\overline{\mu}(a_P G, x)$. In the sequel, if we say matching polynomial without any further attribute, then we mean the signed one. Since every vector a_v, $v \in V$, appears in each homogeneous polynomial of degree $n - 2r$ precisely once (in a factor $\langle a_u, a_v\rangle_P$ if v is covered by the matching and in a factor $\langle x, a_v\rangle_P$ otherwise), the polynomials $\mu(a_P G, x)$ and $\overline{\mu}(a_P G, x)$ are linear in the vectors a_v (and equal to zero, if there exists a $v \in V$ such that $a_v = 0$). Moreover, they satisfy the obvious relation

$$\mu(a_P G, x) \;=\; (-i)^n \cdot \overline{\mu}(a_P G, i \cdot x), \qquad |V| \;=\; n, \quad i \;=\; \sqrt{-1}.$$

Example 1.1. Let $K_n = (V, \binom{V}{2})$ be the complete graph on n vertices such that its complementary graph $\overline{K_n} = (V, \emptyset)$ has no edge at all. Let us attach to each $v \in V$ a vector $a_v \in \mathbf{R}$. Then we have the following expressions for the matching polynomials of the two complementary decorated graphs $a_P K_n$ and $\overline{a_P K_n}$:

$$\begin{aligned}
\mu(\overline{a_P K_n}, x) \;&= \overline{\mu}(\overline{a_P K_n}, x) \;=\; \prod_{v \in V} \langle x, a_v\rangle_P;\\
\mu(a_P K_2, x) \;&= \langle x, a_u\rangle_P \langle x, a_v\rangle_P - \langle a_u, a_v\rangle_P,\\
\overline{\mu}(a_P K_2, x) \;&= \langle x, a_u\rangle_P \langle x, a_v\rangle_P + \langle a_u, a_v\rangle_P,\\
\mu(a_P K_3, x) \;&= \langle x, a_u\rangle_P \langle x, a_v\rangle_P \langle x, a_w\rangle_P - \langle x, a_u\rangle_P \langle a_v, a_w\rangle_P\\
&\qquad - \langle x, a_v\rangle_P \langle a_u, a_w\rangle_P - \langle x, a_w\rangle_P \langle a_u, a_v\rangle_P, \quad \text{etc.}
\end{aligned}$$

Remark 1.1. In physics, the polynomial $\mu(a_P K_n, x)$ is called *Wick* (or *Hermite*) *polynomial*. It is denoted by $\Phi(x|a_1, \ldots, a_n)$ or by $:\prod_{v \in V} \langle x, a_v\rangle_P:$ or by $[\prod_{v \in V} \langle x, a_v\rangle_P]^\sharp$ (see [3],[6],[15]).

Let $G = (V, E)$ and $\overline{G} := (V, \overline{E})$ be *arbitrary* complementary graphs. Let us attach to every $v \in V$ a vector $a_v \in \mathbf{R}$ in order to get two complementary decorated graphs, namely $a_P G = (aV, a_P E)$ and $\overline{a_P G} = (aV, \overline{a_P E})$ (in the classical one-dimensional case, we have $a_v = 1$ for every $v \in V$ as well as $P = 1$). A priori, it is not clear at all whether the matching polynomials $\mu(\overline{a_P G}, x)$ and $\overline{\mu}(\overline{a_P G}, x)$ of the complementary graph $\overline{a_P G}$ are determined by the matching polynomials $\mu(a_P G, x)$ and $\overline{\mu}(a_P G, x)$ of the graph $a_P G$ itself. Lovász [14, 5.18] and Godsil [7], [8] have established such a result in the one-dimensional case. It seems to be difficult, however, to generalize their proofs. Fortunately, the proofs presented in [12] can be generalized almost without any modification. Indeed, we prove that $\overline{\mu}(\overline{a_P G}, x)$ is the Bargmann-Segal-Wiener transform of $\mu(a_P G, x)$, and $\mu(\overline{a_P G}, x)$ is the Wick transform of $\overline{\mu}(a_P G, x)$, i.e.,

$$\overline{\mu}(\overline{a_P G}, x) \;=\; [\mu(a_P G, x)]^{\flat}, \qquad \mu(\overline{a_P G}, x) \;=\; [\overline{\mu}(a_P G, x)]^{\sharp}.$$

More precisely, we establish the identities

$$\begin{aligned} \overline{\mu}(\overline{a_P G}, x) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbf{R}} \mu(a_P G, z) \cdot \mu(x_P K_n, z) \cdot e^{-\langle z, z \rangle_P / 2} \cdot d_P z \\ &= \int_{\mathbf{R}} e^{-\langle z-x, z-x \rangle_P / 2} \cdot \mu(a_P G, z) \cdot d_P z \\ &= e^{-\langle x, x \rangle_P / 2} \cdot \mu(a_P G, \nabla_P) \cdot e^{\langle x, x \rangle_P / 2} \\ &= \exp[\triangle_P / 2] \cdot \mu(a_P G, x), \end{aligned}$$

as well as

$$\begin{aligned} \mu(\overline{a_P G}, x) &= e^{\langle x, x \rangle_P / 2} \cdot \overline{\mu}(a_P G, -\nabla_P) \cdot e^{-\langle x, x \rangle_P / 2} \\ &= \exp[-\triangle_P / 2] \cdot \overline{\mu}(a_P G, x), \end{aligned}$$

where

$$d_P z \;:=\; \sqrt{\det(P)/(2\pi)^N} \cdot d^N z \qquad (P \;=\; 2\pi \cdot Id \quad \Rightarrow \quad d_P z \;=\; d^N z),$$

$$\nabla_P \;:=\; P^{-1} \nabla, \qquad \triangle_P \;:=\; \langle \nabla_P, \nabla_P \rangle_P \;=\; \langle \nabla, \nabla \rangle_{P^{-1}}.$$

Remark 1.2. The operator $\triangle_P$ is nothing else but the general form of what is called an *elliptic operator of second degree.*

Since the space of polynomials is generated (as a vector space) by the matching polynomials, our relations also offer an abundance of possible definitions for the Bargmann-Segal or for the Wick transformation, and one can ask the question whether all of them are well known. Indeed, during the Séminaire Lotharingien de Combinatoire 44, Cartier [2] has made use of the first integral formula in order to define the Bargmann-Segal transformation. He has defined the Wick transformation, however, in a totally combinatorial way with the help of the identity $\mu(a_P K_n, x) = [\prod_{v\in V}\langle x, a_v\rangle_P]^\sharp$. Finally, he has used the identity $\mu(a_P K_n, x) = e^{\langle x,x\rangle_P/2}\cdot\overline{\mu}(\overline{a_P K_n}, -\nabla_P)\cdot e^{-\langle x,x\rangle_P/2}$ in order to define the Wick (i.e. Hermite) polynomials. In his article [3], which, however, treats just the one-dimensional case, he has finally proposed the second integral formula and the second differential formula in order to define the Bargmann-Segal transformation (and the second differential formula for the Wick transformation). Therefore our first differential formula is apparently new as an additional possibility to define the Bargmann-Segal transformation. At least, there cannot be any doubt that matching polynomials provide a better understanding of the respective roles of the two transformations.

Last, but not least, the importance of N-dimensional matching polynomials is underlined by the surprising fact that the matching functions $e^{-\langle x,x\rangle_P/2}\mu(\overline{a_P G}, x)$ and $e^{-\langle x,x\rangle_P/2}\mu(a_P G, x)$ are indeed Fourier transforms of each other.

Example 1.2. Let us suppose that $V = T_1 \uplus \ldots \uplus T_k$, and that $K_{t_1}, \ldots, K_{t_k}$ are complete graphs on the vertex sets $T_1, \ldots, T_k$, respectively. For the matching polynomial of the decorated graph $a_P K_{t_1} \uplus \ldots \uplus o_P K_{t_k}$, we have the obvious identity

$$\begin{aligned}\mu(a_P K_{t_1} \uplus \ldots \uplus o_P K_{t_k}, z) &= \mu(a_P K_{t_1}, z)\cdots\mu(o_P K_{t_k}, z)\\ &= \Big[\prod_{v\in T_1}\langle z, a_v\rangle_P\Big]^\sharp\cdots\Big[\prod_{v\in T_k}\langle z, o_v\rangle_P\Big]^\sharp.\end{aligned}$$

The case $x = 0$ of our second integral identity for the Bargmann-Segal transformation allows us to conclude that

$$\begin{aligned}&\overline{\mu}(\overline{a_P K_{t_1} \uplus \ldots \uplus o_P K_{t_k}}, 0)\\ &= \int_{\mathbf{R}}\Big[\prod_{v\in T_1}\langle z, a_v\rangle_P\Big]^\sharp\cdots\Big[\prod_{v\in T_k}\langle z, o_v\rangle_P\Big]^\sharp\cdot e^{-\langle z,z\rangle_P/2}d_P z\end{aligned}$$

counts the (weighted) number of perfect matchings of the decorated complete multipartite graph $\overline{a_P K_{t_1} \uplus \ldots \uplus o_P K_{t_k}}$. This result corresponds to Formula 3) of Section "Diagrams" in the recent book [15, Chapter 2.2] and to Corollary 8.3.2 of the book [6]. Note that the authors of these books identify all vertices of T_1, all vertices of T_2, ..., and all vertices of T_k, so that they have just k different vertices to consider. The edges inside the blocks T_1, ..., T_k become loops ("self-lines") in this way. In the main result of Section 8.3 of the book [6] by Glimm and Jaffe, these loops carry different scalar products as described in our abstract and in Remark 3.1. The case $k = 2$ is nothing else but the orthogonality of the Wick polynomials, because the (weighted) number of perfect matchings of the decorated complete bipartite graph $\overline{a_P K_{t_1} \uplus o_P K_{t_2}}$ is equal to zero if $t_1 \neq t_2$ and is equal to the permanent $\mathrm{per}(\langle a_u, o_v\rangle_P)$ if $t_1 = t_2$.

We prove all our main theorems with the help of the algebra of set functions. This leads to very short and conceptual proofs, whereas the classical particular cases of our results mentioned in the preceding example were established in the books by Glimm, Jaffe, Malyshev and Minlos by using partial integration, which leads to considerably longer arguments (see, for example, [15, pp. 39-41]). Set functions also contribute to a better understanding of the semi-invariants studied in probability theory, as well as of the bialgebra of partitions introduced by Lando in his article [10]. All this is explained in the following section. Section 3, on the other hand, covers the duality theorems, which we discussed earlier in this introduction.

Finally, we study the divisor $\operatorname{div}[\mu(a_P G, x)]$ in the fourth section. Originally, we had conjectured that the Heilmann–Lieb theorem affirming that all zeroes of $\mu(G, x)$ are real (if $N = 1$) can be generalized to higher dimensions. What we prove is, in fact, a theorem which imposes very restrictive conditions on the class of possible counter-examples. Although this theorem can be refined even further, by using the last theorem of [12], there exist indeed very rare decorated graphs, the matching polynomial of which has no real zero at all, thus refuting our conjecture. We provide a family of such counter-examples. The problem of characterizing all possible counter-examples remains open.

2. Algebraic tools

Let V be a finite set and

$$\begin{gathered} f : 2^V \to A \\ V' \subseteq V \mapsto f(V') \in A \end{gathered}$$

be a *set function*, where A is a *commutative ring* (with 1). Consider the generating function

$$F_f(\nu) \;:=\; \sum_{V' \subseteq V} f(V') \cdot \nu^{V'}, \qquad \nu^{\emptyset} \;:=\; 1,$$

subject to the following multiplication rules ($V', V'' \subseteq V$):

$$\nu^{V'} \cdot \nu^{V''} \;:=\; \nu^{V'+V''}, \quad \text{where}$$

$$V' + V'' \;:=\; \begin{cases} V' \cup V'', & \text{if} \quad V' \cap V'' = \emptyset, \\ \dagger, & \text{if} \quad V' \cap V'' \neq \emptyset, \end{cases} \quad \text{where}$$

$$\dagger + V' \;:=\; \dagger, \quad \dagger + \dagger \;:=\; \dagger, \qquad \text{and} \qquad \nu^{\dagger} \;:=\; 0.$$

The algebra $A[V]$ of those generating functions is not unknown. In fact, we have the isomorphism

$$A[V] \;\simeq\; A[v_1, \ldots, v_n]/\langle v_1^2, \ldots, v_n^2\rangle,$$

if V contains n elements.

REMARK 2.1. If A is a field, $A[V]$ is a local Artinian ring of length $2^{|V|}$.

EXAMPLE 2.1. The product fg of two set functions f and g is defined, for every $V' \subseteq V$, by

$$(fg)(V') \;:=\; \sum_{V'=V''\uplus V'''} f(V'') \cdot g(V''').$$

It follows that

$$F_{fg}(\nu) \;=\; F_f(\nu) \cdot F_g(\nu).$$

For $|V| = \infty$, let $F(V)$ be the partially ordered set of finite subsets of V. We have the canonical projections $p_{V',V''} : A[V'] \to A[V'']$ $(V', V'' \in F(V), V' \supseteq V'')$, and we define

$$A[V] \;:=\; \varprojlim A[V'], \quad V' \in F(V),$$

in order to work with the generating functions of the form

$$F_f(\nu) \;=\; \sum_{V'\in F(V)} f(V') \cdot \nu^{V'}.$$

Let

$$V \;:=\; \sum_{v\in V} \nu^{\{v\}}$$

be the indicator function of the subsets of V of cardinality 1 (the double use of V for the set itself, on the one hand, and for an element of $A[V]$, on the other hand, will never cause any confusion). In the product V^n each subset of cardinality n occurs $n!$ times, so that $V^n/n!$ represents the indicator function of the subsets of the set V of cardinality n. The identity

$$\sum_{n=0}^{\infty} f(n) \cdot V^n/n! \;=\; \sum_{V'\in F(V)} f(|V'|) \cdot \nu^{V'}, \quad f : \mathbb{N} \to A,$$

provides an embedding of the ring $A![[V]]$ of generating functions of exponential type (usually the variable is called x instead of V) into our ring $A[V]$. This embedding is at the origin of (almost?) all applications of $A![[V]]$ to combinatorics, but requires the existence of an infinite combinatorial model depending just on cardinalities. Consequently, $A[V]$ gives more flexibility and allows an algebraic treatment, which perfectly reflects the classical combinatorial operations. In addition, $A[V]$ is ideally suited for computer calculations.

REMARK 2.2. The ring $\mathbb{Z}![[V]]$ is not Noetherian, but it contains the important functions $\exp(V)$ and $\log(1+V)$.

For all $t \in A$ we put $(t \cdot \nu)^{V'} := t^{|V'|} \cdot \nu^{V'}$, $V' \subseteq V$, and therefore,

$$F_f(t\nu) = \sum_{\emptyset \subseteq V' \subseteq V} f(V')\, t^{|V'|}\, \nu^{V'}.$$

It is evident that this definition is compatible with the addition and the multiplication. Most important are the special cases $t = -1$ and $t = 0$: $F_f(0) = F_f(0 \cdot \nu) = f(\emptyset)$.

If $F_f(0) = 0$, then $F_f(\nu)^n/n!$ is defined for any ring A, because a partition into n nonempty subsets can be ordered in $n!$ different ways. Therefore we have an action of $A![[V]]$ on $A[V]$ via the substitution $G(F_f(\nu))$ defined for any $G \in A![[V]]$.

Finally, for every $v \in V$, we use the derivatives ∂^v defined by

$$\partial^v \nu^{V'} := \begin{cases} \nu^{V'}, & \text{if} \quad v \in V', \\ 0, & \text{otherwise.} \end{cases}$$

The product rule

$$\partial^v [F_f(\nu) \cdot F_g(\nu)] = (\partial^v F_f(\nu)) \cdot F_g(\nu) + F_f(\nu) \cdot (\partial^v F_g(\nu))$$

is the algebraic analogue of the most fundamental set theoretic fact:

$$v \in V' \uplus V'' \qquad \Leftrightarrow \qquad v \in V' \quad \text{or} \quad v \in V''.$$

The chain rule

$$\partial^v [G(F_f(\nu))] = G'(F_f(\nu)) \cdot \partial^v F_f(\nu), \quad G \in A![[V]],$$

follows immediately from the product rule.

REMARK 2.3. Under the isomorphism $A[V] \simeq A[v_1, \ldots, v_n]/\langle v_1^2, \ldots, v_n^2 \rangle$, ∂^v does not correspond to $\partial/\partial v_i$, but to $v_i \partial/\partial v_i$. The partial derivative $\partial/\partial v_i$ cannot be defined in $A[V]$.

EXAMPLE 2.2. (The bialgebra of partitions)

Let V be a finite set of cardinality n. Lando [10] has introduced and studied the bialgebra $\mathfrak{B}(V)$, called *bialgebra of partitions*. We have to associate to every $\emptyset \subset V' \subseteq V$ a variable denoted by (V'). As an algebra, $\mathfrak{B}(V)$ is the commutative polynomial algebra generated (over $\mathbb{Z}$) by the $2^n - 1$ variables (V'), where $e := (\emptyset)$ can be considered as a unit element. $\mathfrak{B}(V)$ is graduated if we define the order of the variable (V') to be $|V'|$. With respect to the algebra structure, the comultiplication μ is a homomorphism, which we define by putting

$$(V')_1 \ := \ (V') \otimes e, \qquad (V')_2 \ := \ e \otimes (V'),$$

$$\mu((V')) \ := \ (V')_1 + (V')_2 + \sum_{\emptyset \subset V'' \subset V'} (V'')_1 \cdot (V' \backslash V'')_2$$

for every $\emptyset \subset V' \subseteq V$. An element $b \in \mathfrak{B}(V)$ is called *primitive* if and only if $\mu(b) = b_1 + b_2 = b \otimes e + e \otimes b$. For example, $(\{v\})$ is primitive for every $v \in V$.

In this context, it is very useful to consider the generating functions

$$1 + F_V(\nu) \ := \ (\emptyset) \cdot 1 + \sum_{\emptyset \subset V' \subseteq V} (V') \cdot \nu^{V'},$$

$$[1 + F_{V_1}(\nu)] \ := \ [1 + F_V(\nu)] \otimes e, \qquad [1 + F_{V_2}(\nu)] \ := \ e \otimes [1 + F_V(\nu)].$$

They enable us to obtain an easier definition of the comultiplication,

$$\mu[1 + F_V(\nu)] \ = \ [1 + F_{V_1}(\nu)] \cdot [1 + F_{V_2}(\nu)],$$

which allows us to conclude that

$$\begin{aligned} \mu\Big(\log[1 + F_V(\nu)]\Big) \ &= \ \log\Big(\mu[1 + F_V(\nu)]\Big) \ = \ \log\Big([1 + F_{V_1}(\nu)] \cdot [1 + F_{V_2}(\nu)]\Big) \\ &= \ \log[1 + F_{V_1}(\nu)] \ + \ \log[1 + F_{V_2}(\nu)]. \end{aligned}$$

In other words, every coefficient of $\log[1 + F_V(\nu)]$ is a primitive element of the bialgebra $\mathfrak{B}(V)$. This result is nothing else but the main theorem of the article [10] (see also [11, Chapter 6.1.6]).

EXAMPLE 2.3. (Semi-invariants)

Let V be a finite set of cardinality n. Let us associate to every $v \in V$ a random variable ξ_v, where coincidences $\xi_u = \xi_v$ for $u \neq v$ are not forbidden. In fact, they are even desirable if we want to calculate higher moments, as we will see immediately. Let us put

$$\xi V \ := \ \sum_{v \in V} \xi_v \cdot \nu^{\{v\}} \qquad \Rightarrow \qquad \exp[\xi V] \ = \ 1 + \sum_{\emptyset \subset V' \subseteq V} \Big(\prod_{v \in V'} \xi_v \Big) \cdot \nu^{V'},$$

and let us define

$$1 + M_\xi(\nu) \;:=\; \langle \exp[\xi V] \rangle \;:=\; \mathbb{E}\big(\exp[\xi V]\big) \;=\; 1 + \sum_{\emptyset \subset V' \subseteq V} \mathbb{E}\Big(\prod_{v \in V'} \xi_v\Big) \cdot \nu^{V'},$$

in order to calculate the expectations called *moments*. Evidently, we suppose that they are finite. We define the *semi-invariants* or *cumulants* to be the coefficients of

$$S_\xi(\nu) \;:=\; \log[1 + M_\xi(\nu)]$$

(see [4, Chapter 13] or [17, Chapter II.12]). The reader familiar with the theory will easily see the equivalence of our definition with his own one. We remark that our algebraic tools allow us to derive some other relations between the moments and the semi-invariants, such as

$$1 + M_\xi(\nu) \;=\; \exp[S_\xi(\nu)], \qquad [1 + M_\xi(\nu)] \cdot \partial^v S_\xi(\nu) \;=\; \partial^v M_\xi(\nu),$$

where the very last relation is particularly useful for efficient computer calculations. We have not found it in the literature.

Suppose that $V = T_1 \uplus \ldots \uplus T_k$. Let us put $\xi_{T_i} := \prod_{v \in T_i} \xi_v$ for each $i \in \{1, \ldots, k\}$, and let us calculate all the moments of the family $\{\xi_{T_i}\}$, $i = 1, \ldots, k$, with the help of the semi-invariants of the family $\{\xi_v\}$, $v \in V$. By the formula $1 + M_\xi(\nu) = \exp[S_\xi(\nu)]$, we have to sum over all partitions of V. Each of them consists of several *connected components* with respect to the partition $V = T_1 \uplus \ldots \uplus T_k$, i.e., it defines a partition of the set $\{1, \ldots, k\}$. Consequently, we have to sum over all partitions $\{a_1, \ldots, o_1\} \uplus \ldots \uplus \{a_l, \ldots, o_l\} = \{1, \ldots, k\}$, where the contribution of the block $\{a_1, \ldots, o_1\}$, for example, is the sum over all partitions of the set $B_1 := T_{a_1} \cup \ldots \cup T_{o_1}$ which are *connected* with respect to the partition $B_1 = T_{a_1} \uplus \ldots \uplus T_{o_1}$. The contribution of the block $\{a_1, \ldots, o_1\}$, however, is nothing else but one of the semi-invariants of the family $\{\xi_{T_i}\}$, $i = 1, \ldots, k$ (because of the formula $1 + M_\xi(\nu) = \exp[S_\xi(\nu)]$ *applied to the set* $\{1, \ldots, k\}$). In other words, we have established the fact that the semi-invariants of the family $\{\xi_{T_i}\}$ can be calculated with the help of the semi-invariants of the family $\{\xi_v\}$ by summing "over all connected partitions". This result was proved in [15, pp. 30–32], by induction and some manipulation.

For every $\emptyset \subset V' \subseteq V$, let us now introduce the *Wick polynomials* $:\!(\prod_{v \in V'} \xi_v)\!: \; = (\prod_{v \in V'} \xi_v)^\sharp$ by putting

$$1 + \sum_{\emptyset \subset V' \subseteq V} \Big(\prod_{v \in V'} \xi_v\Big)^\sharp \cdot \nu^{V'} \;:=\; \big(\exp[\xi V]\big)^\sharp \;:=\; \exp[\xi V - S_\xi(\nu)]$$

$$= \frac{\exp[\xi V]}{1 + M_\xi(\nu)} \;=\; \frac{\exp[\xi V]}{\langle \exp[\xi V] \rangle}$$

(see [3, Chapter 3.7] and [15, p. 37]). It is, without doubt, the leitmotiv of this whole article to choose two set functions E_ξ and $\overline{E_\xi}$ such that

$$E_\xi + \overline{E_\xi} \;=\; S_\xi(\nu).$$

The linearity of expectation implies

$$\begin{aligned}\langle \exp[\xi V - E_\xi]\rangle \;&=\; \langle \exp[\xi V]\rangle \cdot \exp[-E_\xi] \;=\; \exp[S_\xi(\nu)] \cdot \exp[-E_\xi] \\ &=\; \exp[\overline{E_\xi}].\end{aligned}$$

Let us suppose finally that the vector $\{\xi_v\}$ $(v \in V)$ follows a Gaußian law with $\langle \xi_v\rangle = 0$ for every $v \in V$ such that the set function $S_\xi(\nu)$ equals zero except on the subsets of V of cardinality two. Let us attach to every edge $\{u, v\}$ of the complete graph $K_n = (V, \binom{V}{2})$ the weight $\langle \xi_u \xi_v\rangle$, and let us choose a partition $E \uplus \overline{E} = \binom{V}{2}$ such that $G = (V, E)$. By putting

$$E_\xi \;:=\; \sum_{\{u,v\}\in E} \langle \xi_u \xi_v\rangle \cdot \nu^{\{u,v\}}, \qquad \overline{E_\xi} \;:=\; \sum_{\{u,v\}\in \overline{E}} \langle \xi_u \xi_v\rangle \cdot \nu^{\{u,v\}},$$

we get indeed $E_\xi + \overline{E_\xi} = S_\xi(\nu)$. If $E = \emptyset$, then our preceding identity implies that $\langle \prod_{v\in V} \xi_v\rangle$ counts the number of perfect matchings of the decorated graph $K_n = \big(V, \binom{V}{2}\big)$. If E is the set of edges both endpoints of which are in the same block of the partition $V = T_1 \uplus \ldots \uplus T_k$, then $\langle \exp[\xi V - E_\xi]\rangle = \exp[\overline{E_\xi}]$ implies that $\langle (\xi_{T_1})^\sharp \cdots (\xi_{T_k})^\sharp\rangle$ counts the number of perfect matchings of the decorated complete multipartite graph $\overline{G} = (V, \overline{E})$ (see Example 1.2). These two results are nothing else but the Formulae 1) and 3) of Section "Diagrams" in [15, Chapter 2.2], respectively.

In the sequel, we will no longer adhere to the probabilistic vision, and we "replace the variance–covariance matrix by a matrix of symmetric bilinear forms."

3. Matching polynomials and duality

Let $a_P G = (aV, a_P E)$ and $\overline{a_P G} = (aV, \overline{a_P E})$ be two decorated complementary graphs (see the Introduction). We can now define the set functions

$$a_P E \;:=\; \sum_{\{u,v\}\in E} \langle a_u, a_v\rangle_P \cdot \nu^{\{u,v\}}, \qquad \overline{a_P E} \;:=\; \sum_{\{u,v\}\in \overline{E}} \langle a_u, a_v\rangle_P \cdot \nu^{\{u,v\}}.$$

We have to pay attention to the fact that, strictly speaking, the expression

$$aV \;:=\; \sum_{v\in V} a_v \cdot \nu^{\{v\}}$$

does not correspond to a set function, because $\mathbf{R}$ is not a ring, but this difficulty disappears as soon as we consider scalar products such as

$$\langle x, aV\rangle_P \;=\; \sum_{v\in V}\langle x, a_v\rangle_P\cdot \nu^{\{v\}}, \qquad \langle aV, aV\rangle_P/2 \;=\; \sum_{\{u,v\}\in\binom{V}{2}}\langle a_u, a_v\rangle_P\cdot \nu^{\{u,v\}}$$

($x \in \mathbf{R}$). The following identity is at the origin of all results of this section.

Fundamental lemma. *We have*

$$\mathbf{a_P E} + \overline{\mathbf{a_P E}} \;=\; \langle \mathbf{aV}, \mathbf{aV}\rangle_\mathbf{P}/\mathbf{2}. \quad \blacksquare$$

For $\emptyset \subset V' \subseteq V$, let us denote by $a_pG[V']$ the decorated subgraph of a_PG induced by V': it is the graph the vertices of which are the elements of V' (with their associated vectors a_v, $v \in V'$) and the edges of which are the edges of a_PG having their two end points in V'. Thus, the set function $\exp[a_PE]$ counts, for every $V' \subseteq V$, the number of perfect matchings of the decorated graph $a_pG[V']$. Therefore the following proposition is an immediate consequence of the definitions of the matching polynomials (see the Introduction).

Proposition 3.1. *We have*

$$1 + \sum_{\emptyset\subset V'\subseteq V}\mu(a_PG[V'], x)\cdot v^{V'} \;=\; \exp[\langle x, aV\rangle_P - a_PE],$$

$$1 + \sum_{\emptyset\subset V'\subseteq V}\overline{\mu}(a_PG[V'], x)\cdot v^{V'} \;=\; \exp[\langle x, aV\rangle_P + a_PE]. \quad \blacksquare$$

The generating functions become ordinary generating functions of exponential type if and only if there is a vector $a \in \mathbf{R}$ such that $a_v = a$, for all $v \in V$, and $G = K_\infty$ (or $G = \overline{K_\infty}$).

Proposition 3.2. *We have*

$$1 + \sum_{n=1}^{\infty}\mu(a_PK_n, x)\cdot V^n/n! \;=\; \exp[\langle x, a\rangle_P\cdot V - \langle a, a\rangle_P\cdot V^2/2],$$

$$1 + \sum_{n=1}^{\infty}\overline{\mu}(a_PK_n, x)\cdot V^n/n! \;=\; \exp[\langle x, a\rangle_P\cdot V + \langle a, a\rangle_P\cdot V^2/2]. \quad \blacksquare$$

Let us start by proving the N-dimensional analogue of Theorem 1.1 of [7, Chapter 1]. A very special case of this theorem served in [18, Chapter 1.1.15], as the main example to underline the usefulness of ordinary generating functions.

PROPOSITION 3.3. *We have*

$$a)\quad \mu(a_P G' \uplus b_P G'', x) \;=\; \mu(a_P G', x) \cdot \mu(b_P G'', x), \qquad a_P G' \;=\; (aV', a_P E'), \quad b_P G'' \;=\; (bV'', b_P E'');$$

$$b)\quad \mu(a_P G, x) \;=\; \mu(a_P G \backslash e, x) - \langle a_u, a_v\rangle_P \cdot \mu(a_P G \backslash uv, x), \quad e = \{u, v\} \in E;$$

$$c)\quad \mu(a_P G, x) \;=\; \langle x, a_v\rangle_P \cdot \mu(a_P G \backslash v, x) - \sum_{\{u,v\}\in E} \langle a_u, a_v\rangle_P \cdot \mu(a_P G \backslash uv, x), \quad v \in V;$$

$$d)\quad \frac{\partial}{\partial x_i}\, \mu(a_P G, x) \;=\; \sum_{v\in V} \langle e_i, a_v\rangle_P \cdot \mu(a_P G \backslash v, x),$$

$$\nabla_P\, \mu(a_P G, x) \;=\; \sum_{v\in V} a_v \cdot \mu(a_P G \backslash v, x),$$

$$\triangle_P\, \mu(a_P G, x) \;=\; \sum_{u\neq v\in V} \langle a_u, a_v\rangle_P \cdot \mu(a_P G \backslash uv, x).$$

The analogue for $\overline{\mu}(a_P G, x)$ *is obtained by replacing all signs* $-$ *by* $+$.

Proof.

$$\begin{aligned} a)\quad & \exp[\langle x, aV' + bV''\rangle_P - a_P E' - b_P E''] \\ &= \exp[\langle x, aV'\rangle_P - a_P E'] \cdot \exp[\langle x, bV''\rangle_P - b_P E'']; \end{aligned}$$

$$\begin{aligned} b)\quad & \exp[\langle x, aV\rangle_P - a_P E] \\ &= \exp[\langle x, aV\rangle_P - a_P(E\backslash e)] \cdot \exp[-\langle a_u, a_v\rangle_P uv] \\ &= \exp[\langle x, aV\rangle_P - a_P(E\backslash e)] \cdot (1 - \langle a_u, a_v\rangle_P uv) \\ &= \exp[\langle x, aV\rangle_P - a_P(E\backslash e)] \;-\; \langle a_u, a_v\rangle_P uv \cdot \exp[\langle x, aV\rangle_P - a_P(E\backslash e)]; \end{aligned}$$

$$\begin{aligned} c)\quad & \partial^v \exp[\langle x, aV\rangle_P - a_P E] \\ &= \exp[\langle x, aV\rangle_P - a_P E] \cdot \partial^v(\langle x, aV\rangle_P - a_P E) \\ &= \exp[\langle x, aV\rangle_P - a_P E] \cdot (\langle x, a_v\rangle_P v - \sum_{\{u,v\}\in E} \langle a_u, a_v\rangle_P uv) \\ &= \langle x, a_v\rangle_P v \cdot \exp[\langle x, aV\rangle_P - a_P E] - \sum_{\{u,v\}\in E} \langle a_u, a_v\rangle_P uv \cdot \exp[\langle x, aV\rangle_P - a_P E]; \end{aligned}$$

$$\begin{aligned} d)\quad \frac{\partial}{\partial x_i}\, \exp[\langle x, aV\rangle_P - a_P E] \;&=\; \exp[\langle x, aV\rangle_P - a_P E] \cdot \frac{\partial}{\partial x_i}\langle x, aV\rangle_P \\ &=\; \langle e_i, aV\rangle_P \cdot \exp[\langle x, aV\rangle_P - a_P E], \end{aligned}$$

$$\nabla_P\, \exp[\langle x, aV\rangle_P - a_P E] \;=\; aV \cdot \exp[\langle x, aV\rangle_P - a_P E],$$

$$\triangle_P\, \exp[\langle x, aV\rangle_P - a_P E] \;=\; \langle aV, aV\rangle_P \cdot \exp[\langle x, aV\rangle_P - a_P E]. \quad \blacksquare$$

Let us leave the preceding recurrences in order to walk into the garden of duality theorems.

Duality theorem for the matching polynomials ($\triangle$).

$$\begin{aligned}\overline{\mu}(\overline{a_P G},x) &= \exp[\triangle_P/2]\cdot\mu(a_P G,x),\\ \mu(\overline{a_P G},x) &= \exp[-\triangle_P/2]\cdot\overline{\mu}(a_P G,x).\end{aligned}$$

Proof. By using the very last identity of the proof of the preceding proposition, we get indeed

$$\begin{aligned}\exp[\langle x,aV\rangle_P+\overline{a_P E}] &= \exp[\langle aV,aV\rangle_P/2]\cdot\exp[\langle x,aV\rangle_P-a_P E]\\ &= \exp[\triangle_P/2]\cdot\exp[\langle x,aV\rangle_P-a_P E].\end{aligned}$$

The differential operator $\exp[-\triangle_P/2]$ is the inverse of $\exp[\triangle_P/2]$. ∎

Duality theorem for the matching polynomials (∇).

$$\begin{aligned}\overline{\mu}(\overline{a_P G},x) &= e^{-\langle x,x\rangle_P/2}\cdot\mu(a_P G,\nabla_P)\cdot e^{\langle x,x\rangle_P/2},\\ \mu(\overline{a_P G},x) &= e^{\langle x,x\rangle_P/2}\cdot\overline{\mu}(a_P G,-\nabla_P)\cdot e^{-\langle x,x\rangle_P/2}.\end{aligned}$$

Proof. By Taylor's theorem we have

$$f(x+a) = \exp[\langle\nabla,a\rangle]\cdot f(x) = \exp[\langle\nabla_P,a\rangle_P]\cdot f(x)$$

for vectors x, a and a formal power series f. It follows

$$\begin{aligned}&\exp[-\langle x,x\rangle_P/2]\cdot\exp[\langle\nabla_P,aV\rangle_P-a_P E]\cdot\exp[\langle x,x\rangle_P/2]\\ &= \exp[-\langle x,x\rangle_P/2]\cdot\exp[-a_P E]\cdot\exp[\langle x+aV,x+aV\rangle_P/2]\\ &= \exp[\langle x,aV\rangle_P+\overline{a_P E}].\end{aligned}$$

The second identity is proved in the same way. ∎

Example 3.1. Let S and T be two disjoint sets. A bipartite graph $G=(S,T;E)$ is called simple if and only if the set of its edges E is a subset of $S\times T$. For such graphs, one defines the complementary bipartite graph by $\overline{G}:=(S,T;\overline{E})$, with $\overline{E}:=(S\times T)\backslash E$. Let us put $N=2$, $A=\mathbb{R}[s,t]$, and let us attach to each $s'\in S$ (respectively $t'\in T$) the vector $a_{s'}:=\binom{0}{1}$ (respectively $a_{t'}:=\binom{1}{0}$). For $x=\binom{s}{t}$, it follows that

$$P = P^{-1} = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad\Rightarrow\quad \langle a_{s'},a_{s'}\rangle_P = \langle a_{t'},a_{t'}\rangle_P = 0,\quad \langle a_{s'},a_{t'}\rangle_P = 1,$$

$$\langle x,a_{s'}\rangle_P = s,\ \ \langle x,a_{t'}\rangle_P = t,\ \ \langle x,x\rangle_P/2 = st,\ \ \nabla_P = \binom{d/dt}{d/ds},\ \ \triangle_P/2 = \frac{d}{ds}\frac{d}{dt}.$$

Therefore the matching polynomials $\mu(a_PG,x)$ and $\overline{\mu}(a_PG,x)$ of the decorated graph a_PG are nothing else but the *symmetric rook polynomials* $\rho(G,s,t)$ and $\overline{\rho}(G,s,t)$, introduced in [13]. More precisely, we have

$$\mu(a_PG,x) \;=\; \sum_{r=0}^{\min(|S|,|T|)} (-1)^r p(G,r)\cdot s^{|S|-r}t^{|T|-r} \;=:\; \rho(G,s,t),$$

$$\overline{\mu}(a_PG,x) \;=\; \sum_{r=0}^{\min(|S|,|T|)} p(G,r)\cdot s^{|S|-r}t^{|T|-r} \;=:\; \overline{\rho}(G,s,t),$$

if $p(G,r)$ is the number of r-matchings of the bipartite graph $G=(S,T;E)$. Since $\langle a_{s'},a_{s'}\rangle_P = \langle a_{t'},a_{t'}\rangle_P = 0$ and $\langle a_{s'},a_{t'}\rangle_P = 1$, for all $s'\in S$ and $t'\in T$, the complementary decorated graph $\overline{a_PG}$ corresponds precisely to the complementary bipartite graph $\overline{G}=(S,T;\overline{E})$, and we have

$$\overline{\mu}(\overline{a_PG},x) \;=\; \overline{\rho}(\overline{G},s,t), \qquad \mu(\overline{a_PG},x) \;=\; \rho(\overline{G},s,t).$$

Consequently, our two duality theorems for the matching polynomials imply

$$\overline{\rho}(\overline{G},s,t) \;=\; \exp\left[\tfrac{d}{ds}\tfrac{d}{dt}\right]\cdot\rho(G,s,t) \;=\; e^{-st}\cdot\rho(G,\tfrac{d}{dt},\tfrac{d}{ds})\cdot e^{st},$$

$$\rho(\overline{G},s,t) \;=\; \exp\left[-\tfrac{d}{ds}\tfrac{d}{dt}\right]\cdot\overline{\rho}(G,s,t) \;=\; e^{st}\cdot\overline{\rho}(G,-\tfrac{d}{dt},-\tfrac{d}{ds})\cdot e^{-st}.$$

These identities are nothing else but Theorems 3.1 and 3.2 of [13].

Let us continue our walk through the garden of duality theorems.

Duality theorem for the matching polynomials ($\int$).

$$\overline{\mu}(\overline{a_PG},x) \;=\; \int_{\mathbf{R}} e^{-\langle z-x,z-x\rangle_P/2}\cdot\mu(a_PG,z)\cdot d_Pz$$
$$\;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbf{R}}\mu(a_PG,z)\cdot\mu(x_PK_n,z)\cdot e^{-\langle z,z\rangle_P/2}\cdot d_Pz.$$

Proof. Using the invariance of the integral with respect to translations, we get:

$$\int_{\mathbf{R}}\exp[-\langle z-x,z-x\rangle_P/2]\cdot\exp[\langle z,aV\rangle_P - a_PE]\cdot d_Pz$$
$$=\;\int_{\mathbf{R}}\exp[-\langle s,s\rangle_P/2]\cdot\exp[\langle s+x,aV\rangle_P - a_PE]\cdot d_Ps$$
$$=\;\exp[\langle x,aV\rangle_P + \overline{a_PE}]\cdot\int_{\mathbf{R}}\exp[-\langle s-aV,s-aV\rangle_P/2]\cdot d_Ps$$
$$=\;\exp[\langle x,aV\rangle_P + \overline{a_PE}]\cdot\int_{\mathbf{R}}\exp[-\langle t,t\rangle_P/2]\cdot d_Pt$$
$$=\;\exp[\langle x,aV\rangle_P + \overline{a_PE}].$$

Let us put $V := 1$, $x := z$ and $a := x$ in Proposition 3.2:

$$1 + \sum_{n=1}^{\infty} \mu(x_P K_n, z)/n! \;=\; \exp[\langle z, x\rangle_P - \langle x, x\rangle_P/2].$$

It follows ($\mu(x_P K_0, z) := 1$):

$$\begin{aligned}
&\int_{\mathbf{R}} e^{-\langle z-x, z-x\rangle_P/2} \cdot \mu(a_P G, z) \cdot d_P z \\
&= \int_{\mathbf{R}} \mu(a_P G, z) \cdot \exp[\langle z, x\rangle_P - \langle x, x\rangle_P/2] \cdot e^{-\langle z,z\rangle_P/2} \cdot d_P z \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbf{R}} \mu(a_P G, z) \cdot \mu(x_P K_n, z) \cdot e^{-\langle z,z\rangle_P/2} \cdot d_P z. \quad \blacksquare
\end{aligned}$$

Since the space of polynomials is generated (as a vector space) by the matching polynomials, the three preceding theorems imply the following corollary (for the identity $\mu(x_P K_n, z) = \Phi(z|x, \ldots, x) = {:}\langle z, x\rangle_P^n{:} = [\langle z, x\rangle_P^n]^\sharp$ see Remark 1.1).

Corollary 3.1. *Let $f(x)$ be an arbitrary polynomial, then we have for the Bargmann-Segal transform $[f(x)]^\flat = f^\flat(x)$ and for the Wick transform $[f(x)]^\sharp = f^\sharp(x)$ the following identities*:

$$\begin{aligned}
f^\flat(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbf{R}} f(z) \cdot \left[\langle z, x\rangle_P^n\right]^\sharp \cdot e^{-\langle z,z\rangle_P/2} \cdot d_P z \\
&= \int_{\mathbf{R}} e^{-\langle z-x, z-x\rangle_P/2} \cdot f(z) \cdot d_P z \\
&= e^{-\langle x,x\rangle_P/2} \cdot f(\nabla_P) \cdot e^{\langle x,x\rangle_P/2} \\
&= \exp[\triangle_P/2] \cdot f(x),
\end{aligned}$$

$$\begin{aligned}
f^\sharp(x) &= e^{\langle x,x\rangle_P/2} \cdot f(-\nabla_P) \cdot e^{-\langle x,x\rangle_P/2} \\
&= \exp[-\triangle_P/2] \cdot f(x). \quad \blacksquare
\end{aligned}$$

Example 3.2. Of course, it is not necessary to restrict ourselves to polynomials. Let us put $f(x) := \sqrt{\det(Q)} \cdot e^{-\langle x,x\rangle_Q/2}$, where $\langle\cdot,\cdot\rangle_Q$ is a second form. The preceding corollary allows us to conclude that

$$\begin{aligned}
\left[\sqrt{\det(Q)} \cdot e^{-\langle x,x\rangle_Q/2}\right]^\flat &= \sqrt{\det(Q)} \cdot \exp[\triangle_P/2] \cdot e^{-\langle x,x\rangle_Q/2} \\
&= \sqrt{\det(Q)} \cdot e^{-\langle x,x\rangle_P/2} \cdot \exp[\triangle_{PQ^{-1}P}/2] \cdot e^{\langle x,x\rangle_P/2} \\
&= \sqrt{\det(R)} \cdot e^{-\langle x,x\rangle_R/2}, \quad R := (P^{-1} + Q^{-1})^{-1}.
\end{aligned}$$

Indeed, we have

$$\begin{aligned}
&\big[\sqrt{\det(Q)}\cdot e^{-\langle x,x\rangle_Q/2}\big]^\flat \\
&= \sqrt{\det(Q)}\cdot\int_{\mathbf{R}} e^{-\langle z-x,z-x\rangle_P/2}\cdot e^{-\langle z,z\rangle_Q/2}\cdot d_P z \\
&= \sqrt{\det(Q)}\cdot\int_{\mathbf{R}} \exp[-\langle z,z\rangle_{P+Q}/2+\langle z,x\rangle_P-\langle x,x\rangle_P/2]\cdot d_P z \\
&= \sqrt{\det(Q)}\cdot\exp[-\langle x,x\rangle_{P-P(P+Q)^{-1}P}/2]\cdot \\
&\quad\sqrt{\tfrac{\det(P)}{\det(P+Q)}}\cdot\int_{\mathbf{R}}\exp\big[-\big\langle z-(P+Q)^{-1}Px,z-(P+Q)^{-1}Px\big\rangle_{P+Q}/2\big]\cdot d_{P+Q}z \\
&= \sqrt{\tfrac{\det(P)\det(Q)}{\det(P+Q)}}\cdot\exp[-\langle x,x\rangle_{P-P(P+Q)^{-1}P}/2]\cdot\int_{\mathbf{R}}\exp[-\langle t,t\rangle_{P+Q}/2]\cdot d_{P+Q}t \\
&= \sqrt{\det(P^{-1}+Q^{-1})^{-1}}\cdot\exp[-\langle x,x\rangle_{(P^{-1}+Q^{-1})^{-1}}/2],
\end{aligned}$$

as $P-P(P+Q)^{-1}P=P(P+Q)^{-1}Q=(Q^{-1}(P+Q)P^{-1})^{-1}=(P^{-1}+Q^{-1})^{-1}$. The identity $\sqrt{\det(Q)}\cdot\exp[\triangle_P/2]\cdot e^{-\langle x,x\rangle_Q/2}=\sqrt{\det(R)}\cdot e^{-\langle x,x\rangle_R/2}$ appears in Mehta's book [16] (see the pages 87–89 there for a different proof).

We remark that polynomials have a decomposition into homogeneous parts, and each homogeneous part is uniquely determined by its values on the ellipsoid S_P^{N-1} given by the equation $\langle x,x\rangle_P=1$. Let f be homogeneous of degree n, i.e., $f(r\cdot\omega)=r^n\cdot f(\omega)$, $\omega\in S_P^{N-1}$, and let $d_P\omega$ be such that $d_Px=c(N)\cdot r^{N-1}\cdot dr\cdot d_P\omega$ and $\int_{S_P^{N-1}}d_P\omega=1$. We have the following corollary.

Corollary 3.2. *Let n be even. Then,*

$$\begin{aligned}
&N\cdot(N+2)\cdot(N+4)\cdots(N+n-2)\cdot\int_{S_P^{N-1}}f(\omega)\cdot d_P\omega \\
&= \int_{\mathbb{R}^N}f(x)\cdot e^{-\langle x,x\rangle_P/2}\cdot d_Px \\
&= f(\nabla_P)\cdot e^{\langle x,x\rangle_P/2}\Big|_{x=0} \\
&= \exp[\triangle_P/2]\cdot f(x)\Big|_{x=0},
\end{aligned}$$

and if $f(x)=\mu(\overline{a_PK_n},x)$, the result is the number $\overline{\mu}(a_PK_n,0)$ of perfect matchings of a_PK_n. ∎

The identities of the preceding corollary are easy consequences of the results of the recent article [5] by Folland. However, they are not stated explicitly there. In addition, we have the impression that our proofs are shorter, although Folland has used ideas of Bargmann and Nelson to simplify the proofs of the article [1], which treats the same subject.

EXAMPLE 3.3. We have

$$\int_{S^{N-1}} \langle x, \alpha\rangle^{2p} \cdot d\omega \;=\; \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2p-1)}{N(N+2)(N+4)\ldots(N+2p-2)} \cdot \langle \alpha, \alpha\rangle^{p},$$

because every perfect matching of the graph αK_{2p} contains p edges (to which we had attached the weight $\langle\alpha,\alpha\rangle$), and the number of perfect matchings of K_{2p} equals $1\cdot 3\cdot 5\cdots(2p-1)$ (see [19, théorème 3.2 and 3.6]). We also have

$$\int_{S^{N-1}} \langle x, \alpha_1\rangle^{2}\langle x, \alpha_2\rangle^{2} \cdot d\omega \;=\; \frac{\langle\alpha_1,\alpha_1\rangle\langle\alpha_2,\alpha_2\rangle + 2\cdot\langle\alpha_1,\alpha_2\rangle^2}{N\cdot(N+2)},$$

because the complete graph $\overline{\overline{\alpha_1 K_2} \uplus \overline{\alpha_2 K_2}}$ contains three perfect matchings: two of weight $\langle\alpha_1,\alpha_2\rangle\langle\alpha_1,\alpha_2\rangle$ and one of weight $\langle\alpha_1,\alpha_1\rangle\langle\alpha_2,\alpha_2\rangle$ (see [19, 6.5]).

EXAMPLE 3.4. By definition, a polynomial $f(x)$ is *harmonic* if and only if $\triangle_P f(x) = 0$. Let $f(x)$ be a homogeneous polynomial of degree $n > 0$ which is harmonic. The preceding corollary allows us to conclude that

$$N(N+2)\cdots(N+n-2)\int_{S_P^{N-1}} f(\omega)\cdot d_P\omega \;=\; \exp[\triangle_P/2]\cdot f(x)\Big|_{x=0} \;=\; f(0) \;=\; 0.$$

Since every polynomial $f(x)$ is harmonic if and only if all its homogeneous components are harmonic, the classical identity

$$\int_{S_P^{N-1}} f(\omega)\cdot d_P\omega \;=\; \int_{S_P^{N-1}} f(0)\cdot d_P\omega \;=\; f(0)$$

for every harmonic polynomial follows immediately.

Let us continue our study of matching polynomials.

SCALAR PRODUCT FORMULA. *For two decorated graphs $a_P G' = (aV', a_P E')$ and $b_P G'' = (bV'', b_P E'')$ we have*

$$\begin{aligned}\overline{\mu}(\overline{a_P G' \uplus b_P G''}, 0) \;&=\; \int_{\mathbf{R}} \mu(a_P G', x)\cdot\mu(b_P G'', x)\cdot e^{-\langle x,x\rangle_P/2}\cdot d_P x \\ &=\; \overline{\mu}(\overline{a_P G'}, \nabla_P)\cdot\overline{\mu}(\overline{b_P G''}, x)\Big|_{x=0}.\end{aligned}$$

Proof. The first identity is an obvious consequence of the case $x = 0$ of the preceding theorem. In order to establish the second identity, we use, once again, Taylor's theorem:

$$\begin{aligned}
&\exp[\langle \nabla_P, aV' \rangle_P + \overline{a_P E'}] \cdot \exp[\langle x, bV'' \rangle_P + \overline{b_P E''}]\Big|_{x=0} \\
&= \exp[\overline{a_P E'}] \cdot \exp[\langle x + aV', bV'' \rangle_P + \overline{b_P E''}]\Big|_{x=0} \\
&= \exp[\langle aV', bV'' \rangle_P + \overline{a_P E'} + \overline{b_P E''}]. \quad \blacksquare
\end{aligned}$$

Corollary 3.3. *Let $f(x)$ and $g(x)$ be two polynomials and let $f^\flat(x)$ and $g^\flat(x)$ be their respective Bargmann-Segal transforms defined with the help of one of the formulae of Corollary 3.1. Then we have*

$$\int_{\mathbf{R}} f(x) \cdot g(x) \cdot e^{-\langle x,x \rangle_P/2} \cdot d_P x \;=\; f^\flat(\nabla_P) \cdot g^\flat(x)\Big|_{x=0}. \quad \blacksquare$$

The duality theorem $(\int)$ implies

$$\mu(\overline{a_P G}, y) \;=\; (-i)^n \cdot \overline{\mu}(\overline{a_P G}, yi) \;=\; (-i)^n \cdot \int_{\mathbf{R}} e^{-\langle x-yi, x-yi \rangle_P/2} \cdot \mu(a_P G, x) \cdot d_P x.$$

$(|V| = n,\ i = \sqrt{-1})$. This proves the following theorem.

Duality theorem for the matching polynomial $(\mathbb{C})$.

$$e^{-\langle y,y \rangle_P/2} \mu(\overline{a_P G}, y) \;=\; (-i)^n \cdot \int_{\mathbf{R}} e^{i\langle x,y \rangle_P} \cdot e^{-\langle x,x \rangle_P/2} \mu(a_P G, x) \cdot d_P x. \quad \blacksquare$$

Since the matching function $e^{-\langle x,x \rangle_P/2}\mu(a_P G, x)$ has the same parity as n, the last theorem of this section follows.

Duality theorem for the matching polynomial $(\mathbb{R})$.

$$\begin{aligned}
&e^{-\langle y,y \rangle_P/2} \mu(\overline{a_P G}, y) \cdot (-1)^{n/2} \\
&= \int_{\mathbf{R}} \cos\langle x, y \rangle_P \cdot e^{-\langle x,x \rangle_P/2} \mu(a_P G, x) \cdot d_P x, \qquad n \quad \textit{even},
\end{aligned}$$

$$\begin{aligned}
&e^{-\langle y,y \rangle_P/2} \mu(\overline{a_P G}, y) \cdot (-1)^{(n-1)/2} \\
&= \int_{\mathbf{R}} \sin\langle x, y \rangle_P \cdot e^{-\langle x,x \rangle_P/2} \mu(a_P G, x) \cdot d_P x, \qquad n \quad \textit{odd}.
\end{aligned}$$

In other words, the matching functions of $\overline{a_P G}$ and of $a_P G$ are, up to sign, real Fourier transforms of each other. $\blacksquare$

REMARK 3.1. Since all our duality theorems rely just on the fundamental lemma $a_P E + \overline{a_P E} = \langle aV, aV\rangle_P/2$, it is natural to associate to each $\{u,v\} \in \binom{V}{2}$ two scalar products $Q(\{u,v\})$ and $\overline{Q}(\{u,v\})$ such that

$$\langle a_u, a_v\rangle_{Q(\{u,v\})} + \langle a_u, a_v\rangle_{\overline{Q}(\{u,v\})} \;=\; \langle a_u, a_v\rangle_P,$$

or just two numbers $q(\{u,v\})$ and $\overline{q}(\{u,v\})$ such that $q(\{u,v\}) + \overline{q}(\{u,v\}) = \langle a_u, a_v\rangle_P$. We have preferred the classical context of the complementary graph because it is sufficient for most applications.

Another way to extend our results consists in replacing the real coefficients of our vectors a_v, $v \in V$, by complex coefficients, for example. Moreover, complex coefficients are admissible for the matrix P under the condition that it remains *symmetric*. In other words, we can have $P = R + i\cdot Q$ ($i = \sqrt{-1}$) with real symmetric matrices R and Q. In this manner, we can see that the two identities of the duality theorems ∇ and $\triangle$ can be reduced to each other by replacing P by $-P$. In order to guarantee the convergence of the (real!) integrals, it is, however, indispensable that $\Re e\ P = R > 0$, i.e., the matrix R must correspond to a scalar product.

4. The zeroes of $\mu(\mathbf{a_p G}, \mathbf{x})$

Let us admit weights $w_{uv} \in \mathbb{R}\backslash\{0\}$ (in [9], the condition $w_{uv} > 0$ was imposed) for every edge $\{u,v\} \in E$, such that it contributes the factor $w_{uv}\langle a_u, a_v\rangle_P$ (as an edge of a matching) instead of $\langle a_u, a_v\rangle_P$, and let us denote by $a_P G_w = (aV, a_P E_w)$ the graph with these weights.

Let $\mathrm{div}[\mu(a_p G_w, x)] \in \mathrm{Div}(\mathbf{R})$ be the (effective) divisor of $\mu(a_P G_w, x)$, but we consider it in the projective space $\mathbb{P}\mathbf{R}$ ($= \mathbb{P}^N(\mathbb{R})$; unless we consider it even in $\mathbb{P}^N(\mathbb{C})$). The generating function of the homogenized matching polynomials $\mu(a_P G_w, x, x_0)$ can be written in the form

$$1 + \sum_{\emptyset \subset V' \subseteq V} \mu(a_P G_w[V'], x, x_0)\cdot v^{V'} \;=\; \exp[\langle x, aV\rangle_P - x_0^2 a_P E_w].$$

In the so-called improper hyperplane $x_0 = 0$, we have

$$\mu(a_P G_w, x, 0) \;=\; 0 \qquad \Leftrightarrow \qquad \prod_{v\in V}\langle x, a_v\rangle_P \;=\; 0.$$

Since $\mu(a_P G_w, x, x_0)$ is linear in every a_v, $v \in V$, we can (and we will) suppose for a moment (without changing $\mathrm{div}[\mu(a_P G_w, x, x_0)]$) that $\langle a_v, a_v\rangle_P = 1$ for every $v \in V$ (if there exists a $v \in V$ with $a_v = 0$, then $\mu(a_P G_w, x, x_0) \equiv 0$) and that there is no $u \neq v$ with $a_u = -a_v$ (we could consider the a_v's, $v \in V$, as points of $\mathbb{P}^{N-1}(\mathbb{R})$). Let

$$A \;=\; \{a_v | v \in V\}, \qquad \text{and let} \quad V \;=\; \biguplus_{a\in A} V_a$$

such that, for every $a \in A$, we have $a_v = a$ for each $v \in V_a$. Then,

$$\operatorname{div}[\mu(a_P G_w, x, 0)] \ = \ \sum_{a \in A} |V_a| \cdot V(\langle x, a \rangle_P).$$

Let $a^* \in A$, and let us consider $x^* \in \mathbf{R} \backslash \{\mathbf{0}\}$ such that $\langle x^*, a^* \rangle_P = 0$ and $\langle x^*, a \rangle_P \neq 0$ for each $a \in A \backslash a^*$. The Taylor expansion of $\mu(a_P G_w, x, x_0)$ at $(x^*, 0)$ up to the order $|V_{a^*}|$ can be written in the form

$$\mu(a_P G_w[V_{a^*}], x, x_0) \cdot \prod_{v \notin V_{a^*}} \langle x^*, a_v \rangle_P.$$

This implies the following theorem.

Theorem 4.1. *If, for every $a \in A$, the one dimensional matching polynomial $\mu(G_w[V_a], x)$ has $|V_a|$ different real zeroes (for example, because all weights of the edges of the graph $G_w[V_a]$ are positive and $G_w[V_a]$ contains a Hamiltonian path, see [9], Theorem 4.2), and if $V_i \subset \mathbf{R}$ are the (irreducible) components of $\operatorname{div}[\mu(a_P G_w, x)]$, then*

$$\sum_i \deg(V_i) \ = \ \deg[\mu(a_P G_w, x)] \ = \ |V|. \quad \blacksquare$$

Remark 4.1. It is not possible to extend this theorem by continuity as in one dimension, because ($\mathbb{R}$ is closed in $\mathbb{C}$ but) $\mathbf{R}$ is not closed in $\mathbb{P}\mathbf{R}$. More subtle conditions can be obtained by continuing the Taylor expansion (see the very last theorem of [12]), but it seems preferable to present them in a different context.

Theorem 4.2. *Every zero of $\mu(a_P G_w, x)$ is close to at least one hyperplane $V(\langle x, a_v \rangle_P)$, $v \in V$. More precisely, let $|a_P G_w|$ be the graph obtained by replacing all weights $w_{uv} \langle a_u, a_v \rangle_P$ of the edges $\{u, v\}$ by $|w_{uv} \langle a_u, a_v \rangle_P|$, let B be bigger than all zeroes of the one dimensional matching polynomial $\mu(|a_P G_w|, x)$ (see [9], Theorem 4.3, for good estimations of B), and suppose that $|\langle x, a_v \rangle_P| \geq B$ for every $v \in V$. Then, for each $v \in V' \subseteq V$, we have*

$$\left| \frac{\mu(a_P G_w[V'], x)}{\mu(a_P G_w[V' \backslash v], x)} \right| \ \geq \ \frac{\mu(|a_P G_w[V']|, B)}{\mu(|a_P G_w[V' \backslash v]|, B)}$$

and, in particular, $\mu(a_P G_w[V'], x) \neq 0$.

Proof. The proof (by induction) is the same as that of Theorem 4.5 in [9]. $\blacksquare$

Since affine regular polygons do not enjoy such a great popularity, we will put (without loss of generality!) $P = \mathrm{Id}$ (and $N = 2$) for a moment. Let $w : \mathbb{Z}/n\mathbb{Z} \to \mathbb{R}$ be such that $w(k) = w(-k)$ for every $k \in \mathbb{Z}/n\mathbb{Z}$, $w(0) = 0$, and let $G_w = (\mathbb{Z}/n\mathbb{Z}, E_w)$ be the graph with $\{i, j\} \in E_w$ if and only if $w(i-j) \neq 0$, and, in that case, $w_{ij} := w(i-j)$, $i, j \in \mathbb{Z}/n\mathbb{Z}$. If we denote by aG_w the decoration of G_w given by

$$a_k \ := \ \begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} \cos(k \cdot 2\pi/n) \\ \sin(k \cdot 2\pi/n) \end{pmatrix},$$

for every $k \in \mathbb{Z}/n\mathbb{Z}$, then we have the following surprise.

Proposition 4.1. *We have*

$$\mu(aG_w, x, y) \ = \ \prod_{k=0}^{n-1} \big\langle \tbinom{x}{y}, a_k \big\rangle, \qquad \textit{if } n \textit{ is odd.}$$

Proof. We have indeed

$$\begin{aligned} \mu(aG_w, 0, y) \ &= \ - \sum_{\{k,0\} \in E_w} w(k) \cdot \langle a_k, a_0 \rangle \cdot \mu(aG_w \backslash \{k, 0\}, 0, y) \\ &= \ 0, \end{aligned}$$

since the symmetry $k \longleftrightarrow -k$ preserves $w_{ij} \langle a_i, a_j \rangle$ and changes the sign of $\langle \binom{0}{y}, a_k \rangle$, and since the number of those last expressions is odd for every matching. It follows that $\mu(aG_w, x, y) = x \cdot p(x, y)$, and, by symmetry, $\mu(aG_w, x, y) = \prod_{k=0}^{n-1} \langle \binom{x}{y}, a_k \rangle \cdot q(x, y)$. The proof is completed by comparing the terms of degree n. ∎

If n is even, the situation becomes more complicated (because $a_k = -a_{k+n/2}$). We restrict ourselves to the case where there is just a single $(d, -d) \neq (0, 0)$ with $w := w(d) = w(-d) \neq 0$. In that case, aG_w is the disjoint union of $\gcd(n, d)$ cycles, and $\mu(aG_w, x, y)$ is the product of their matching polynomials. Therefore we obtain a more readable result if we suppose $\gcd(n, d) = 1$.

Proposition 4.2. *Let n be even, $n > 2$, and* $\gcd(n, d) = 1$. *Then*

$$\begin{aligned} \mu(aG_w, x, y) \ &= \ \prod_{k=0}^{n-1} \big\langle \tbinom{x}{y}, a_k \big\rangle \quad + \quad 2 \cdot (-w \cdot \langle a_d, a_0 \rangle)^{n/2} \\ &= \ (-1)^{n/2} \left[\prod_{k=0}^{n/2-1} \big\langle \tbinom{x}{y}, a_k \big\rangle^2 \quad + \quad 2 \cdot [w \cdot \cos(d \cdot 2\pi/n)]^{n/2} \right]. \end{aligned}$$

Proof. As in the preceding proof, we use a sign-reversing involution. We have

$$\begin{aligned}\mu(aG_w,0,y) &= -w\cdot\langle a_d,a_0\rangle\cdot[\mu(aG_w\backslash\{d,0\},0,y)+\mu(aG_w\backslash\{-d,0\},0,y)]\\ &= 2\cdot(-w\cdot\langle a_d,a_0\rangle)^{n/2}.\end{aligned}$$

Indeed, let us consider a matching of the cycle aG_w, and let us walk, starting from 0, in both directions of aG_w (at the same time and with the same speed!) until there will be the first vertex not covered by the matching, and let us apply the symmetry $k \longleftrightarrow -k$ for all vertices visited so far. Since there is just a single expression of the type $\langle\binom{0}{y},a_k\rangle$ (for the uncovered vertex), the sign changes, and, finally, there will just remain the contributions of the two perfect matchings of aG_w. It follows that $\mu(aG_w,x,y)-2\cdot(-w\cdot\langle a_d,a_0\rangle)^{n/2}=x\cdot p(x,y)$. The symmetry $\mu(aG_w,-x,y)=\mu(aG_w,x,y)$ implies $p(-x,y)=-p(x,y)$, i.e. $p(x,y)=x\cdot q(x,y)$; and, once again, by symmetry, we obtain

$$\mu(aG_w,x,y)-2\cdot(-w\cdot\langle a_d,a_0\rangle)^{n/2} = \prod_{k=0}^{n/2-1}\left\langle\binom{x}{y},a_k\right\rangle^2 \quad\cdot\quad r(x,y).$$

The proof is completed by comparing the terms of degree n. ∎

The main reason for having considered the preceding propositions is the following highly surprising corollary.

Corollary 4.1. *For every $N>1$ and every P there exist ordinary (i.e., without weights) decorated graphs a_PG (rarissime!) such that $\mu(a_PG,x)\neq 0$ for every $x\in\mathbf{R}$.* ∎

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