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Algèbres de Lie/*Lie Algebras*
(Combinatoire/*Combinatorics*)

On the combinatorics of the graph-complex

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Abstract. In their recent preprint [3] Kontsevich and Shoikhet have introduced two graph-complexes: the complex on the even (resp. odd) space in order to study the cohomology of the Lie algebra Ham_0 (resp. Ham_0^{odd}) of Hamiltonian vector fields vanishing at the origin on the infinite-dimensional even (resp. odd) space. We construct an isomorphism between those two graph-complexes, proving in particular that their cohomologies coincide. This solves a problem posed by Shoikhet. ⓒ Académie des Sciences/Elsevier, Paris

Sur la combinatoire du complexe de graphes

Résumé. *Dans leur prépublication récente [3], Kontsevich et Shoikhet ont introduit deux complexes de graphes : le complexe sur l'espace pair (resp. impair) pour étudier la cohomologie de l'algèbre de Lie Ham_0 (resp. Ham_0^{odd}) des champs vectoriels hamiltoniens sans terme constant sur l'espace pair (resp. impair) de dimension infinie. Nous construisons un isomorphisme entre ces deux complexes de graphes, démontrant notamment que leur cohomologies coïncident. Ceci résoud un problème posé par Shoikhet.* ⓒ Académie des Sciences/Elsevier, Paris

Version française abrégée

Le problème de calculer les groupes de cohomologie de beaucoup d'algèbres de Lie peut être ramené au calcul des groupes de cohomologie de différents complexes de graphes (voir [1],[2],[3]). Un *complexe (cohomologique) de graphes* est un complexe

$$0 \longrightarrow \mathbb{C} \xrightarrow{\;0\;} C^1 \xrightarrow{\;\delta\;} C^2 \xrightarrow{\;\delta\;} C^3 \xrightarrow{\;\delta\;} \ldots$$

où C^n est l'espace vectoriel engendré par les *classes d'isomorphie* de paires (G, or_G) telles que G est un (multi)graphe de n sommets et or_G est une *orientation* d'un espace vectoriel réel (ou plutôt d'un ensemble fini) associé à G. Un *(multi)graphe* $G = (H; V, E)$ est un ensemble de *demi-arêtes* H ($|H| = 2m$) muni de deux partitions V et E. Les n blocs $v \in V$ sont appelés *sommets* tandis que les m blocs $e \in E$ de la deuxième partition sont des sous-ensembles de H de cardinalité deux,

Note présentée par Michel DUFLO.

appelés *arêtes*. Une arête est une *boucle* si et seulement si ses deux demi-arêtes font partie du même sommet. Le *degré* d'un sommet $v \in V$ est sa cardinalité, c'est-à-dire le nombre de ses demi-arêtes. On dit que v est *pair* (resp. *impair*) si et seulement si son degré est pair (resp. impair). Notons Q (resp. O) l'ensemble des sommets pairs (resp. impairs) de G.

On *oriente* un ensemble fini (ou bien l'espace réel engendré par cet ensemble) en le munissant d'une *forme alternante* (multilinéaire) $[\ \cdot\ ]$ prenant ses valeurs dans l'ensemble $\{1, -1\}$. Évidemment, le nombre d'orientations différentes vaut deux. L'union disjointe de plusieurs ensembles orientés est naturellement *pseudo-orientée* (i.e. orientée par rapport à la partition) par le produit ordinaire des formes alternantes des blocs de la partition. Une *pseudo-orientation* d'un ensemble partitionné est alors une forme alternante par rapport à toutes les permutations respectant la partition. Évidemment, le nombre de pseudo-orientations est aussi égal à deux. L'orientation d'un graphe dite *paire* est la pseudo-orientation de l'union disjointe $V \uplus (\biguplus_{e \in E} e)$; autrement dit, on oriente l'ensemble V et, pour chaque arête $e \in E$, l'ensemble des deux demi-arêtes de e (voir [2],[3],[4]). L'orientation d'un graphe dite *impaire*, cependant, est la pseudo-orientation de l'union disjointe $Q \uplus (\biguplus_{v \in V} v)$, ce qui veut dire que l'on oriente l'ensemble Q des sommets pairs et, pour chaque sommet $v \in V$, l'ensemble des demi-arêtes de v (voir [3]).

Un *automorphisme* P du graphe G est une permutation P_H de l'ensemble des demi-arêtes H préservant les structures de V et de E. Voilà pourquoi P agit sur l'orientation paire (resp. impaire) par une multiplication avec un nombre noté $\Theta_{even}(P) \in \{1, -1\}$ (resp. $\Theta_{odd}(P)$). Pour la permutation T des deux demi-arêtes d'une boucle, on a, par exemple, $\Theta_{even}(T) = \Theta_{odd}(T) = -1$, puisque l'orientation d'une seule arête (pour l'orientation paire) et d'un seul sommet (pour l'orientation impaire) est renversée. De façon plus générale, nous démontrerons pour chaque automorphisme P que $\Theta_{even}(P) = \Theta_{odd}(P) = \mathrm{sign}(P_H) \cdot \mathrm{sign}(P_V)$, si P_V désigne la permutation induite sur l'ensemble des sommets de G. On impose la relation $(G, -\mathrm{or}_G) = -(G, \mathrm{or}_G)$, de sorte que $(G, \mathrm{or}_G) = 0$ si et seulement si G a un automorphisme tel que $\Theta_{even}(P) = \Theta_{odd}(P) = -1$, comme dans le cas des boucles (d'où $C^1 = 0$). Par conséquent, les générateurs du complexe de graphes sont (à une multiplication par ± 1 près) les mêmes pour l'orientation paire et impaire. En fait, nous montrerons que ces deux orientations correspondent de façon naturelle à une orientation ordinaire de l'ensemble $H \uplus V$, que nous appelons *universelle*.

Cette correspondance permet également de comparer les différentielles, définies par Kontsevich et Shoikhet [3] de la manière suivante : Pour $\delta_{even}(G)$ et $\delta_{odd}(G)$ on prend toujours la somme sur *tous* les « gonflages » de *tous* les sommets de G, où un gonflage d'un sommet veut dire que l'ensemble de ses demi-arêtes est partitionné (de façon non-ordonnée) en deux blocs (ce sont des sommets nouveaux), qui sont reliés par une arête nouvelle (c'est-à-dire chaque bloc nouveau obtient une demi-arête nouvelle, et ces deux demi-arêtes nouvelles forment une arête). Évidemment, chaque gonflage augmente le nombre de sommets d'une unité. Soit $G = (H; V, E)$ avec $V = \{v_1, v_2, \ldots, v_n\}$; $E = \{e_1, \ldots, e_m\}$; $v_1 = \{h_{a_1}, \ldots, h_{x_1}, h_{y_1}, \ldots, h_{b_1}\}$, $v_2 = \{h_{a_2}, \ldots, h_{b_2}\}$, $\ldots$, $v_n = \{h_{a_n}, \ldots, h_{b_n}\}$; $e_1 = \{h_{i_1}, h_{t_1}\}$, $\ldots$, $e_m = \{h_{i_m}, h_{t_m}\}$; et $H = \biguplus_{k=1}^{n} v_k = \biguplus_{k=1}^{m} e_k$. Considérons un gonflage de v_1 avec $v_1' = \{h_{a_1}, \ldots, h_{x_1}\}$, $v_1'' = \{h_{y_1}, \ldots, h_{b_1}\}$ et avec la nouvelle arête $\{h_1', h_1''\}$. Pour l'orientation, Kontsevich et Shoikhet [3] définissent alors

$$[v_1', v_1'', v_2, \ldots, v_n] \cdot [h_1', h_1''] \prod_{k=1}^{m} [h_{i_k}, h_{t_k}] \ := \ [v_1, v_2, \ldots, v_n] \cdot \prod_{k=1}^{m} [h_{i_k}, h_{t_k}] \quad \text{sur l'espace pair et}$$

$$[v_1', v_1'', v_2, \ldots, v_n]_Q \cdot [h_1', h_{a_1}, \ldots, h_{x_1}][h_1'', h_{y_1}, \ldots, h_{b_1}] \prod_{k=2}^{n} [h_{a_k}, \ldots, h_{b_k}]$$

$$:= \ [v_1, v_2, \ldots, v_n]_Q \cdot \prod_{k=1}^{n} [h_{a_k}, \ldots, h_{b_k}] \quad \text{sur l'espace impair,}$$

où $[\,\cdot\,]_Q$ signifie qu'il faut supprimer tous les sommets impairs entre les crochets. Nous démontrons le théorème suivant ($o := |O|$, $n := |V|$).

THÉORÈME. - *Du point de vue de l'orientation universelle, il suffit de multiplier l'orientation de chaque graphe G par $(-1)^{n-o/2}$ pour obtenir un isomorphisme entre l'orientation paire et impaire qui est compatible avec les différentielles. C'est pourquoi le complexe sur l'espace pair et le complexe sur l'espace impair sont isomorphes.*

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1. Introduction

The underlying vector space of the Lie algebra $\mathrm{Ham}_0(N)$ of Hamiltonian vector fields vanishing at the origin *on the even space* is equal to $\bigoplus_{d\geq 2} S^dV$, where V is the defining $2N$-dimensional representation of $\mathrm{sp}(2N)$, $S^2V = \mathrm{sp}(2N)$, etc. It is classical (see [1] and [2]) that the cochain complex

$$C^\bullet_{Lie}(\mathrm{Ham}_0(N);\mathbb{C}) = \bigoplus_{d_2\geq 0,\, d_3\geq 0,\,\dots} \bigwedge\nolimits^{d_2}(S^2V)\wedge\bigwedge\nolimits^{d_3}(S^3V)\wedge\cdots$$

is quasi-isomorphic to its sp-invariant part $[C^\bullet_{Lie}(\mathrm{Ham}_0(N);\mathbb{C})]^{sp}$. Let us consider $\bigwedge^{d_2}(S^2V)\wedge\bigwedge^{d_3}(S^3V)\wedge\cdots$ as a quotient space of $(V^{\otimes 2})^{\otimes d_2}\otimes(V^{\otimes 3})^{\otimes d_3}\otimes\cdots$. If $2d_2+3d_3+\cdots =: 2m$ is even, then any perfect matching on $H := \{1,2,\dots,2m\}$ defines a (multi)graph $G=(H;V,E)$ with $2m$ half-edges in H, m edges in E and $n := d_2+d_3+\cdots$ vertices in V, d_2 of which have degree 2, d_3 of which have degree 3, etc. There exists a canonical symplectic invariant in $\bigwedge^2 V\hookrightarrow V^{\otimes 2}$, that can be chosen in $V_i\otimes V_j$ for every edge $\{i,j\}\in E$ ($i,j\in H$) to obtain an invariant in $V^{\otimes 2m}$. By the main theorem of invariant theory those tensors form a basis of $[V^{\otimes 2m}]^{sp}$ and there are no invariants for $(2d_2+3d_3+\cdots)$ odd, if $2N=\dim(V)$ is sufficiently large. The (multi)graphs $G=(H;V,E)$ carry the *even orientation* corresponding to the exterior product over the set of vertices (in the definition of the cochain complex) and to the exterior products over the set of half-edges of any edge (in the definition of the canonical symplectic invariants). In fact, the consideration of those oriented graphs modulo the action of symmetric groups provides a bijection with the elements of the complex $[C^\bullet_{Lie}(\mathrm{Ham}_0(N);\mathbb{C})]^{sp}$. The differentials of this complex can be easily expressed in graph-theoretical terms (see Section 3). Therefore the cohomology of the resulting *graph-complex on the even space* is equal to the cohomology of Hamiltonian vector fields vanishing at the origin on the infinite-dimensional even space.

The underlying vector space of the Lie algebra $\mathrm{Ham}_0^{odd}(N)$ of Hamiltonian vector fields vanishing at the origin *on the odd space* is equal to $\bigoplus_{d\geq 2}\bigwedge^d V$, where V is the defining N-dimensional representation of $\mathrm{sp}^{odd}(N)=\mathfrak{o}(N)$, $\bigwedge^2 V=\mathrm{sp}^{odd}(N)$, etc. The main difference from the even case is that the canonical invariant lies in $S^2V\hookrightarrow V^{\otimes 2}$ and that the cochain complex is given by

$$C^\bullet_{Lie}(\mathrm{Ham}_0^{odd}(N);\mathbb{C}) = \bigoplus_{d_2\geq 0,\, d_3\geq 0,\,\dots} \bigwedge\nolimits^{d_2}_{super}(\bigwedge\nolimits^2 V)\wedge_{super}\bigwedge\nolimits^{d_3}_{super}(\bigwedge\nolimits^3 V)\wedge_{super}\cdots,$$

where $\wedge_{super}$ denotes the graded exterior product (which is anticommutative if the degrees of both factors are even and commutative otherwise). Therefore the cohomology of Hamiltonian vector fields vanishing at the origin on the infinite-dimensional odd space is equal to the cohomology of the *graph-complex on the odd space*, which differs from the graph-complex on the even space just by its orientation. In fact, the (multi)graphs $G=(H;V,E)$ carry the *odd orientation* corresponding to the graded exterior product over the set of vertices and to the exterior products over the set of half-edges of any vertex, and the differentials of the graph-complex on the odd space can be expressed in graph-theoretical terms with the help of the odd orientation.

With respect to this situation Shoikhet suggested that *"you obtain at first look a different graph-complex, but our idea with Kontsevich was that the two cohomologies coincide - and to prove it by something like an isomorphism of orientations. ... So, up to me, a really interesting problem in this direction is to prove that the even and the odd graph-complexes give the same orientation."*

We solve this problem in the next two sections using still another orientation, which we call *universal* and which is more classical from the point of view of algebraic topology.

2. The generators of the graph complexes on the even space and on the odd space

A (multi)graph $G = (H; V, E)$ is a set of *half-edges* H consisting of $2m$ elements, together with two partitions of this set into disjoint unions of subsets. The n blocks of the first partition $v \in V$ are called *vertices* whereas the m blocks of the second partition $e \in E$ are two-element subsets of H called *edges*. The *degree* of a vertex $v \in V$ is the number of its half-edges and v is called *even* (resp. *odd*) iff its degree is even (resp. odd). Let $[\,\cdot\,]$ be an alternating (multilinear) form on (the real space generated by) the disjoint union of all half-edges $h_1, h_2, \ldots, h_{2m}$ and all vertices $v_1, v_2, \ldots, v_n$. The form $[\,\cdot\,]$ is uniquely defined by determining the value $[h_1, h_2, \ldots, h_{2m}; v_1, v_2, \ldots, v_n] = [v_1, v_2, \ldots, v_n; h_1, h_2, \ldots, h_{2m}] \in \{1, -1\}$. A (multi)graph with such a form will be called *universally oriented*. In order to determine $[h_1, h_2, \ldots, h_{2m}; v_1, v_2, \ldots, v_n]$ we can write the elements $h_1, h_2, \ldots, h_{2m}; v_1, v_2, \ldots, v_n$ in different orders, all of which will start with the half-edges.

One natural order makes use of the block structure determined on the set of half-edges H by the set of edges E, i.e. we consider $[h_{i_1}, h_{t_1}; h_{i_2}, h_{t_2}; \ldots; h_{i_m}, h_{t_m}; v_{j_1}, v_{j_2}, \ldots, v_{j_n}]$, where $\{h_{i_1}, h_{t_1}\}$, $\{h_{i_2}, h_{t_2}\}$, ..., $\{h_{i_m}, h_{t_m}\}$ are the m edges of the graph G. Permuting whole edges does not change the value of $[h_{i_1}, h_{t_1}; h_{i_2}, h_{t_2}; \ldots; h_{i_m}, h_{t_m}; v_{j_1}, v_{j_2}, \ldots, v_{j_n}]$ whereas permuting two half-edges within an edge does change this value. Therefore, orienting a graph universally is canonically the same as orienting the set V of vertices and orienting, for every edge $e \in E$, the set of half-egdes of e. This corresponds to the orientation in the case of the graph-complex on the even space (see [2],[3],[4]).

Another natural order makes use of the block structure determined on the set of half-edges H by the set Q ($|Q| = q$) of even vertices and by the set O ($|O| = o$) of odd vertices. In other words, we consider

$$(*) \quad w := [h_{a_1}, \ldots, h_{b_1}; \ldots; h_{a_q}, \ldots, h_{b_q}; h_{c_1}, \ldots, h_{d_1}; \ldots; h_{c_o}, \ldots, h_{d_o}; v_{f_1}, \ldots, v_{f_o}; v_{g_1}, \ldots, v_{g_q}],$$

where $\{h_{a_1}, \ldots, h_{b_1}\}$, ..., $\{h_{a_q}, \ldots, h_{b_q}\}$ as well as $v_{g_1}, \ldots, v_{g_q}$ are the q even vertices of G and where $v_{f_1} = \{h_{c_1}, \ldots, h_{d_1}\}$, ..., $v_{f_o} = \{h_{c_o}, \ldots, h_{d_o}\}$ are its o odd vertices. Permuting whole even vertices (i.e. their sets of half-edges) does not change the value of w. But permuting whole odd vertices (i.e. their sets of half-edges) does not change w either, because in that case the vertices $v_{f_1}, \ldots, v_{f_o}$ have to be permuted in the same way by our convention. On the other hand, permutations within whole vertices (i.e. within their sets of half-edges) as well as permutations of $Q = \{v_{g_1}, \ldots, v_{g_q}\}$ do change w. Therefore, orienting a graph universally is equivalent to orienting the set Q of even vertices and for every vertex $v \in V$ orienting the set of half-egdes of v. This corresponds to the orientation in the case of the graph-complex on the odd space (see [3]).

An *automorphism* P of the graph G is an arbitrary permutation P_H of the set H which preserves the structures of V and E. In particular, it induces permutations P_Q of Q, P_O of O and P_V of the set V of vertices. The automorphism P acts on the even (resp. odd) orientation by a multiplication by

$$\Theta_{even}(P) = \operatorname{sign}(P_V) \cdot \prod_{e \in E} \epsilon_P(e), \quad \Theta_{odd}(P) = \operatorname{sign}(P_Q) \cdot \prod_{v \in V} \epsilon_P(v), \quad \text{respectively,}$$

where $\epsilon_P(e)$ equals 1 (resp. -1) if and only if P is sign-preserving (resp. sign-reversing) on (the set of half-edges of) e, and where $\epsilon_P(v)$ equals 1 (resp. -1) if and only if P is sign-preserving (resp. sign-reversing) on (the set of half-edges of) v. Finally, P acts on the universal orientation by a multiplication by $\mathrm{sign}(P_H) \cdot \mathrm{sign}(P_V)$. Now our established correspondence between those three orientations implies:

$$\Theta_{even}(P) \;=\; \Theta_{odd}(P) \;=\; \mathrm{sign}(P_H) \cdot \mathrm{sign}(P_V).$$

The n-dimensional generators of the different graph-complexes are always *equivalence classes* of pairs (G, or_G), where G is a graph with n vertices (whose degrees are all greater than one), and where or_G is one of our orientations. One imposes the relation $(G, -\mathrm{or}_G) = -(G, \mathrm{or}_G)$, so that $(G, \mathrm{or}_G) = 0$ if and only if G has an automorphism whose sign is -1. We have established the following theorem.

THEOREM 1. - *There is a canonical correspondence between the generators of the graph-complex on the even space, the generators of the graph-complex on the odd space and the generators of the graph-complex using the universal orientation.* $\square$

3. The differentials of the graph complexes on the even space and on the odd space

In order to get an isomorphism of complexes we have to verify whether our canonical correspondences between the generators are compatible with the differentials. Let us first define a differential for the universal orientation. It is essentially given by the dual of the operation of contracting each edge. More precisely, some vertex $v_s \in V$ has to be split up into two vertices v_s' and v_s'', i.e. the set of its half-edges has to be partitioned. Moreover, a new edge $\{h_s', h_s''\}$ will be introduced, where $h_s' \in v_s'$ and $h_s'' \in v_s''$. Finally, the orientation can be defined by

$$\begin{aligned} &[h_1, h_2, \ldots, h_{2m}, h_s', h_s''; v_1, v_2, \ldots, v_{s-1}, v_s', v_s'', v_{s+1}, \ldots, v_n] \\ &:= \; (-1)^{s-1}[h_1, h_2, \ldots, h_{2m}; v_1, v_2, \ldots, v_{s-1}, v_s, v_{s+1}, \ldots, v_n]. \end{aligned}$$

It is evident that this definition is well defined and leads in fact to a differential δ (for the differential, the sum over all splittings is taken; if we split twice in the same way but in a different order, then the corresponding terms cancel, i.e. $\delta^2 = 0$).

The expression of the definition in terms of the block structure determined by the set of edges of the graph G is immediate. It leads precisely to the definition of [3] (and [2]) in the case of the graph-complex on the even space (corresponding to the cochain complex $C^\bullet_{Lie}(\mathrm{Ham}_0; \mathbb{C})$).

In order to find the expression for the definition in terms of the block structure determined by the set of vertices of G, it is sufficient to consider what happens to w (see equation $(*)$ in the preceding section) if we partition v_{f_1} or v_{g_1} (these vertices are both on odd places because the number o of odd vertices of G is even). First suppose that $v_{f_1} = \{h_{c_1}, \ldots, h_{x_1}, h_{y_1}, \ldots, h_{d_1}\}$ and that the half-edges of v_{f_1} are partitioned into the blocks $v_{f_1}' = \{h_{f_1}', h_{c_1}, \ldots, h_{x_1}\}$ and $v_{f_1}'' = \{h_{f_1}'', h_{y_1}, \ldots, h_{d_1}\}$, where $\{h_{f_1}', h_{f_1}''\}$ is the new edge. If $|v_{f_1}'|$ is odd and $|v_{f_1}''|$ is even, then, according to our universal orientation, w is transformed into

$$\begin{aligned} &[h_{a_1}, \ldots, h_{b_1}; \ldots; h_{a_q}, \ldots, h_{b_q}; h_{f_1}', h_{f_1}''; h_{c_1}, \ldots, h_{x_1}; h_{y_1}, \ldots, h_{d_1}; h_{c_2}, \ldots, h_{d_2}; \ldots; h_{c_o}, \ldots, h_{d_o}; \\ &v_{f_1}', v_{f_1}''; v_{f_2}, \ldots, v_{f_o}; v_{g_1}, \ldots, v_{g_q}] \;= \\ &[h_{a_1}, \ldots, h_{b_1}; \ldots; h_{a_q}, \ldots, h_{b_q}; h_{f_1}'', h_{y_1}, \ldots, h_{d_1}; h_{f_1}', h_{c_1}, \ldots, h_{x_1}; h_{c_2}, \ldots, h_{d_2}; \ldots; h_{c_o}, \ldots, h_{d_o}; \\ &v_{f_1}', v_{f_2}, \ldots, v_{f_o}; v_{f_1}'', v_{g_1}, \ldots, v_{g_q}] \cdot (-1). \end{aligned}$$

If $|v'_{f_1}|$ is even and $|v''_{f_1}|$ is odd, we get

$$[h_{a_1},\ldots,h_{b_1};\ldots;h_{a_q},\ldots,h_{b_q};h'_{f_1},h_{c_1},\ldots,h_{x_1};h''_{f_1},h_{y_1},\ldots,h_{d_1};h_{c_2},\ldots,h_{d_2};\ldots;h_{c_o},\ldots,h_{d_o};$$
$$v''_{f_1},v_{f_2},\ldots,v_{f_o};v'_{f_1},v_{g_1},\ldots,v_{g_q}]\cdot(-1).$$

Now suppose that $v_{g_1}=\{h_{a_z},\ldots,h_{x_z},h_{y_z},\ldots,h_{b_z}\}$ and that the half-edges of v_{g_1} are partitioned into the blocks $v'_{g_1}=\{h'_{g_1},h_{a_z},\ldots,h_{x_z}\}$ and $v''_{g_1}=\{h''_{g_1},h_{y_z},\ldots,h_{b_z}\}$. If $|v'_{g_1}|$ and $|v''_{g_1}|$ are odd, then

$$[h_{a_1},\ldots,h_{b_1};\ldots;h'_{g_1},h''_{g_1};h_{a_z},\ldots,h_{x_z};h_{y_z},\ldots,h_{b_z};\ldots;h_{a_q},\ldots,h_{b_q};h_{c_1},\ldots,h_{d_1};\ldots;h_{c_o},\ldots,h_{d_o};$$
$$v_{f_1},\ldots,v_{f_o};v'_{g_1},v''_{g_1};v_{g_2},\ldots,v_{g_q}]\ =$$
$$[h_{a_1},\ldots,h_{b_1};\ldots;h_{a_{z-1}},\ldots,h_{b_{z-1}};h_{a_{z+1}},\ldots,h_{b_{z+1}};\ldots;h_{a_q},\ldots,h_{b_q};$$
$$h'_{g_1},h_{a_z},\ldots,h_{x_z};h''_{g_1},h_{y_z},\ldots,h_{b_z};h_{c_1},\ldots,h_{d_1};\ldots;h_{c_o},\ldots,h_{d_o};$$
$$v'_{g_1},v''_{g_1},v_{f_1},\ldots,v_{f_o};v_{g_2},\ldots,v_{g_q}].$$

Finally, if $|v'_{g_1}|$ and $|v''_{g_1}|$ are even, we get

$$[h_{a_1},\ldots,h_{b_1};\ldots;h'_{g_1},h_{a_z},\ldots,h_{x_z};h''_{g_1},h_{y_z},\ldots,h_{b_z};\ldots;h_{a_q},\ldots,h_{b_q};h_{c_1},\ldots,h_{d_1};\ldots;h_{c_o},\ldots,h_{d_o};$$
$$v_{f_1},\ldots,v_{f_o};v'_{g_1},v''_{g_1},v_{g_2},\ldots,v_{g_q}]\cdot(-1).$$

If there were no factors -1, the preceding relations would be precisely those used by Kontsevich and Shoikhet [3] for defining their differential of the graph-complex on the odd space (corresponding to the cochain complex $C^\bullet_{Lie}(\mathrm{Ham}_0^{odd};\mathbb{C})$). In order to get rid of those factors -1, however, *it suffices to use the isomorphism which multiplies every graph G (i.e. its orientation) by $(-1)^{n-o/2}$, if G has n vertices, o of which are odd.* In fact, the multiplication by $(-1)^n$ provides a minus for every differential, whereas the multiplication by $(-1)^{o/2}$ (note that o is always even) kills this minus if two new odd vertices are created. Therefore we have proved the following theorem.

THEOREM 2. - *The cohomological graph-complex on the even space, the cohomological graph-complex on the odd space and the universally oriented cohomological graph-complex are isomorphic. In particular, their cohomologies coincide.* □

By the classical arguments reproduced in the introduction this implies our final corollary, a surprising duality relation which (according to Feigin) seems to be difficult to derive by different means.

COROLLARY. - *The cohomologies of the Lie algebras Ham_0 and Ham_0^{odd} of Hamiltonian vector fields vanishing at the origin on the infinite-dimensional even and odd spaces coincide.* □

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