KNUTH'S COHERENT PRESENTATIONS OF PLACTIC MONOIDS OF TYPE A

NOHRA HAGE PHILIPPE MALBOS

Abstract – We construct finite coherent presentations of plactic monoids of type A. Such coherent presentations express a system of generators and relations for the monoid extended in a coherent way to give a family of generators of the relations amongst the relations. Such extended presentations are used for representations of monoids, in particular, it is a way to describe actions of monoids on categories. Moreover, a coherent presentation provides the first step in the computation of a categorical cofibrant replacement of a monoid. Our construction is based on a rewriting method introduced by Squier that computes a coherent presentation from a convergent one. We compute a finite coherent presentation of a plactic monoid from its column presentation and we reduce it to a Tietze equivalent one having Knuth's generators.

1. INTRODUCTION

Plactic monoids. The structure of plactic monoids appeared in the combinatorial study of Young tableaux by Schensted [21] and Knuth [12]. The *plactic monoid* of rank $n > 0$, denoted by $\mathbf{P}_n$, is generated by the set $\{1, \ldots, n\}$ and subject to the *Knuth relations*:

$$zxy = xzy \quad \text{for} \quad 1 \leqslant x \leqslant y < z \leqslant n, \qquad yzx = yxz \quad \text{for} \quad 1 \leqslant x < y \leqslant z \leqslant n.$$

For instance, the monoid $\mathbf{P}_2$ is generated by $\{1, 2\}$ and submitted to the relations $211 = 121$ and $221 = 212$. The Knuth presentation of the monoid $\mathbf{P}_3$ has 3 generators and 8 relations. Lascoux and Schützenberger used the plactic monoid in order to prove the Littlewood-Richardson rule for the decomposition of tensor products of irreducible modules over the Lie algebra of n by n matrices, [22, 16]. The structure of plactic monoids has several applications in algebraic combinatorics and representation theory [15, 16, 14, 5] and several works have generalised the notion of tableaux to classical Lie algebras [1, 25, 10, 19, 23].

Syzygies of Knuth's relations. The aim of this work is to give an algorithmic method for the syzygy problem of finding all independent irreducible algebraic relations amongst the Knuth relations. A 2-*syzygy* for a presentation of a monoid is a relation amongst relations. For instance, using the Knuth relations there are two ways to prove the equality $2211 = 2121$ in the monoid $\mathbf{P}_2$, either by applying the first Knuth relation $211 = 121$ or the second relation $221 = 212$. This two equalities are related by a syzygy. Starting with a monoid presentation, we would like to compute all syzygies for this presentation and in particular to compute a family of generators for the syzygies. For instance, we will prove that in rank 2 the two Knuth relations form a unique generating syzygy for the Knuth relations. For rank greater than 3, the syzygies problem is difficult due to the combinatorial complexity of the relations. In commutative algebra, the theory of Gröbner bases gives algorithms to compute bases for linear syzygies. By a similar method, the syzygy problem for presentation of monoids can be algorithmically solved using *convergent rewriting systems*.

This work is partially supported by the French National Research Agency, ANR-13-BS02-0005-02.

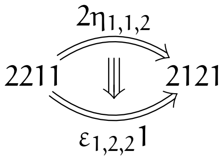
1. Introduction

Rewriting and plactic monoids. Study presentations from a rewriting approach consists in the orientation of the relations, then called *reduction rules*. For instance, the relations of the monoid $\mathbf{P}_2$ can be oriented with respect to the lexicographic order as follows

$$\eta_{1,1,2} : 211 \Rightarrow 121 \qquad \varepsilon_{1,2,2} : 221 \Rightarrow 212.$$

In a monoid presented by a rewriting system, two words are equal if they are related by a zig-zag sequence of applications of reductions rules. A rewriting system is convergent if the reduction relation induced by the rules is well-founded and if it satisfies the *confluence property*. This means that any reductions starting on a same word can be extended to end on a same reduced word. Recently plactic monoids were investigated by rewriting methods [13, 2, 4, 9, 3].

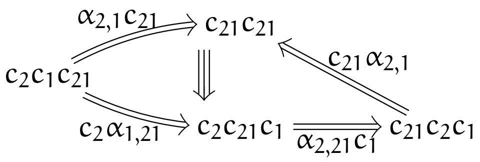
Coherent presentations of plactic monoids. We give a categorical description of 2-syzygies of presentations of the monoid $\mathbf{P}_n$ using *coherent presentations*. Such a presentation extends the notion of a presentation of the monoid by globular homotopy generators taking into account the relations amongst the relations. We compute a coherent presentation of the monoid $\mathbf{P}_n$ using the homotopical completion procedure introduced in [8, 6]. Such a procedure extends the Knuth-Bendix completion procedure [11], by keeping track of homotopy generators created when adding rules during the completion. Its correctness is based on the Squier theorem, [24], which states that a convergent presentation of a monoid extended by the homotopy generators defined by the confluence diagrams induced by *critical branchings* forms a coherent convergent presentation. The notion of critical branching describes the overlapping of two rules on a same word. For instance, the Knuth presentation of the monoid $\mathbf{P}_2$ is convergent. It can be extended into a coherent presentation with a unique globular homotopy generator described by the following 3-cell corresponding to the unique critical branching of the presentation between the rules $\eta_{1,1,2}$ and $\varepsilon_{1,2,2}$:



The Knuth presentation of the monoid $\mathbf{P}_3$ is not convergent, but it can be completed by adding 3 relations to get a presentation with 27 3-cells corresponding to the 27 critical branchings. For the monoid $\mathbf{P}_4$ we have 4 1-cells and 20 2-cells, for $\mathbf{P}_5$ we have 5 1-cells and 40 2-cells and for $\mathbf{P}_6$ we have 6 1-cells and 70 2-cells. However, in the last three cases, the completion is infinite and another approach is necessary to compute a finite generating family for syzygies of the Knuth presentation.

The column presentation. Kubat and Okniński showed in [13] that for rank $n > 3$, a finite convergent presentation of the monoid $\mathbf{P}_n$ cannot be obtained by completion of the Knuth presentation with the degree lexicographic order. Then Bokut, Chen, Chen and Li in [2] and Cain, Gray and Malheiro in [4] constructed with independent methods a finite convergent presentation by adding column generators to the Knuth presentation. However, on the one hand, the proof given in [4] does not give explicitly the critical branchings of the presentation which does not permit to use the homotopical completion procedure. On the other hand, the construction in [2] gave an explicit description of the critical branchings of the presentation, but this does not allow to get explicitly the relations amongst the relations.

The Knuth coherent presentation. We construct a coherent presentation of the monoid $\mathbf{P}_n$ that extends the Knuth presentation in two steps. The first step consists in giving an explicit description of the critical branchings of the column presentation. The column presentation of the plactic monoid has one generator c_u for each column u, that is, a word $u = x_p \ldots x_1$ such that $x_p > \ldots > x_1$. Given two columns u and v, using the Schensted algorithm, we compute the Schensted tableau $P(uv)$ associated to the word uv. One proves that the planar representation of the tableau $P(uv)$ contains at most two columns. If the planar representation is not the tableau obtained as the concatenation of the two columns u and v, one defines a rule $\alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$ where w and w' are respectively the left and right columns (with one of them possibly empty). We show that the column presentation can be extended into a *coherent column presentation* whose any 3-cell has at most an hexagonal form. For instance, the column presentation for the monoid $\mathbf{P}_2$ has generators c_1, c_2, c_{21}, with the rules $\alpha_{2,1} : c_2 c_1 \Rightarrow c_{21}$, $\alpha_{1,21} : c_1 c_{21} \Rightarrow c_{21} c_1$ and $\alpha_{2,21} : c_2 c_{21} \Rightarrow c_{21} c_2$. This presentation has only one critical branching:



and thus the 3-cell of the extended coherent presentation is reduced to this 3-cell defined by this confluence diagram. Note that for column presentations of the monoids $\mathbf{P}_3$, $\mathbf{P}_4$ and $\mathbf{P}_5$ we count respectively 7, 15 and 31 generators, 22, 115 and 531 relations, 42, 621 and 6893 3-cells.

The second step aimed at to reduce the coherent column presentation using Tietze transformations that coherently eliminates redundant column generators and defining relations to the *Knuth coherent presentation* giving syzygies of the Knuth presentation. For instance, if we apply this Tietze transformation on the column coherent presentation of the monoid $\mathbf{P}_2$, we prove that the Knuth coherent presentation of $\mathbf{P}_2$ on the generators c_1, c_2 and the relations $\eta_{1,1,2}$, $\varepsilon_{1,2,2}$ has a unique generating 3-cell $2\eta_{1,1,2} \Rrightarrow \varepsilon_{1,2,2}1$ described above.

Organisation of the article. The polygraphical description of string rewriting systems that we will use in this work is briefly recalled in Section 2.1, we refer the reader to [7] for a deeper presentation. In Section 2.2, we define the Knuth 2-polygraph that corresponds to the Knuth relations oriented with respect to the lexicographic order. In Section 2.3, we recall the column presentation introduced in [4]. The proof given in [4] for the convergence of this presentation consists in showing that this presentation has the unique normal form property. We give another proof of the confluence by showing the confluence of all the critical branchings of the column presentation. In Section 3, we recall the notion of coherent presentation of a monoid and we show the first main result of this article, that extends the column presentation into a coherent presentation, Theorem 3.2.2. In Section 4, we reduce the coherent column presentation into a coherent presentation that extends the Knuth presentation and that gives all syzygies of the Knuth's relations, Theorem 4.4.7. Finally, we explicit a procedure that computes a family of generating syzygies for any plactic monoids of type A.

2. Presentation of plactic monoids by rewriting

In this preliminary section, we recall rewriting notions and some presentations and constructions of plactic monoids used in this article.

2.1. Presentations of monoids by two-dimensional polygraphs

2.1.1. Two-dimensional polygraphs. In this article, we deal with presentations of monoids by rewriting systems, described by 2-polygraphs with only 0-cell denoted by $\bullet$. Such a 2-*polygraph* Σ is given by a pair (Σ_1, Σ_2), where Σ_1 is a set and Σ_2 is a *globular extension* of the free monoid Σ_1^*, that is a set of 2-cells $\beta : u \Rightarrow v$ relating 1-cells in Σ_1^*, where u and v denote the *source* and the *target* of β, respectively denoted by $s_1(\beta)$ and $t_1(\beta)$. If there is no possible confusion, Σ_2 will denote the 2-polygraph itself. Recall that a 2-*category* (resp. $(2,1)$-*category*) is a category enriched in categories (resp. in groupoids). When two 1-cells, or 2-cells, f and g of a 2-category are 0-composable (resp. 1-composable), we denote by fg (resp. $f \star_1 g$) their 0-composite (resp. 1-composite). We will denote by Σ_2^* (resp. $\Sigma_2^\top$) the 2-category (resp. $(2,1)$-category) freely generated by the 2-polygraph Σ, see [7, Section 2.4.] for expended definitions.

The *monoid presented by a* 2-*polygraph* Σ, denoted by $\overline{\Sigma}$, is defined as the quotient of the free monoid Σ_1^* by the congruence generated by the set of 2-cells Σ_2. A presentation of a monoid **M** is a 2-polygraph whose presented monoid is isomorphic to **M**. Two 2-polygraphs are *Tietze equivalent* if they present isomorphic monoids.

2.1.2. Tietze transformations of 2-polygraphs. A 2-cell β of a 2-polygraph Σ is *collapsible*, if $t_1(\beta)$ is a 1-cell of Σ_1 and the 1-cell $s_1(\beta)$ does not contain $t_1(\beta)$, then $t_1(\beta)$ is called *redundant*. Recall from [6, 2.1.1.], that an *elementary Tietze transformation* of a 2-polygraph Σ is a 2-functor with domain $\Sigma_2^\top$ that belongs to one of the following four transformations:

i) adjunction $\iota_\beta^1 : \Sigma_2^\top \to \Sigma_2^\top[x](\beta)$ of a redundant 1-cell x with its collapsible 2-cell β.

ii) elimination $\pi_\beta : \Sigma_2^\top \to (\Sigma_1 \setminus \{x\}, \Sigma_2 \setminus \{\beta\})^\top$ of a redundant 1-cell x with its collapsible 2-cell β.

iii) adjunction $\iota_\beta : \Sigma_2^\top \to \Sigma_2^\top(\beta)$ of a redundant 2-cell β.

iv) elimination $\pi_{(\gamma,\beta)} : \Sigma_2^\top \to \Sigma_2^\top/(\gamma, \beta)$ of a redundant 2-cell β.

If Σ and Υ are 2-polygraphs, a *Tietze transformation* from Σ to Υ is a 2-functor $F : \Sigma^\top \to \Upsilon^\top$ that decomposes into sequence of elementary Tietze transformations. Two 2-polygraphs are Tietze equivalent if, and only if, there exists a Tietze transformation between them [6, Theorem 2.1.3.].

Given a 2-polygraph Σ and a 2-cell $\gamma_1 \star_1 \gamma \star_1 \gamma_2$ in $\Sigma_2^\top$, the *Nielsen transformation* $\kappa_{\gamma\leftarrow\beta}$ is the Tietze transformation that replaces in the $(2,1)$-category $\Sigma_2^\top$ the 2-cell γ by a 2-cell $\beta : s_1(\gamma_1) \Rightarrow t_1(\gamma_2)$. When γ_2 is identity, we will denote by $\kappa'_{\gamma\leftarrow\beta}$ the Nielsen transformation which, given a 2-cell $\gamma_1 \star_1 \gamma$ in $\Sigma_2^\top$, replaces the 2-cell γ by a 2-cell $\beta : s_1(\gamma_1) \Rightarrow t_1(\gamma)$.

2.1.3. Convergence. A *rewriting step* of a 2-polygraph Σ is a 2-cell of Σ_2^* with shape $w\beta w'$, where β is a 2-cell of Σ_2 and w and w' are 1-cells of Σ_1^*. A *rewriting sequence* of Σ is a finite or infinite sequence of rewriting steps. A 1-cell u of Σ_1^* is a *normal form* if there is no rewriting step with source u. The 2-polygraph Σ *terminates* if it has no infinite rewriting sequence.

A *branching* of the 2-polygraph Σ is a non ordered pair (f, g) of 2-cells of Σ_2^* such that $s_1(f) = s_1(g)$. A branching (f, g) is *local* if f and g are rewriting steps. A branching is *aspherical* if it is of the form (f, f), for a rewriting step f and *Peiffer* when it is of the form (fv, ug) for rewriting steps f and g with $s_1(f) = u$ and $s_1(g) = v$. The *overlapping* branchings are the remaining local branchings. An overlapping local branching is *critical* if it is minimal for the order $\sqsubseteq$ generated by the relations $(f, g) \sqsubseteq (wfw', wgw')$, given for any local branching (f, g) and any possible 1-cells w and w' of the category Σ_1^*. A branching (f, g) is *confluent* if there exist 2-cells f' and g' in Σ_2^* such that $s_1(f') = t_1(f)$, $s_1(g') = t_1(g)$ and $t_1(f') = t_1(g')$. We say that a 2-polygraph Σ is *confluent* if all of its branchings are confluent. It is *convergent* if it terminates and it is confluent. In that case, every 1-cell u of Σ_1^* has a unique normal form.

2.2. Plactic monoids

2.2.1. Rows, columns and tableaux. For $n > 0$, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$ totally ordered by $1 < 2 < \ldots < n$. A *row* is a non-decreasing 1-cell $x_1 \ldots x_k$ in the free monoid $[n]^*$, *i.e.*, with $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_k$. A *column* is a decreasing 1-cell $x_p \ldots x_1$ in $[n]^*$, *i.e.*, with $x_p > \ldots > x_2 > x_1$. We will denote by $\mathrm{col}(n)$ the set of non-empty columns in $[n]^*$. We denote by $\ell(w)$ (resp. $\ell^{\mathrm{nds}}(w)$) the length of a 1-cell w (resp. the length of the longest non-decreasing subsequence in w). A row $x_1 \ldots x_k$ *dominates* a row $y_1 \ldots y_l$, and we denote $x_1 \ldots x_k \rhd y_1 \ldots y_l$, if $k \leqslant l$ and $x_i > y_i$, for $1 \leqslant i \leqslant k$. Any 1-cell w in $[n]^*$ has a unique decomposition as a product of rows of maximal length $u_1 \ldots u_k$. Such a 1-cell w is a *tableau* if $u_1 \rhd u_2 \rhd \ldots \rhd u_k$. We will write tableaux in a planar form, with the rows placed in order of domination from bottom to top and left-justified as in [5]. The *degree lexicographic order* is the total order on $\mathrm{col}(n)$, denoted by $\preccurlyeq_{\mathrm{deglex}}$, and defined by $u \preccurlyeq_{\mathrm{deglex}} v$ if $\ell(u) < \ell(v)$ or $\ell(u) = \ell(v)$ and $u <_{\mathrm{lex}} v$, for all u and v in $\mathrm{col}(n)$, where $<_{\mathrm{lex}}$ denotes the lexicographic order on $[n]^*$.

2.2.2. Schensted's algorithm. The *Schensted algorithm* computes for each 1-cell w in $[n]^*$ a tableau denoted by $P(w)$, called the *Schensted tableau* of w and constructed as follows, [21]. Given u a tableau written as a product of rows of maximal length $u = u_1 \ldots u_k$ and y in $[n]$, it computes the tableau $P(uy)$ as follows. If $u_k y$ is a row, the result is $u_1 \ldots u_k y$. If $u_k y$ is not a row, then suppose $u_k = x_1 \ldots x_l$ with x_i in $[n]$ and let j minimal such that $x_j > y$, then the result is $P(u_1 \ldots u_{k-1} x_j) v_k$, where $v_k = x_1 \ldots x_{j-1} y x_{j+1} \ldots x_l$. The tableau $P(w)$ is computed from the empty tableau and iteratively applying the Schensted algorithm. In this way, $P(w)$ is the row reading of the planar representation of the tableau computed by the Schensted algorithm. The number of columns in $P(w)$ is equal to $\ell^{\mathrm{nds}}(w)$, [21]. We will denote by $C(w)$ the *column reading* of the tableau $P(w)$, obtained by reading $P(w)$ column-wise from bottom to top and from left to right. We denote by $C_r(w)$ (resp. $C_l(w)$) the reading of the last right (resp. first left) column of the tableau $P(w)$.

2.2.3. Knuth's 2-polygraph and the plactic congruence. The *plactic monoid of rank* n, denoted by $\mathbf{P}_n$, is the quotient of the free monoid $[n]^*$ by the congruence $\sim_{\mathrm{plax}(n)}$, defined by $u \sim_{\mathrm{plax}(n)} v$ if $P(u) = P(v)$. The *Knuth 2-polygraph of rank* n is the 2-polygraph, denoted by $\mathrm{Knuth}_2(n)$, whose set of 1-cells is $[n]$ and the set of 2-cells is

$$\{\, zxy \overset{\eta_{x,y,z}}{\Longrightarrow} xzy \mid 1 \leqslant x \leqslant y < z \leqslant n \,\} \cup \{\, yzx \overset{\varepsilon_{x,y,z}}{\Longrightarrow} yxz \mid 1 \leqslant x < y \leqslant z \leqslant n \,\}. \tag{1}$$

The congruence on the free monoid $[n]^*$ generated by the 2-polygraph $\mathrm{Knuth}_2(n)$ is called the *plactic congruence of rank* n and the 2-polygraph $\mathrm{Knuth}_2(n)$ is a presentation of the monoid $\mathbf{P}_n$, [12, Theorem 6]. Each plactic congruence class contains exactly one tableau, [20, Proposition 5.2.3], and for any 1-cell w, we have that $w = C(w)$ holds in $\mathbf{P}_n$, [20, Problem 5.2.4].

2.3. Column presentation

We recall some presentations of the plactic monoid $\mathbf{P}_n$ obtained by adding new generators. In particular, we recall the column presentation of the monoid $\mathbf{P}_n$ introduced in [4] which is finite and convergent.

2.3.1. Columns as generators. Let us denote by $\mathrm{Col}_1(n) = \{c_u \mid u \in \mathrm{col}(n)\}$ the set of *column generators* of the monoid $\mathbf{P}_n$ and by

$$C_2(n) = \left\{ c_{x_p} \ldots c_{x_1} \overset{\gamma_u}{\Longrightarrow} c_u \;\middle|\; u = x_p \ldots x_1 \in \mathrm{col}(n) \;\text{ with } \ell(u) \geqslant 2 \right\}$$

the set of the defining relations for the column generators. We denote by $\mathrm{Knuth}_2^c(n)$ the 2-polygraph whose set of 1-cells is $\{c_1, \ldots, c_n\}$ and whose set of 2-cells is given by

$$\left\{ c_z c_x c_y \overset{\eta^c_{x,y,z}}{\Longrightarrow} c_x c_z c_y \;\middle|\; 1 \leqslant x \leqslant y < z \leqslant n \right\} \cup \left\{ c_y c_z c_x \overset{\varepsilon^c_{x,y,z}}{\Longrightarrow} c_y c_x c_z \;\middle|\; 1 \leqslant x < y \leqslant z \leqslant n \right\}.$$

By definition, this 2-polygraph is Tietze equivalent to the 2-polygraph $\mathrm{Knuth}_2(n)$. In the sequel, we will identify the 2-polygraphs $\mathrm{Knuth}_2^c(n)$ and $\mathrm{Knuth}_2(n)$.

Let us define the 2-polygraph $\mathrm{Knuth}_2^{cc}(n)$, whose set of 2-cells is $C_2(n) \cup \mathrm{Knuth}_2^c(n)$. The 2-polygraph $\mathrm{Knuth}_2^{cc}(n)$ is a presentation of the monoid $\mathbf{P}_n$. Indeed, we add to the 2-polygraph $\mathrm{Knuth}_2^c(n)$ all the column generators c_u, for all $u = x_p \ldots x_1$ in $\mathrm{col}(n)$ such that $\ell(u) \geqslant 2$, and the corresponding collapsible 2-cell $\gamma_u : c_{x_p} \ldots c_{x_1} \Rightarrow c_u$.

2.3.2. Pre-column presentation. Let us define the 2-polygraph $\mathrm{PreCol}_2(n)$ whose set of 1-cells is $\mathrm{Col}_1(n)$ and the set of 2-cells is

$$\mathrm{PreCol}_2(n) = \mathrm{PC}_2(n) \cup \left\{ c_x c_u \overset{\alpha'_{x,u}}{\Longrightarrow} c_{xu} \mid xu \in \mathrm{col}(n) \;\text{ and }\; 1 \leqslant x \leqslant n \right\},$$

where

$$\mathrm{PC}_2(n) = \left\{ c_x c_{zy} \overset{\alpha'_{x,zy}}{\Longrightarrow} c_{zx} c_y \mid 1 \leqslant x \leqslant y < z \leqslant n \right\} \cup \left\{ c_y c_{zx} \overset{\alpha'_{y,zx}}{\Longrightarrow} c_{yx} c_z \mid 1 \leqslant x < y \leqslant z \leqslant n \right\}.$$

2.3.3. Proposition. *For $n > 0$, the 2-polygraph $\mathrm{PreCol}_2(n)$ is a presentation of the monoid $\mathbf{P}_n$, called the* pre-column presentation of $\mathbf{P}_n$.

Proof. We proceed in two steps. The first step consists to prove that the 2-polygraph

$$\mathrm{CPC}_2(n) := \langle \;\mathrm{Col}_1(n) \;\mid\; C_2(n) \cup \mathrm{PC}_2(n) \;\rangle$$

is Tietze equivalent to the 2-polygraph $\mathrm{Knuth}_2^{cc}(n)$. For $1 \leqslant x \leqslant y < z \leqslant n$, consider the following critical branching

$$\begin{array}{l} \qquad \overset{\eta^c_{x,y,z}}{\nearrow\!\!\!\nearrow} c_x c_z c_y \overset{c_x \gamma_{zy}}{\Longrightarrow} c_x c_{zy} \\ c_z c_x c_y \\ \qquad \underset{\gamma_{zx} c_y}{\searrow\!\!\!\searrow} c_{zx} c_y \end{array}$$

of the 2-polygraph $\mathrm{Knuth}_2^{cc}(n)$. Let consider the Tietze transformation

$$\kappa_{\eta^c_{x,y,z} \leftarrow \alpha'_{x,zy}} : \mathrm{Knuth}_2^{cc}(n)^\top \longrightarrow \mathrm{Knuth}_2^{cc}(n)^\top/(\eta^c_{x,y,z} \leftarrow \alpha'_{x,zy}),$$

that substitutes the 2-cell $\alpha'_{x,zy} : c_x c_{zy} \Rightarrow c_{zx} c_y$ to the 2-cell $\eta^c_{x,y,z}$, for every $1 \leqslant x \leqslant y < z \leqslant n$. We denote by $T_{\eta \leftarrow \alpha'}$ the successive applications of the Tietze transformation $\kappa_{\eta^c_{x,y,z} \leftarrow \alpha'_{x,zy}}$, for every $1 \leqslant x \leqslant y < z \leqslant n$, with respect to the lexicographic order on the triples (x, y, z) induced by the total order on $[n]$.

Similarly, we study in the same way the critical branching $(\varepsilon^c_{x,y,z}, c_y \gamma_{zx})$ of the 2-polygraph $\mathrm{Knuth}_2^{cc}(n)$, for every $1 \leqslant x < y \leqslant z \leqslant n$, by introducing the Tietze transformation $\kappa_{\varepsilon^c_{x,y,z} \leftarrow \alpha'_{y,zx}}$ from $\mathrm{Knuth}_2^{cc}(n)^\top$ to $\mathrm{Knuth}_2^{cc}(n)^\top/(\varepsilon^c_{x,y,z} \leftarrow \alpha'_{y,zx})$. We denote by $T_{\varepsilon \leftarrow \alpha'}$ the successive applications of this Tietze transformation with respect to the lexicographic order on the triples (x, y, z) induced by the total order on $[n]$. In this way, we obtain a Tietze transformation $T_{\eta,\varepsilon \leftarrow \alpha'}$ from $\mathrm{Knuth}_2^{cc}(n)^\top$ to $\mathrm{CPC}_2(n)^\top$ given by the composite $T_{\eta \leftarrow \alpha'} \circ T_{\varepsilon \leftarrow \alpha'}$.

In a second step, we prove that the 2-polygraph $\mathrm{PreCol}_2(n)$ is Tietze equivalent to the 2-polygraph $\mathrm{CPC}_2(n)$. Let $x_p \ldots x_1$ be a column with $\ell(x_p \ldots x_1) > 2$ and define $\alpha'_{y,x} := \gamma_{yx} : c_y c_x \Rightarrow c_{yx}$, for every $x < y$. Consider the following critical branching

$$\begin{array}{ccc} & \overset{c_{x_p}\gamma_{x_{p-1}\ldots x_1}}{\Longrightarrow} & c_{x_p} c_{x_{p-1}\ldots x_1} \\ c_{x_p} c_{x_{p-1}} \ldots c_{x_1} & & \\ & \underset{\gamma_{x_p\ldots x_1}}{\Longrightarrow} & c_{x_p\ldots x_1} \end{array}$$

of the 2-polygraph $\mathrm{CPC}_2(n)$ and the following Tietze transformation

$$\kappa'_{\gamma_{x_p\ldots x_1} \leftarrow \alpha'_{x_p,x_{p-1}\ldots x_1}} : \mathrm{CPC}_2(n)^\top \longrightarrow \mathrm{CPC}_2(n)^\top/(\gamma_{x_p\ldots x_1} \leftarrow \alpha'_{x_p,x_{p-1}\ldots x_1}),$$

that substitutes the 2-cell $\alpha'_{x_p,x_{p-1}\ldots x_1}$ to the 2-cell $\gamma_{x_p\ldots x_1}$, for each column $x_p \ldots x_1$ such that $p > 2$. Starting from the 2-polygraph $\mathrm{CPC}_2(n)$, we apply successively the Tietze transformation $\kappa'_{\gamma_{x_p\ldots x_1} \leftarrow \alpha'_{x_p,x_{p-1}\ldots x_1}}$, for every column $x_p \ldots x_1$ such that $\ell(x_p \ldots x_1) > 2$, from the bigger to the smaller one with respect to the total order $\preccurlyeq_{\mathrm{deglex}}$. The composite

$$T_{\gamma \leftarrow \alpha'} = \kappa'_{\gamma_{x_3x_2x_1} \leftarrow \alpha'_{x_3,x_2x_1}} \circ \ldots \circ \kappa'_{\gamma_{x_n\ldots x_1} \leftarrow \alpha'_{x_n,x_{n-1}\ldots x_1}},$$

gives us a Tietze transformation from $\mathrm{CPC}_2(n)^\top$ to $\mathrm{PreCol}_2(n)^\top$. □

2.3.4. Column presentation. Let $n > 0$. Given columns $u = x_p \ldots x_1$ and $v = y_q \ldots y_1$ in $\mathrm{col}(n)$, the length $\ell^{\mathrm{nds}}(uv)$ of the longest non-decreasing subsequence of uv is lower or equal to 2 [4, Lemma 3.1.]. We will use graphical notations depending on whether the tableau $P(uv)$ consists in two columns:

- **i)** we will denote $u \frown v$ if the planar representation of $P(uv)$ is a tableau, that is, $p \geqslant q$ and $x_i \leqslant y_i$, for any $i \leqslant q$,
- **ii)** we will denote $u^\times v$ in all the other cases, that is, when $p < q$ or $x_i > y_i$, for some $i \leqslant q$.

In the case **ii)**, we will denote $u \overset{\times_1}{} v$ if the tableau $P(uv)$ has one column and we will denote $u \overset{\times_2}{} v$ if the tableau $P(uv)$ has two columns. For every columns u and v in $\mathrm{col}(n)$ such that $u \overset{\times}{} v$, we define a 2-cell

$$\alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$$

where

i) $w = uv$ and $c_{w'} = 1$, if $u \overset{\times_1}{} v$,

ii) w and w' are respectively the left and right columns of the tableau $P(uv)$, if $u \overset{\times_2}{} v$.

Let us denote by $\mathrm{Col}_2(n)$ the 2-polygraph whose set of 1-cells is $\mathrm{Col}_1(n)$ and the set of 2-cells is

$$\mathrm{Col}_2(n) \;=\; \left\{\, c_u c_v \overset{\alpha_{u,v}}{\Longrightarrow} c_w c_{w'} \;\middle|\; u, v \in \mathrm{col}(n) \text{ and } u \overset{\times}{} v \,\right\}. \qquad (2)$$

The 2-polygraph $\mathrm{Col}_2(n)$ is a finite convergent presentation of the monoid $\mathbf{P}_n$ [4, Theorem 3.4], called the *column presentation* of the monoid $\mathbf{P}_n$. Note that Schensted's algorithm that computes a tableau $P(w)$ from a 1-cell w, corresponds to the leftmost reduction path in $\mathrm{Col}_2^*(n)$ from w to its normal form $P(w)$, that is, the reduction paths obtained by applying the rules of $\mathrm{Col}_2(n)$ starting from the left. In particular, we have

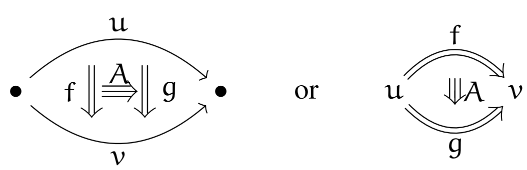
2.3.5. Lemma. *For any $u_1, \ldots, u_n$ in $\mathrm{col}(n)$, the length of the leftmost rewriting path in $\mathrm{Col}_2(n)^*$ from $u_1 u_2 \ldots u_n$ to its normal form $P(u_1 u_2 \ldots u_n)$ is at most n.*

3. Coherent column presentation

In this section, we begin by recalling the notion of coherent presentations of monoids from [6]. In a second part, using the homotopical completion procedure, we construct a coherent presentation of the monoid $\mathbf{P}_n$ starting from its column presentation.

3.1. Coherent presentations of monoids

3.1.1. $(3,1)$-polygraph. A $(3,1)$-*polygraph* is a pair (Σ_2, Σ_3) made of a 2-polygraph Σ_2 and a globular extension Σ_3 of the $(2,1)$-category $\Sigma_2^\top$, that is a set of 3-cells $A : f \Rrightarrow g$ relating 2-cells f and g in $\Sigma_2^\top$, respectively denoted by $s_2(A)$ and $t_2(A)$ and satisfying the globular relations $s_1 s_2(A) = s_1 t_2(A)$ and $t_1 s_2(A) = t_1 t_2(A)$. Such a 3-cell can be represented with the following globular shape:



We will denote by $\Sigma_3^\top$ the free $(3,1)$-category generated by the $(3,1)$-polygraph (Σ_2, Σ_3). A pair (f, g) of 2-cells of $\Sigma_2^\top$ such that $s_1(f) = s_1(g)$ and $t_1(f) = t_1(g)$ is called a 2-*sphere* of $\Sigma_2^\top$.

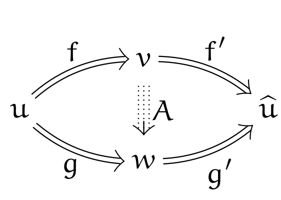
3.1.2. Coherent presentations of monoids. An *extended presentation* of a monoid **M** is a $(3,1)$-polygraph whose underlying 2-polygraph is a presentation of the monoid **M**. A *coherent presentation of* **M** is an extended presentation Σ of **M** such that the cellular extension Σ_3 is a *homotopy basis* of the $(2,1)$-category $\Sigma_2^\top$, that is, for every 2-sphere γ of $\Sigma_2^\top$, there exists a 3-cell in $\Sigma_3^\top$ with boundary γ.

3.1.3. Tietze transformations of $(3,1)$-polygraphs. We recall the notion of Tietze transformation from [6, Section 2.1]. Let Σ be a $(3,1)$-polygraph. A 3-cell A of Σ is called *collapsible* if $t_2(A)$ is in Σ_2 and $s_2(A)$ is a 2-cell of the free $(2,1)$-category over $(\Sigma_2 \setminus \{t_2(A)\})^\top$, then $t_2(A)$ is called *redundant*. An *elementary Tietze transformation* of a $(3,1)$-polygraph Σ is a 3-functor with domain $\Sigma_3^\top$ that belongs to one of the following operations:

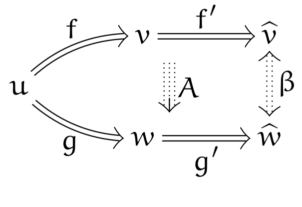
- **i)** adjunction ι_α^1 and elimination π_α of a 2-cell α as described in 2.1.2.
- **ii)** coherent adjunction $\iota_A^2 : \Sigma_3^\top \to \Sigma_3^\top(\alpha)(A)$ of a redundant 2-cell α with its collapsible 3-cell A.
- **iii)** coherent elimination $\pi_A : \Sigma_3^\top \to \Sigma_3^\top/A$ of a redundant 2-cell α with its collapsible 3-cell A.
- **iv)** coherent adjunction $\iota_A : \Sigma_3^\top \to \Sigma_3^\top(A)$ of a redundant 3-cell A.
- **v)** coherent elimination $\pi_{(B,A)} : \Sigma_3^\top \to \Sigma_3^\top/(B,A)$ of a redundant 3-cell A, that maps A to B.

For $(3,1)$-polygraphs Σ and Υ, a *Tietze transformation from Σ to Υ* is a 3-functor $F : \Sigma_3^\top \to \Upsilon_3^\top$ that decomposes into a sequence of elementary Tietze transformations. Two $(3,1)$-polygraphs Σ and Υ are *Tietze-equivalent* if there exists an equivalence of 2-categories $F : \Sigma_2^\top/\Sigma_3 \to \Upsilon_2^\top/\Upsilon_3$ and the presented monoids $\overline{\Sigma}_2$ and $\overline{\Upsilon}_2$ are isomorphic. Two $(3,1)$-polygraphs are Tietze equivalent if, and only if, there exists a Tietze transformation between them, [6, Theorem 2.1.3.].

3.1.4. Homotopical completion procedure. Following [6, Section 2.2], we recall the homotopical completion procedure that produces a coherent convergent presentation from a terminating presentation. Given a terminating 2-polygraph Σ, equipped with a total termination order $\preceq$, the homotopical completion of Σ is the $(3,1)$-polygraph obtained from Σ by successive applications of the Knuth-Bendix completion procedure, [11], and the Squier construction, [24]. Explicitly, for any critical branching (f,g) of Σ, if (f,g) is confluent one adds a dotted 3-cell A:



where $\widehat{u}$ is a normal form, and if the critical branching (f,g) is not confluent one add a 2-cell β and a 3-cell A:



where the 2-cell β is directed from a normal form $\widehat{v}$ of v to a normal form $\widehat{w}$ of w if $\widehat{w} \prec \widehat{v}$ and from $\widehat{w}$ to $\widehat{v}$ otherwise. The adjunction of 2-cells can create new critical branchings, possibly generating the adjunction of additional 2-cells and 3-cells in the same way. This defines an increasing sequence of $(3,1)$-polygraphs, whose union is called a *homotopical completion* of Σ. Following [24, Theorem 5.2], such a homotopical completion of Σ is a coherent convergent presentation of the monoid $\overline{\Sigma}$.

3.2. Column coherent presentation

Using the homotopical completion procedure, we extend the 2-polygraph $\mathrm{Col}_2(n)$ into a coherent presentation of the monoid $\mathbf{P}_n$.

3.2.1. Column coherent presentation. The presentation $\mathrm{Col}_2(n)$ has exactly one critical branching of the form

$$\begin{array}{ccc} & \overset{\alpha_{u,v}c_t}{\Longrightarrow} & c_e c_{e'} c_t \\ c_u c_v c_t & & \\ & \underset{c_u\alpha_{v,t}}{\Longrightarrow} & c_u c_w c_{w'} \end{array} \qquad (3)$$

for any u, v, t in $\mathrm{col}(n)$ such that $u \,{}^{\times}\, v \,{}^{\times}\, t$, where e and e' (resp. w and w') denote the two columns of the tableau $\mathrm{P}(uv)$ (resp. $\mathrm{P}(vt)$). We prove in this section that all of these critical branchings are confluent and that all the confluence diagrams of these branchings are of the following form:

$$\begin{array}{ccccccc} & \overset{\alpha_{u,v}c_t}{\Longrightarrow} & c_e c_{e'} c_t & \overset{c_e\alpha_{e',t}}{\Longrightarrow} & c_e c_b c_{b'} & \overset{\alpha_{e,b}c_{b'}}{\Longrightarrow} & \\ c_u c_v c_t & & & \Downarrow \mathcal{X}_{u,v,t} & & & c_a c_d c_{b'} \\ & \underset{c_u\alpha_{v,t}}{\Longrightarrow} & c_u c_w c_{w'} & \underset{\alpha_{u,w}c_{w'}}{\Longrightarrow} & c_a c_{a'} c_{w'} & \underset{c_a\alpha_{a',w'}}{\Longrightarrow} & \end{array} \qquad (4)$$

where a and a' (resp. b and b') denote the two columns of the tableau $\mathrm{P}(uw)$ (resp. $\mathrm{P}(e't)$) and a, d, b' are the three columns of the tableau $\mathrm{P}(uvt)$, which is a normal form for the 2-polygraph $\mathrm{Col}_2(n)$. Note that in some cases described below, one or further columns e', w', a' and b' can be empty. In those cases some indicated 2-cells α in the confluence diagram correspond to identities.

Let us denote by $\mathrm{Col}_3(n)$ the extended presentation of the monoid $\mathbf{P}_n$ obtained from $\mathrm{Col}_2(n)$ by adjunction of one 3-cell $\mathcal{X}_{u,v,t}$ of the form (4), for every columns u, v and t such that $u \,{}^{\times}\, v \,{}^{\times}\, t$.

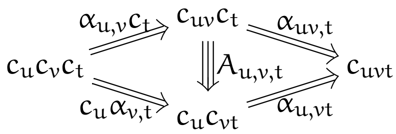
3.2.2. Theorem. *For $n > 0$, the $(3,1)$-polygraph $\mathrm{Col}_3(n)$ is a coherent presentation of the monoid $\mathbf{P}_n$.*

The extended presentation $\mathrm{Col}_3(n)$ is called the *column coherent presentation* of the monoid $\mathbf{P}_n$. The rest of this section consists in a constructive proof of Theorem 3.2.2, that makes explicit all possible forms of 3-cells. Another arguments are given in Remark 3.2.7. Our proof is based on the following arguments. The presentation $\mathrm{Col}_2(n)$ is convergent, thus using the homotopical completion procedure described in 3.1.4, it suffices to prove that the 3-cells $\mathcal{X}_{u,v,t}$ with $u \,{}^{\times}\, v \,{}^{\times}\, t$ form a family of generating confluences for the presentation $\mathrm{Col}_2(n)$. There are four possibilities for the critical branching (3) depending on the following four cases:

$$u^{\times 1} v^{\times 1} t, \qquad u^{\times 2} v^{\times 1} t, \qquad u^{\times 1} v^{\times 2} t, \qquad u^{\times 2} v^{\times 2} t.$$

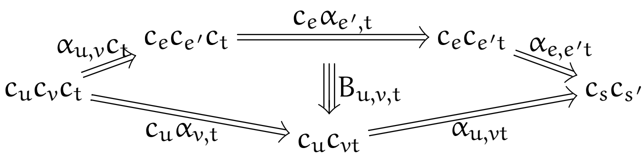
Each of these cases is examined in the following four lemmas, where $u = x_p \ldots x_1$, $v = y_q \ldots y_1$ and $t = z_l \ldots z_1$ denote columns of length p, q and l respectively.

3.2.3. Lemma. *If* $u^{\times 1} v^{\times 1} t$ *, we have the following confluent critical branching:*



Proof. By hypothesis uv and vt are columns, then uvt is a column. Thus $uv^{\times 1}t$ and $u^{\times 1}vt$ and there exist 2-cells $\alpha_{uv,t}$ and $\alpha_{u,vt}$ in $\mathrm{Col}_2(n)$ making the critical branching (3) confluent, where $e = uv$, $w = vt$ and e', w' are the empty column. □

3.2.4. Lemma. *If* $u^{\times 2} v^{\times 1} t$ *, we have the following confluent critical branching:*



(5)

where e *and* e' *(resp.* s *and* s'*) denote the two columns of the tableau* $P(uv)$ *(resp.* $P(uvt)$*).*

Proof. By hypothesis, vt is a column and $y_1 > z_l$. The tableau $P(uv)$ consists of two columns, that we will denote e and e', then $\ell^{\mathrm{nds}}(uv) = 2$ and $x_1 \leqslant y_q$. We have $u^{\times 2} v$, so that we distinguish the following possible three cases.

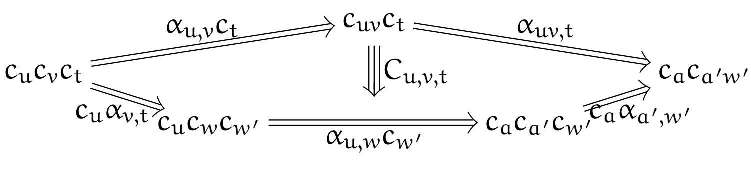
Case 1: $p \geqslant q$ and $x_{i_0} > y_{i_0}$ for some $1 \leqslant i_0 \leqslant q$. Suppose that $i_0 = 1$, that is, $x_1 > y_1$. We consider y_j the biggest element of the column v such that $x_1 > y_j$, then the smallest element of the column e' is y_{j+1}. By hypothesis, the word vt is a column, in particular $y_{j+1} > z_l$. It follows that $e't$ is a column. Suppose that $i_0 > 1$, then $x_1 \leqslant y_1$ and the smallest element of e' is y_1. Since $y_1 > z_l$ by hypothesis, the word $e't$ is a column. Hence, in all cases, $e't$ is a column and there is a 2-cell $\alpha_{e',t} : c_{e'}c_t \Rightarrow c_{e't}$.

Case 2: $p < q$ and $x_i \leqslant y_i$ for any $1 \leqslant i \leqslant p$. We have $e = y_q \ldots y_{p+1} x_p \ldots x_1$ and $e' = y_p \ldots y_1$. By hypothesis, $y_1 > z_l$, hence $e't$ is a column and there is a 2-cell $\alpha_{e',t} : c_{e'}c_t \Rightarrow c_{e't}$.

Case 3: $p < q$ and $x_{i_0} > y_{i_0}$ for some $1 \leqslant i_0 \leqslant p$. With the same arguments of Case 1, the smallest element of e' is y_1 or y_{j+1}, where y_j is the biggest element of the column v such that $y_j < x_1$. Hence, $e't$ is a column and there is a 2-cell $\alpha_{e',t} : c_{e'}c_t \Rightarrow c_{e't}$.

In each case, we have $\ell^{\mathrm{nds}}(uv) = 2$, hence $\ell^{\mathrm{nds}}(uvt) = 2$. Thus the tableau $P(uvt)$ consists of two columns, that we denote s and s' and there is a 2-cell $\alpha_{u,vt} : c_u c_{vt} \Rightarrow c_s c_{s'}$. Moreover, to compute the tableau $P(uvt)$, one begins by computing $P(uv)$ and after by introducing the elements of the column t on the tableau $P(uv)$. As $C(uv) = ee'$, we have $P(uvt) = P(P(uv)t) = P(ee't)$. Hence $C(ee't) = ss'$ and there is a 2-cell $\alpha_{e,e't}$ which yields the confluence diagram (5). □

3.2.5. Lemma. *If* $u^{\times 1} v^{\times 2} t$ *, we have the following confluent critical branching:*



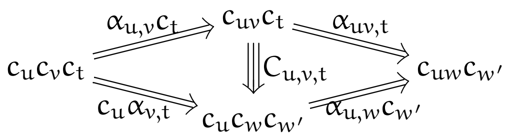
(6)

where w *and* w' *(resp.* a *and* a'*) denote the two columns of the tableau* $P(vt)$ *(resp.* $P(uw)$*).*

Proof. By hypothesis, uv is a column hence $x_1 > y_q$. Moreover, the tableau $P(vt)$ consists of two columns w and w', then $\ell^{\mathrm{nds}}(vt) = 2$, hence $y_1 \leqslant z_l$. We have $v \overset{\times 2}{\ } t$, so that we distinguish the three possible following cases.

Case 1: $q \geqslant l$ and $y_{i_0} > z_{i_0}$ for some $1 \leqslant i_0 \leqslant l$. Let us denote $w = w_r \dots w_1$ and $w' = w'_{r'} \dots w'_1$. Since $q \geqslant l$, we have $w_r = y_q$. By hypothesis, $x_1 > y_q$. Then the word uw is a column. As a consequence, there is a 2-cell $\alpha_{u,w} : c_u c_w \Rightarrow c_{uw}$. In addition, the column w appears to the left of w' in the planar representation of the tableau $P(vt)$, that is, $\ell(w) \geqslant \ell(w')$ and $w_i \leqslant w'_i$ for any $i \leqslant \ell(w')$. Then $\ell(uw) \geqslant \ell(w')$. We set $uw = \xi_{\ell(uw)} \dots \xi_1$ and we have $\xi_i \leqslant w'_i$ for any $i \leqslant \ell(w')$. Then $uw \widehat{\ } w'$ and $c_{uw} c_{w'}$ is a normal form.

On the other hand, the tableau $P(vt)$ consists of two columns, hence $\ell^{\mathrm{nds}}(vt) = 2$. As a consequence, $\ell^{\mathrm{nds}}(uvt) = 2$ and the tableau $P(uvt)$ consists of two columns. Since $q \geqslant l$, we have $C(uvt) = uww'$, hence the two columns of $P(uvt)$ are uw and w'. Then there is a 2-cell $\alpha_{uv,t} : c_{uv} c_t \Rightarrow c_{uw} c_{w'}$ which yields the confluence of the critical branching on $c_u c_v c_t$, as follows



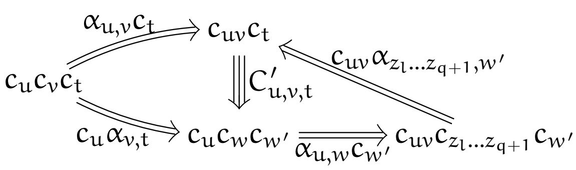
(7)

Case 2: $q < l$ and $y_i \leqslant z_i$ for any $i \leqslant q$. We have $w = z_l \dots z_{q+1} y_q \dots y_1$ and $w' = z_q \dots z_1$. There are two cases along uw is a column or not.

Case 2. A. If $x_1 > z_l$, then uw is a column. Hence, there is a 2-cell $\alpha_{u,w} : c_u c_w \Rightarrow c_{uw}$. Moreover, using Schensted's algorithm we prove that $C_l(uvt) = uw$ and $C_r(uvt) = w'$. Thus there is a 2-cell $\alpha_{uv,t} : c_{uv} c_t \Rightarrow c_{uw} c_{w'}$ which yields the confluence diagram (7).

Case 2. B. If $x_1 \leqslant z_l$, then $\ell^{\mathrm{nds}}(uw) = 2$ and $P(uw)$ consists of two columns, that we denote by a and a'. Then there is a 2-cell $\alpha_{u,w} : c_u c_w \Rightarrow c_a c_{a'}$. In addition, by Schensted's algorithm, we deduce that $a' = z_{i_k} \dots z_{i_1}$, with $q + 1 \leqslant i_1 < \dots < i_k \leqslant l$. We have $a'w' = z_{i_k} \dots z_{i_1} z_q \dots z_1$. Since all the elements of a' are elements of t and bigger than z_q, we have $z_{i_1} > z_q$. It follows that $a'w'$ is a column and there is a 2-cell $\alpha_{a',w'} : c_{a'} c_{w'} \Rightarrow c_{a'w'}$.

In the other hand, we have two cases whether $uv \overset{\times}{\ } t$ or $uv \widehat{\ } t$. Suppose $uv \overset{\times}{\ } t$. By Schensted's algorithm, we have $C_l(uvt) = a$ and $C_r(uvt) = a'w'$. Hence there is a 2-cell $\alpha_{uv,t} : c_{uv} c_t \Rightarrow c_a c_{a'w'}$, which yields the confluence of Diagram (6). Suppose $uv \widehat{\ } t$. Then we obtain $C(uw) = uvz_l \dots z_{q+1}$, and $C(z_l \dots z_{q+1} w') = t$. Hence there is a 2-cell $\alpha_{z_l \dots z_{q+1}, w'}$ yielding the confluence diagram



Case 3: $q < l$ and $y_{i_0} > z_{i_0}$ for some $1 \leqslant i_0 \leqslant q$. We compute the columns w and w' of the tableau $P(vt)$. If the biggest element of the column w is y_q, then we obtain the same confluent branching

as in Case 1. If the first element of w is z_l, then one obtains the same confluent critical branchings as in Case 2. □

3.2.6. Lemma. *If* $u^{\times 2}v^{\times 2}t$ *, we have the following confluent critical branching:*

$$\begin{array}{ccccccc} & & c_ec_{e'}c_t & \xRightarrow{c_e\alpha_{e',t}} & c_ec_bc_{b'} & & \\ & \overset{\alpha_{u,v}c_t}{\nearrow} & & \Downarrow D_{u,v,t} & & \overset{\alpha_{e,b}c_{b'}}{\searrow} & \\ c_uc_vc_t & & & & & & c_ac_dc_{b'} \\ & \underset{c_u\alpha_{v,t}}{\searrow} & & & & \underset{c_a\alpha_{a',w'}}{\nearrow} & \\ & & c_uc_wc_{w'} & \xRightarrow[\alpha_{u,w}c_{w'}]{} & c_ac_{a'}c_{w'} & & \end{array} \qquad (8)$$

where e, e' *(resp.* w, w'*) denote the two columns of the tableau* $P(uv)$ *(resp.* $P(vt)$*) and* a, a' *(resp.* b, b'*) denote the two columns of the tableau* $P(uw)$ *(resp.* $P(e't)$*).*

Proof. By hypothesis, $\ell^{\mathrm{nds}}(uv) = 2$ and $\ell^{\mathrm{nds}}(vt) = 2$, hence $x_1 \leqslant y_q$ and $y_1 \leqslant z_l$. In addition, since $u^{\times 2}v$, the tableau $P(uw)$ consists of two columns, that we denote by a and a'. Thus there is a 2-cell $\alpha_{u,w} : c_uc_w \Rightarrow c_ac_{a'}$. Moreover, as $u^{\times 2}v$ and $v^{\times 2}t$, we have

$$((p < q) \text{ or } (x_{i_0} > y_{i_0} \text{ for some } i_0 \leqslant q)) \quad \text{and} \quad ((q < l) \text{ or } (y_{j_0} > z_{j_0} \text{ for some } j_0 \leqslant l)),$$

thus we consider the following cases.

Case 1: $p < q < l$ and $y_i \leqslant z_i$, for all $i \leqslant q$, and $x_i \leqslant y_i$, for all $i \leqslant p$. We have

$$w = z_l \dots z_{q+1}y_q \dots y_1, \quad w' = z_q \dots z_1, \quad e = y_q \dots y_{p+1}x_p \dots x_1 \quad \text{and} \quad e' = y_p \dots y_1.$$

Since $z_l \geqslant y_1$, the tableau $P(e't)$ consists of two columns, that we denote by b and b'. Thus there is a 2-cell $\alpha_{e',t} : c_{e'}c_t \Rightarrow c_bc_{b'}$. In addition, we have

$$b = z_l \dots z_{p+1}y_p \dots y_1, \quad b' = z_p \dots z_1, \quad a = z_l \dots z_{q+1}y_q \dots y_{p+1}x_p \dots x_1 \quad \text{and} \quad a' = y_p \dots y_1.$$

Since $z_q \geqslant y_1$, the tableau $P(a'w')$ consists of two columns, that we denote by d and d'. Thus there is a 2-cell $\alpha_{a',w'} : c_{a'}c_{w'} \Rightarrow c_dc_{d'}$. Since $z_l \geqslant x_1$, the tableau $P(eb)$ consists of two columns, that we denote by s and s'. Then there is a 2-cell $\alpha_{e,b} : c_ec_b \Rightarrow c_sc_{s'}$. In the other hand, we have

$$d = z_q \dots z_{p+1}y_p \dots y_1, \; d' = z_p \dots z_1, \; s = z_l \dots z_{q+1}y_q \dots y_{p+1}x_p \dots x_1 \text{ and } s' = z_q \dots z_{p+1}y_p \dots y_1.$$

Hence $a = s$, $d = s'$ and $d' = b'$ which yields the confluence diagram (8).

Case 2: $\begin{cases} q < l \text{ and } y_i \leqslant z_i \text{ for all } i \leqslant q \\ p \geqslant q \text{ and } x_{i_0} > y_{i_0} \text{ for some } i_0 \leqslant q \end{cases}$ or $\begin{cases} q < l \text{ and } y_i \leqslant z_i \text{ for all } i \leqslant q \\ p < q \text{ and } x_{i_0} > y_{i_0} \text{ for some } i_0 \leqslant p \end{cases}$

We have $w = z_l \dots z_{q+1}y_q \dots y_1$ and $w' = z_q \dots z_1$. Using Schensted's algorithm the smallest element of the column a' is an element of v. Since z_q is greater or equal than each element of v, the tableau $P(a'w')$ consists of two columns, that we denote by d and d'.

On the other hand, all the elements of e' are elements of v. Since z_l is bigger than each element of v, the tableau $P(e't)$ consists of two columns, that we denote by b and b'. Thus there is a 2-cell $\alpha_{e',t} : c_{e'}c_t \Rightarrow c_bc_{b'}$. Hence, we consider two cases depending on whether or not $c_ec_bc_{b'}$ is a tableau. Suppose $c_ec_bc_{b'}$ is a tableau. The column e does not contain elements from the column t, then during inserting the column w into the column u, we can only insert some elements of $y_q \dots y_1$ into u and we obtain $a = e$. Since $c_e c_b c_{b'}$ is the unique tableau obtained from $c_u c_v c_t$ and $a = e$, we obtain $C(a'w') = bb'$. As a consequence, there is a 2-cell $\alpha_{a',w'} : c_{a'} c_{w'} \Rightarrow c_b c_{b'}$ yielding the following confluence diagram:

$$\begin{array}{ccccc} & & c_e c_{e'} c_t & \overset{c_e \alpha_{e',t}}{\Longrightarrow} & c_e c_b c_{b'} \\ & \overset{\alpha_{u,v} c_t}{\nearrow\!\!\nearrow} & \Downarrow D^{(1)}_{u,v,t} & & \Uparrow c_a \alpha_{a',w'} \\ c_u c_v c_t & & & & \\ & \underset{c_u \alpha_{v,t}}{\searrow\!\!\searrow} & c_u c_w c_{w'} & \underset{\alpha_{u,w} c_{w'}}{\Longrightarrow} & c_a c_{a'} c_{w'} \end{array} \tag{9}$$

Suppose $c_e c_b c_{b'}$ is not a tableau. The first element of the column b is z_l. The smallest element of the column e is either x_1 or y_j, where y_j is the biggest element of the column v such that $y_j < x_1$. By hypothesis the tableau $P(uw)$ consists of two columns, then $x_1 \leqslant z_l$. In addition, z_l is greater than each element of v then $y_j \leqslant z_l$. Hence, in all cases, the tableau $P(eb)$ consists of two columns. On the other hand, using Schensted's algorithm, we have $a' = z_{i_k} \dots z_{i_1} y_{j_{k'}} \dots y_{j_1}$ with $q + 1 \leqslant i_1 < \dots < i_k \leqslant l$, $1 \leqslant j_1 < \dots < j_{k'} \leqslant q$ and we have $e' = y_{j_{k'}} \dots y_{j_1}$. In addition, we have $b' = d' = z_{i_{k''}} \dots z_{i_1}$ with $1 \leqslant i_1 < \dots < i_{k''} \leqslant q$ and $C(eb) = ad$. Hence there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_a c_d$ which yields the confluence diagram (8).

Case 3: $\begin{cases} q \geqslant l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leqslant l \\ p < q \text{ and } x_i \leqslant y_i \text{ for all } i \leqslant p \end{cases}$ or $\begin{cases} q < l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leqslant q \\ p < q \text{ and } x_i \leqslant y_i \text{ for all } i \leqslant p \end{cases}$

We have $e = y_q \dots y_{p+1} x_p \dots x_1$ and $e' = y_p \dots y_1$. Since $y_1 \leqslant z_l$, the tableau $P(e't)$ consists of two columns, that we denote by b and b'. The first element of the column b is either z_l or y_p which are bigger or equal to x_1, then the tableau $P(eb)$ consists of two columns, that we denote by s and s'. Suppose $l \leqslant p$. By Schensted's insertion algorithm, we have $C(e't) = bw'$ and $w = y_q \dots y_{p+1} b$. On the other hand, since $x_p < y_{p+1}$, we have $P(uw) = P(u(y_q \dots y_{p+1} b)) = P(eb)$. Hence, there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_a c_{a'}$ which yields the confluence diagram:

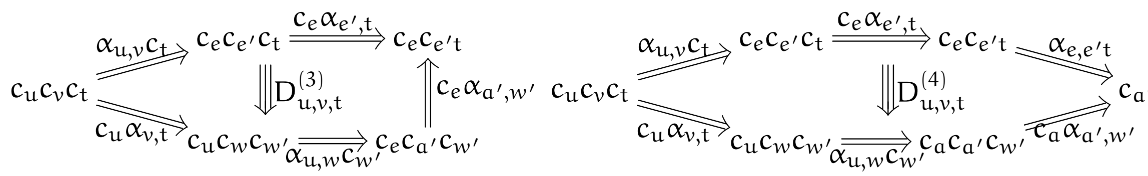
$$\begin{array}{ccccc} & & c_e c_{e'} c_t & \overset{c_e \alpha_{e',t}}{\Longrightarrow} & c_e c_b c_{w'} \\ & \overset{\alpha_{u,v} c_t}{\nearrow\!\!\nearrow} & \Downarrow D^{(2)}_{u,v,t} & & \Downarrow \alpha_{e,b} c_{w'} \\ c_u c_v c_t & & & & \\ & \underset{c_u \alpha_{v,t}}{\searrow\!\!\searrow} & c_u c_w c_{w'} & \underset{\alpha_{u,w} c_{w'}}{\Longrightarrow} & c_a c_{a'} c_{w'} \end{array} \tag{10}$$

For $l > p$, we consider two cases depending on whether or not the first element of the column b is y_p. If this element is y_p, then when computing the tableau $P(vt)$ no element of the column t is inserted in $y_q \dots y_{p+1}$. Hence we have $w = y_q \dots y_{p+1} b$ and $b' = w'$. On the other hand, by Schensted's insertion procedure we have $P(uw) = P(eb)$. Hence, there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_a c_{a'}$ which yields the confluence diagram (10). Suppose that the first element of the column b is z_l. Then when computing the tableau $P(vt)$ some elements of the column t are inserted in $y_q \dots y_{p+1}$. In this case, we have that the column w' contains more elements than b' and that $c_s c_{s'} c_{b'}$ is a tableau. Moreover, by Schensted's insertion procedure, we have $a = s$. Since $c_s c_{s'} c_{b'}$ is the unique tableau obtained from $c_u c_v c_t$ and $a = s$, we obtain that $C(a'w') = s'b'$. As a consequence, there is a 2-cell $\alpha_{a',w'} : c_{a'} c_{w'} \Rightarrow c_{s'} c_{b'}$ which yields the confluence diagram (8).

Case 4: $\begin{cases} q \geqslant l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leqslant l \\ p \geqslant q \text{ and } x_{j_0} > y_{j_0} \text{ for some } j_0 \leqslant q \end{cases}$ or $\begin{cases} q \geqslant l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leqslant q \\ p < q \text{ and } x_{j_0} > y_{j_0} \text{ for some } j_0 \leqslant p \end{cases}$

or $\begin{cases} q < l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leqslant q, \\ p \geqslant q \text{ and } x_{j_0} > y_{j_0} \text{ for some } j_0 \leqslant q. \end{cases}$ or $\begin{cases} q < l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leqslant q \\ p < q \text{ and } x_{j_0} > y_{j_0} \text{ for some } j_0 \leqslant p \end{cases}$

By Lemma 3.2.4, the last term of e' is y_1 or y_{j+1}, where y_j is the biggest element of v such that $y_j < x_1$. Suppose that the last term of e' is y_1. Since $z_l \geqslant y_1$, the tableau $P(e't)$ consists of two columns. Furthermore, if the last term of e' is y_{j+1}, then we consider two cases: $z_l \geqslant y_{j+1}$ or $z_l < y_{j+1}$. Suppose $z_l < y_{j+1}$, then the tableau $P(e't)$ consists of one column $e't$. We consider two cases depending on whether or not $c_e c_{e't}$ is a tableau. With the same arguments of Case 2, we obtain a confluence diagram of the following forms:



Suppose the tableau $P(e't)$ consists of two columns. Using the same arguments as in Case 2 and Case 3, we obtain a confluence diagram of the form $D_{u,v,t}$, $D^{(1)}_{u,v,t}$ or $D^{(2)}_{u,v,t}$. □

3.2.7. Remark, [17]. Thanks to a private communication by Lecouvey, Lemma 2.3.5 and an involution on tableaux can be used to prove the confluence of the critical branching (3) as follows. Let u be a column in $\mathrm{col}(n)$ of length p. Schützenberger introduced the *involution* of u, denoted by u^*, as the column of length $n-p$ obtained by taking the complement of the elements of u. More generally, let $u_1 \ldots u_r$ be the column reading of a tableau, then $(u_1 \ldots u_r)^* = u_r^* \ldots u_1^*$ and $u_r^* \ldots u_1^*$ is also the column reading of a tableau. Moreover, if w is the column reading of a Young tableau, then we have $P(w^*) = P(w)^*$. In particular, for three columns c_u, c_v and c_t in $\mathrm{Col}_1(n)$, we have $P(c_t^* c_v^* c_u^*) = P(c_u c_v c_t)^*$, see [18].

By Lemma 2.3.5, $c_a c_d c_{b'}$ is a normal form of $c_u c_v c_t$, that is, $P(c_u c_v c_t) = c_a c_d c_{b'}$. Then to prove the confluence of the 3-cell (3), it is sufficient to show that $P(c_u c_v c_t) = c_a C(c_{a'} c_{w'})$. We have

$$c_u c_v c_t \overset{c_u \alpha_{v,t}}{\Longrightarrow} c_u C(c_v c_t) = c_u c_w c_{w'} \overset{\alpha_{u,w} c_{w'}}{\Longrightarrow} C(c_u c_w) c_{w'} = c_a c_{a'} c_{w'} \overset{c_a \alpha_{a',w'}}{\Longrightarrow} c_a C(c_{a'} c_{w'}).$$

By applying the involution on tableaux, we obtain

$$c_t^* c_v^* c_u^* \Longrightarrow C(c_t^* c_v^*) c_u^* = c_{w'}^* c_w^* c_u^* \Longrightarrow c_{w'}^* C(c_w^* c_u^*) = c_{w'}^* c_{a'}^* c_a^* \Longrightarrow C(c_{w'}^* c_{a'}^*) c_a^*.$$

By Lemma 2.3.5, we have $P(c_t^* c_v^* c_u^*) = C(c_{w'}^* c_{a'}^*) c_a^*$. Since $P(c_t^* c_v^* c_u^*) = P(c_u c_v c_t)^*$, we deduce that $P(c_u c_v c_t)^* = C(c_{w'}^* c_{a'}^*) c_a^*$. Finally, by applying the involution on tableaux, we obtain $P(c_u c_v c_t) = c_a C(c_{a'} c_{w'})$. Note that this construction does not give the explicit forms of the 2-sources and the 2-targets of the confluence diagrams of the critical branchings as doing in lemmas above.

4. Reduction of the coherent presentation

In this section, we begin by recalling the homotopical reduction procedure from [6, Section 2.3.]. We explicit all the reduction steps that we need to reduce the coherent presentation $\mathrm{Col}_3(n)$ into a smaller finite coherent presentation of the monoid $\mathbf{P}_n$ that extends the Knuth presentation.

4.1. Homotopical reduction procedure

4.1.1. Homotopical reduction procedure. Let Σ be a $(3,1)$-polygraph. A 3-*sphere* of the $(3,1)$-category $\Sigma_3^\top$ is a pair (f,g) of 3-cells of $\Sigma_3^\top$ such that $s_2(f) = s_2(g)$ and $t_2(f) = t_2(g)$. A *collapsible part of* Σ is a triple $(\Gamma_2, \Gamma_3, \Gamma_4)$ made of a family Γ_2 of 2-cells of Σ, a family Γ_3 of 3-cells of Σ and a family Γ_4 of 3-spheres of $\Sigma_3^\top$, such that the following conditions are satisfied:

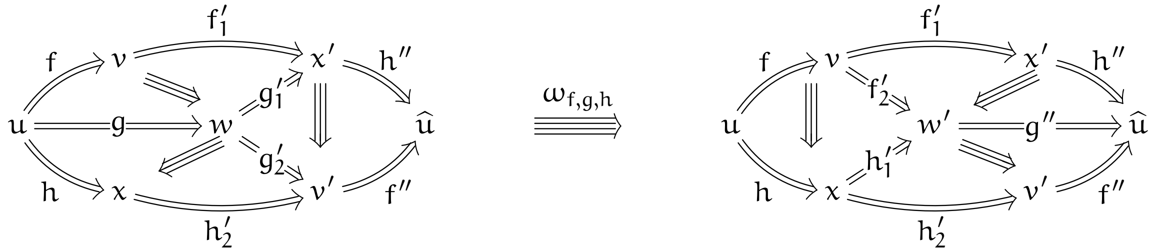
i) every γ of every Γ_k is collapsible, that is, $t_{k-1}(\gamma)$ is in Σ_{k-1} and $s_{k-1}(\gamma)$ does not contain $t_{k-1}(\gamma)$,

ii) no cell of Γ_2 (resp. Γ_3) is the target of a collapsible 3-cell of Γ_3 (resp. 3-sphere of Γ_4),

iii) there exists a well-founded order on the cells of Σ such that, for every γ in every Γ_k, $t_{k-1}(\gamma)$ is strictly greater than every generating $(k-1)$-cell that occurs in the source of γ.

The *homotopical reduction* of the $(3,1)$-polygraph Σ with respect to a collapsible part Γ is the Tietze transformation, denoted by R_Γ, from the $(3,1)$-category $\Sigma_3^\top$ to the $(3,1)$-category freely generated by the $(3,1)$-polygraph obtained from Σ by removing the cells of Γ and all the corresponding redundant cells. We refer the reader to [6, 2.3.1] for details on the definition of the Tietze transformation R_Γ defined by well-founded induction as follows. For any γ in Γ, we have $R_\Gamma(t(\gamma)) = R_\Gamma(s(\gamma))$ and $R_\Gamma(\gamma) = 1_{R_\Gamma(s(\gamma))}$. In any other cases, the transformation R_Γ acts as an identity.

4.1.2. Generating triple confluences. A *local triple branching* of a 2-polygraph Σ is a triple (f,g,h) of rewriting steps of Σ with a common source. An *aspherical* triple branchings have two of their 2-cells equal. A *Peiffer* triple branchings have at least one of their 2-cells that form a Peiffer branching with the other two. The *overlap* triple branchings are the remaining local triple branchings. Local triple branchings are ordered by inclusion of their sources and a minimal overlap triple branching is called *critical*. If Σ is a coherent and convergent $(3,1)$-polygraph, a *triple generating confluence of* Σ is a 3-sphere



where (f,g,h) is a triple critical branching of the 2-polygraph Σ_2 and the other cells are obtained by confluence, see [6, 2.3.2] for details.

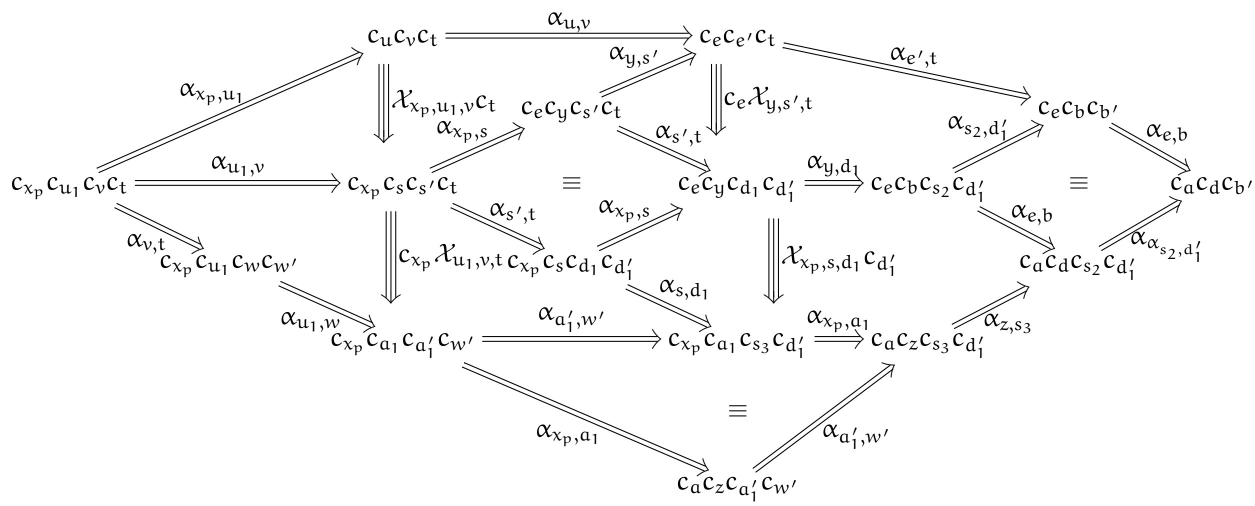
4.1.3. Homotopical reduction of the polygraph $\mathrm{Col}_3(n)$**.** In the rest of this section, we apply three steps of homotopical reduction on the $(3,1)$-polygraph $\mathrm{Col}_3(n)$. As a first step, we apply in 4.2 a homotopical reduction on the $(3,1)$-polygraph $\mathrm{Col}_3(n)$ with a collapsible part defined by some of the generating triple confluences of the 2-polygraph $\mathrm{Col}_2(n)$. In this way, we reduce the coherent presentation $\mathrm{Col}_3(n)$ of the monoid $\mathbf{P}_n$ into the coherent presentation $\overline{\mathrm{Col}}_3(n)$ of $\mathbf{P}_n$, whose underlying 2-polygraph is $\mathrm{Col}_2(n)$ and the 3-cells $\mathcal{X}_{u,v,t}$ are those of $\mathrm{Col}_3(n)$, but with $\ell(u) = 1$. We reduce in 4.3 the coherent presentation $\overline{\mathrm{Col}}_3(n)$ into a coherent presentation $\mathrm{PreCol}_3(n)$ of $\mathbf{P}_n$, whose underlying

2-polygraph is $\mathrm{PreCol}_2(n)$. This reduction is given by a collapsible part defined by a set of 3-cells of $\overline{\mathrm{Col}}_3(n)$. In a final step, we reduce in 4.4 the coherent presentation $\mathrm{PreCol}_3(n)$ into a coherent presentation $\mathrm{Knuth}_3(n)$ of $\mathbf{P}_n$ whose underlying 2-polygraph is $\mathrm{Knuth}_2(n)$. By [6, Theorem 2.3.4], all these homotopical reductions preserve coherence. That is, the $(3,1)$-polygraph $\mathrm{Col}_3(n)$ being a coherent presentation of $\mathbf{P}_n$, the $(3,1)$-polygraphs $\overline{\mathrm{Col}}_3(n)$ and $\mathrm{Knuth}_3(n)$ are coherent presentations of $\mathbf{P}_n$.

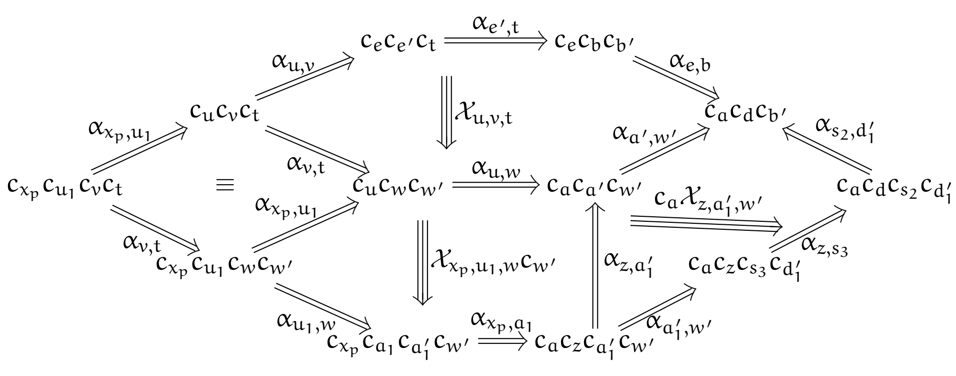
4.2. A reduced column presentation

We apply the homotopical reduction procedure in order to reduce the $(3,1)$-polygraph $\mathrm{Col}_3(n)$ using the generating triple confluences.

4.2.1. Generating triple confluences of $\mathrm{Col}_2(n)$**.** Consider the homotopical reduction procedure on the $(3,1)$-polygraph $\mathrm{Col}_3(n)$ defined using the collapsible part made of generating triple confluences. By Theorem 3.2.2, the family of 3-cells $\mathcal{X}_{u,v,t}$ given in (4) and indexed by columns u, v and t in $\mathrm{col}(n)$ such that $u^{\times}v^{\times}t$ forms a homotopy basis of the $(2,1)$-category $\mathrm{Col}_2(n)^{\top}$. Let us consider such a triple (u,v,t) with $\ell(u) \geqslant 2$. Let x_p be in $[n]$ such that $u = x_p u_1$ with u_1 in $\mathrm{col}(n)$. There is a critical triple branching with source $c_{x_p}c_{u_1}c_v c_t$. Let us show that the confluence diagram induced by this triple branching is represented by the 3-sphere $\Omega_{x_p,u_1,v,t}$ whose source is the following 3-cell



and whose target is the following 3-cell



In the generating triple confluence, some columns may be empty and thus the indicated 2-cells α may be identities. To facilitate the reading of the diagram, we have omitted the context of the 2-cells α.

The 3-sphere $\Omega_{x_p,u_1,v,t}$ is constructed as follows. We have $x_p \,{}^{\times 1}\, u_1$ and $u_1 \,{}^{\times}\, w$, thus $\mathcal{X}_{x_p,u_1,w}$ is either of the form $A_{x_p,u_1,w}$ or $C_{x_p,u_1,w}$. Let us denote by a_1 and a'_1 the two columns of the tableau $P(u_1 w)$. The 3-cell $\mathcal{X}_{x_p,u_1,w}$ being confluent, we have $C(x_p a_1) = az$ with z in $[n]$ and $C(z a'_1) = a'$. In addition, from $z \,{}^{\times}\, a'_1$ and $a'_1 \,{}^{\times}\, w'$, we deduce that $\mathcal{X}_{z,a'_1,w'}$ is either of the form $A_{z,a'_1,w'}$ or $C_{z,a'_1,w'}$. From $x_p \,{}^{\times 1}\, u_1$ and $u_1 \,{}^{\times}\, v$, we deduce that $\mathcal{X}_{x_p,u_1,v}$ is either of the form $A_{x_p,u_1,v}$ or $C_{x_p,u_1,v}$. Let us denote by s and s' the two columns of the tableau $P(u_1 v)$. The 3-cell $\mathcal{X}_{x_p,u_1,v}$ being confluent, we obtain that $C(x_p s) = ey$ with y in $[n]$ and $C(ys') = e'$. From $y \,{}^{\times 1}\, s'$ and $s' \,{}^{\times}\, t$, we deduce that $\mathcal{X}_{y,s',t}$ is either of the form $A_{y,s',t}$ or $C_{y,s',t}$. Denote by d_1 and d'_1 the two columns of the tableau $P(s't)$. The 3-cell $\mathcal{X}_{y,s',t}$ being confluent and $C(e't) = bb'$, we have $C(yd_1) = bs_2$ and $C(s_2 d'_1) = b'$. On the other hand, the 3-cell $\mathcal{X}_{u_1,v,t}$ is confluent, then we have $C(sd_1) = a_1 s_3$ and $C(a'_1 w') = s_3 d'_1$. Finally, since the 3-cell $\mathcal{X}_{x_p,s,d_1}$ is confluent, we obtain $C(zs_3) = ds_2$.

4.2.2. Reduced coherent column presentation. Let us define by $\overline{\mathrm{Col}}_3(n)$ the extended presentation of the monoid $\mathbf{P}_n$ obtained from $\mathrm{Col}_2(n)$ by adjunction of one family of 3-cells $\mathcal{X}_{x,v,t}$ of the form (4), for every 1-cell x in $[n]$ and columns v and t in $\mathrm{col}(n)$ such that $x \,{}^{\times}\, v \,{}^{\times}\, t$. The following result shows that this reduced presentation is also coherent.

4.2.3. Proposition. *For $n > 0$, the $(3,1)$-polygraph $\overline{\mathrm{Col}}_3(n)$ is a coherent presentation of the monoid $\mathbf{P}_n$.*

Proof. Let Γ_4 be the collapsible part made of the family of 3-sphere $\Omega_{x_p,u_1,v,t}$, indexed by x_p in $[n]$ and u_1, v, t in $\mathrm{col}(n)$ such that $u \,{}^{\times}\, v \,{}^{\times}\, t$ and $u = x_p u_1$. On the 3-cells of $\mathrm{Col}_3(n)$, we define a well-founded order $\lhd$ by

- **i)** $A_{u,v,t} \lhd C_{u,v,t} \lhd B_{u,v,t} \lhd D_{u,v,t}$,
- **ii)** if $\mathcal{X}_{u,v,t} \in \{A_{u,v,t}, B_{u,v,t}, C_{u,v,t}, D_{u,v,t}\}$ and $u' \preccurlyeq_{\text{deglex}} u$, then $\mathcal{X}_{u',v',t'} \lhd \mathcal{X}_{u,v,t}$,

for any u, v, t in $\mathrm{col}(n)$ such that $u \,{}^{\times}\, v \,{}^{\times}\, t$. By construction of the 3-sphere $\Omega_{x_p,u_1,v,t}$, its source contains the 3-cell $\mathcal{X}_{u_1,v,t}$ and its target contains the 3-cell $\mathcal{X}_{u,v,t}$ with $\ell(u_1) < \ell(u)$. Up to a Nielsen transformation, the homotopical reduction R_{Γ_4} applied on the $(3,1)$-polygraph $\mathrm{Col}_3(n)$ with respect to Γ_4 and the order $\lhd$ give us the $(3,1)$-polygraph $\overline{\mathrm{Col}}_3(n)$. In this way, the presentation $\overline{\mathrm{Col}}_3(n)$ is a coherent presentation of the monoid $\mathbf{P}_n$. □

4.3. Pre-column coherent presentation

We reduce the coherent presentation $\overline{\mathrm{Col}}_3(n)$ into a coherent presentation whose underlying 2-polygraph is $\mathrm{PreCol}_2(n)$. This reduction is obtained using the homotopical reduction R_{Γ_3} on the $(3,1)$-polygraph $\overline{\mathrm{Col}}_3(n)$ whose collapsible part Γ_3 is defined by

$$
\begin{aligned}
\Gamma_3 = \{\, & A_{x,v,t} \mid x \in [n],\ v,t \in \mathrm{col}(n) \text{ such that } x^{\times 1} v^{\times 1} t \,\} \\
& \cup \{\, B_{x,v,t} \mid x \in [n],\ v,t \in \mathrm{col}(n) \text{ such that } x^{\times 2} v^{\times 1} t \,\} \\
& \cup \{\, C_{x,v,t} \mid x \in [n],\ v,t \in \mathrm{col}(n) \text{ such that } x^{\times 1} v^{\times 2} t \,\},
\end{aligned}
$$

and the well-founded order defined as follows. Given u and v in $\mathrm{col}(n)$ such that $u^{\times}v$. We define a well-founded order $\lhd$ on the 2-cells of $\mathrm{Col}_2(n)$ as follows

$$\alpha_{u',v'} \lhd \alpha_{u,v} \quad \text{if} \quad \begin{cases} \ell(uv) > \ell(u'v') & \text{or} \\ \ell(uv) = \ell(u'v') & \text{and} \quad \begin{cases} \ell(u) > \ell(C_r(u'v')) \text{ or} \\ \ell(u) \leqslant \ell(C_r(u'v')) \text{ and } u' \preccurlyeq_{\mathrm{rev}} u \end{cases} \end{cases}$$

for any columns u, v, u' and v' in $\mathrm{col}(n)$ such that $u^{\times}v$ and $u'^{\times}v'$, where $\preccurlyeq_{\mathrm{rev}}$ is the total order on $\mathrm{col}(n)$ defined by $u \preccurlyeq_{\mathrm{rev}} v$ if $\ell(u) > \ell(v)$ or $\ell(u) = \ell(v)$ and $u <_{\mathrm{lex}} v$, for all u and v in $\mathrm{col}(n)$.

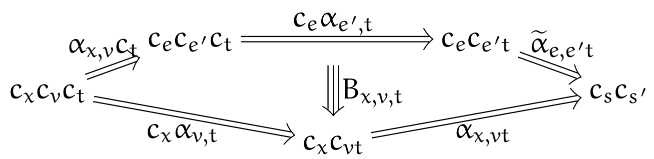
4.3.1. The homotopical reduction R_{Γ_3}. Consider the well-founded order $\lhd$ on the 2-cells of $\mathrm{Col}_2(n)$ and the well-founded order $\lhd$ on 3-cells defined in the proof of Proposition 4.2.3. The reduction R_{Γ_3} induced by these orders can be decomposed as follows. For any x in $[n]$ and columns v, t such that $x^{\times 1}v^{\times 1}t$, we have $\alpha_{x,v} \lhd \alpha_{xv,t}$, $\alpha_{v,t} \lhd \alpha_{xv,t}$ and $\alpha_{x,vt} \lhd \alpha_{xv,t}$. The reduction R_{Γ_3} removes the 2-cell $\alpha_{xv,t}$ together with the 3-cell $A_{x,v,t}$ defined in Lemma 3.2.3. By iterating this reduction on the length of the column v, we reduce all the 2-cells of $\mathrm{Col}_2(n)$ to the following set of 2-cells

$$\{\ \alpha_{u,v} \mid \ell(u) \geqslant 1,\ \ell(v) \geqslant 2 \text{ and } u^{\times 2}v\ \} \cup \{\ \alpha_{u,v} \mid \ell(u) = 1,\ \ell(v) \geqslant 1 \text{ and } u^{\times 1}v\ \}. \tag{11}$$

For any x in $[n]$ and columns v, t such that $x^{\times 1}v^{\times 2}t$, consider the 3-cell $C_{x,v,t}$ defined in Lemma 3.2.5. The 2-cells $\alpha_{x,v}$, $\alpha_{v,t}$, $\alpha_{x,w}$ and $\alpha_{a',w'}$ are smaller than $\alpha_{xv,t}$ for the order $\lhd$. The reduction R_{Γ_3} removes the 2-cell $\alpha_{xv,t}$ together with the 3-cell $C_{x,v,t}$. By iterating this reduction on the length of v, we reduce the set of 2-cells given in (11) to the following set:

$$\{\ \alpha_{u,v} \mid \ell(u) = 1,\ \ell(v) \geqslant 2 \text{ and } u^{\times 2}v\ \} \cup \{\ \alpha_{u,v} \mid \ell(u) = 1,\ \ell(v) \geqslant 1 \text{ and } u^{\times 1}v\ \}. \tag{12}$$

For any x in $[n]$ and columns v, t such that $x^{\times 2}v^{\times 1}t$, consider the following 3-cell:



where e, e', s and s' are defined in Lemma 3.2.4. Note that $\widetilde{\alpha}_{e,e't}$ is the 2-cell in (12) obtained from the 2-cell $\alpha_{e,e't}$ by the previous step of the homotopical reduction by the 3-cell $C_{x,v,t}$. Having x in $[n]$, by definition of α we have e' in $[n]$. The 2-cells $\alpha_{x,v}$, $\alpha_{e',t}$, $\alpha_{v,t}$ and $\widetilde{\alpha}_{e,e't}$ being smaller than $\alpha_{x,vt}$ for the order $\lhd$, we can remove the 2-cells $\alpha_{x,vt}$ together with the 3-cell $B_{x,v,t}$. By iterating this reduction on the length of the column t, we reduce the set (12) to the following set

$$\{\ \alpha_{u,v} \mid \ell(u) = 1,\ \ell(v) = 2 \text{ and } u^{\times 2}v\ \} \cup \{\ \alpha_{u,v} \mid \ell(u) = 1,\ \ell(v) \geqslant 1 \text{ and } u^{\times 1}v\ \}. \tag{13}$$

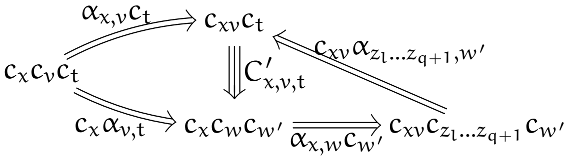
4.3.2. Lemma. *The set of* 2*-cells defined in (13) is equal to* $\mathrm{PreCol}_2(n)$.

Proof. By definition of $\mathrm{PreCol}_2(n)$, it is sufficient to prove that

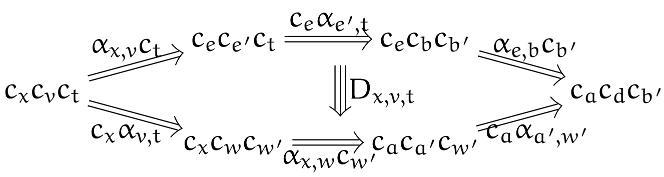
$$\mathrm{PC}_2(n) = \{\ \alpha_{u,v} : c_uc_v \Rightarrow c_wc_{w'} \mid \ell(u) = 1,\ \ell(v) = 2 \text{ and } u^{\times 2}v\ \}.$$

Consider the 2-cells $\alpha_{u,v}$ in $\mathrm{Col}_2(n)$ such that $\ell(u) = 1$, $\ell(v) = 2$ and $u^{\times 2}v$. Suppose that $v = xx'$ with $x > x'$ in $[n]$. Since $u^{\times 2}v$, we obtain that $u \leqslant x$. Hence, we have two cases to consider. If $u \leqslant x'$, then $C(uv) = (xu)x'$. Hence, the 2-cell $\alpha_{u,v}$ is equal to the 2-cell $\alpha'_{u,xx'} : c_uc_{xx'} \Rightarrow c_{xu}c_{x'}$. In the other case, if $x' < u$, then $C(uv) = (ux')x$. Hence the 2-cell $\alpha_{u,v}$ is equal to $\alpha'_{u,xx'} : c_uc_{xx'} \Rightarrow c_{ux'}c_x$. □

4.3.3. Pre-column coherent presentation. The homotopical reduction R_{Γ_3}, defined in 4.3.1, reduces the coherent presentation $\overline{\mathrm{Col}}_3(n)$ into a coherent presentation of the monoid $\mathbf{P}_n$. The set of 2-cells of this coherent presentation is given by (13), which is $\mathrm{PreCol}_2(n)$ by Lemma 4.3.2. Let us denote by $\mathrm{PreCol}_3(n)$ the extended presentation of the monoid $\mathbf{P}_n$ obtained from $\mathrm{PreCol}_2(n)$ by adjunction of the 3-cells of type $R_{\Gamma_3}(C'_{x,v,t})$ where



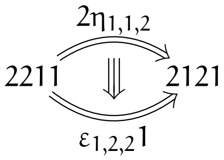
with $x^{\times 1}v^{\times 2}t$, and the 3-cells of type $R_{\Gamma_3}(D_{x,v,t})$ where



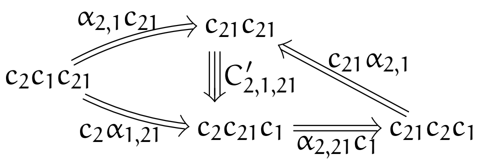
with $x^{\times 2}v^{\times 2}t$. The homotopical reduction R_{Γ_3} eliminates the 3-cells of $\overline{\mathrm{Col}}_3(n)$ of the form $A_{x,v,t}$, $B_{x,v,t}$ and $C_{x,v,t}$, which are not of the form $C'_{x,v,t}$. We have then proved the following result.

4.3.4. Theorem. *For* $n > 0$*, the* $(3,1)$*-polygraph* $\mathrm{PreCol}_3(n)$ *is a coherent presentation of the monoid* $\mathbf{P}_n$.

4.3.5. Example: coherent presentation of monoid $\mathbf{P}_2$. The 2-polygraph $\mathrm{Knuth}_2(2)$ has for 2-cells $\eta_{1,1,2} : 211 \Rightarrow 121$ and $\varepsilon_{1,2,2} : 221 \Rightarrow 212$. It is convergent with only one critical branching with source the 1-cell 2211. This critical branching is confluent:



Following the homotopical completion procedure given in 3.1.4, the 2-polygraph extended by the previous 3-cell is a coherent presentation of the monoid $\mathbf{P}_2$. Consider the column presentation $\mathrm{Col}_2(2)$ of the monoid $\mathbf{P}_2$ with 1-cells c_1, c_2 and c_{21} and 2-cells $\alpha_{2,1}$, $\alpha_{1,21}$ and $\alpha_{2,21}$. The coherent presentation $\mathrm{Col}_3(2)$ has only one 3-cell



It follows that the $(3,1)$-polygraphs $\overline{\mathrm{Col}}_3(2)$ and $\mathrm{Col}_3(2)$ coincide. Moreover, in this case the set Γ_3 is empty and the homotopical reduction R_{Γ_3} is the identity and thus $\mathrm{PreCol}_3(2)$ is also equal to $\mathrm{Col}_3(2)$.

4.3.6. Example: coherent presentation of monoid $\mathbf{P}_3$. For the monoid $\mathbf{P}_3$, the Knuth presentation has 3 generators and 8 relations. It is not convergent, but it can be completed by adding 3 relations. The obtained presentation has 27 3-cells corresponding to the 27 critical branchings. The column coherent presentation $\mathrm{Col}_3(3)$ of $\mathbf{P}_3$ has 7 generators, 22 relations and 42 3-cells. The coherent presentation $\overline{\mathrm{Col}}_3(3)$ has 7 generators, 22 relations and 34 3-cells. After applying the homotopical reduction R_{Γ_3}, the coherent presentation $\mathrm{PreCol}_3(3)$ admits 7 generators, 22 relations and 24 3-cells. We give in 4.4.10 the values of number of cells of the $(3,1)$-polygraphs $\overline{\mathrm{Col}}_3(n)$ and $\mathrm{PreCol}_3(n)$ for plactic monoids of rank $n \leqslant 10$.

4.4. Knuth's coherent presentation

We reduce the coherent presentation $\mathrm{PreCol}_3(n)$ into a coherent presentation of the monoid $\mathbf{P}_n$ whose underlying 2-polygraph is $\mathrm{Knuth}_2(n)$. We proceed in three steps developed in the next sections.

Step 1. We apply the inverse of the Tietze transformation $T_{\gamma\leftarrow\alpha'}$, that coherently replaces the 2-cells $\gamma_{x_p\ldots x_1}$ by the 2-cells $\alpha'_{x_p,x_{p-1}\ldots x_1}$, for each column $x_p\ldots x_1$ such that $\ell(x_p\ldots x_1) > 2$.

Step 2. We apply the inverse of the Tietze transformation $T_{\eta,\varepsilon\leftarrow\alpha'}$, that coherently replaces the 2-cells $\alpha'_{x,zy}$ by $\eta^c_{x,y,z}$, for every $1 \leqslant x \leqslant y < z \leqslant n$, and the 2-cells $\alpha'_{y,zx}$ by $\varepsilon^c_{x,y,z}$, for every $1 \leqslant x < y \leqslant z \leqslant n$.

Step 3. Finally for each column $x_p\ldots x_1$, we coherently eliminate the generator $c_{x_p\ldots x_1}$ together with the 2-cell $\gamma_{x_p\ldots x_1}$ with respect to the order $\preccurlyeq_{\text{deglex}}$.

4.4.1. Step 1. The Tietze transformation $T_{\gamma\leftarrow\alpha'} : \mathrm{CPC}_2(n)^\top \to \mathrm{PreCol}_2(n)^\top$ defined in Proposition 2.3.3 substitutes a 2-cell $\alpha'_{x_p,x_{p-1}\ldots x_1} : c_{x_p}c_{x_{p-1}\ldots x_1} \Longrightarrow c_{x_p\ldots x_1}$ to the 2-cell $\gamma_{x_p\ldots x_1}$ in $C_2(n)$, from the bigger column to the smaller one with respect to the total order $\preccurlyeq_{\text{deglex}}$.

We consider the inverse of this Tietze transformation $T^{-1}_{\gamma\leftarrow\alpha'} : \mathrm{PreCol}_2(n)^\top \to \mathrm{CPC}_2(n)^\top$ that substitutes the 2-cell $\gamma_{x_p\ldots x_1} : c_{x_p}\ldots c_{x_1} \Longrightarrow c_{x_p\ldots x_1}$ to the 2-cell $\alpha'_{x_p,x_{p-1}\ldots x_1} : c_{x_p}c_{x_{p-1}\ldots x_1} \Longrightarrow c_{x_p\ldots x_1}$, for each column $x_p\ldots x_1$ such that $\ell(x_p\ldots x_1) > 2$ with respect to the order $\preccurlyeq_{\text{deglex}}$.

Let us denote by $\mathrm{CPC}_3(n)$ the $(3,1)$-polygraph whose underlying 2-polygraph is $\mathrm{CPC}_2(n)$, and the set of 3-cells is defined by

$$\{\, T^{-1}_{\gamma\leftarrow\alpha'}(R_{\Gamma_3}(C'_{x,v,t})) \quad \text{for } x^{\times 1}v^{\times 2}t \,\} \cup \{\, T^{-1}_{\gamma\leftarrow\alpha'}(R_{\Gamma_3}(D_{x,v,t})) \quad \text{for } x^{\times 2}v^{\times 2}t \,\}.$$

In this way, we extend the Tietze transformation $T^{-1}_{\gamma\leftarrow\alpha'}$ into a Tietze transformation between the $(3,1)$-polygraphs $\mathrm{PreCol}_3(n)$ and $\mathrm{CPC}_3(n)$. The $(3,1)$-polygraph $\mathrm{PreCol}_3(n)$ being a coherent presentation of the monoid $\mathbf{P}_n$ and the Tietze transformation $T^{-1}_{\gamma\leftarrow\alpha'}$ preserves the coherence property, hence we have the following result.

4.4.2. Lemma. *For $n > 0$, the monoid $\mathbf{P}_n$ admits $\mathrm{CPC}_3(n)$ as a coherent presentation.*

4.4.3. Step 2. The Tietze transformation $T_{\eta,\varepsilon\leftarrow\alpha'}$ from $\mathrm{Knuth}_2^{cc}(n)^\top$ into $\mathrm{CPC}_2(n)^\top$ defined in the proof of Proposition 2.3.3 replaces the 2-cells $\eta^c_{x,y,z}$ and $\varepsilon^c_{x,y,z}$ in $\mathrm{Knuth}_2^{cc}(n)$ by composite of 2-cells in $\mathrm{CPC}_2(n)$.

Let us consider the inverse of this Tietze transformation $T^{-1}_{\eta,\varepsilon\leftarrow\alpha'} : \mathrm{CPC}_2(n)^\top \longrightarrow \mathrm{Knuth}_2^{cc}(n)^\top$. making the following transformations. For every $1 \leqslant x \leqslant y < z \leqslant n$, $T^{-1}_{\eta,\varepsilon\leftarrow\alpha'}$ substitutes the

2-cell $\eta^{c}_{x,y,z} : c_z c_x c_y \Rightarrow c_x c_z c_y$ to the 2-cell $\alpha'_{x,zy}$. For every $1 \leqslant x < y \leqslant z \leqslant n$, $T^{-1}_{\eta,\varepsilon\leftarrow\alpha'}$ substitutes the 2-cell $\varepsilon^{c}_{x,y,z} : c_y c_z c_x \Rightarrow c_y c_x c_z$ to the 2-cell $\alpha'_{y,zx}$.

Let us denote by $\mathrm{Knuth}^{cc}_3(n)$ the $(3,1)$-polygraph whose underlying 2-polygraph is $\mathrm{Knuth}^{cc}_2(n)$ and whose set of 3-cells is

$$\{\, T^{-1}_{\eta,\varepsilon\leftarrow\alpha'}(T^{-1}_{\gamma\leftarrow\alpha'}(R_{\Gamma_3}(C'_{x,v,t}))) \quad \text{for } x^{\times 1}v^{\times 2}t \,\} \cup \{\, T^{-1}_{\eta,\varepsilon\leftarrow\alpha'}(T^{-1}_{\gamma\leftarrow\alpha'}(R_{\Gamma_3}(D_{x,v,t}))) \quad \text{for } x^{\times 2}v^{\times 2}t \,\}.$$

We extend the Tietze transformation $T^{-1}_{\eta,\varepsilon\leftarrow\alpha'}$ into a Tietze transformation between $(3,1)$-polygraphs

$$T^{-1}_{\eta,\varepsilon\leftarrow\alpha'} : \mathrm{CPC}_3(n)^{\top} \longrightarrow \mathrm{Knuth}^{cc}_3(n)^{\top},$$

where the $(3,1)$-polygraph $\mathrm{CPC}_3(n)$ is a coherent presentation of the monoid $\mathbf{P}_n$ and the Tietze transformation $T^{-1}_{\eta,\varepsilon\leftarrow\alpha'}$ preserves the coherence property, hence we have the following result.

4.4.4. Lemma. *For $n > 0$, the monoid $\mathbf{P}_n$ admits $\mathrm{Knuth}^{cc}_3(n)$ as a coherent presentation.*

4.4.5. Step 3. Finally, in order to obtain the Knuth coherent presentation, we perform an homotopical reduction, obtained using the homotopical reduction R_{Γ_2} on the $(3,1)$-polygraph $\mathrm{Knuth}^{cc}_3(n)$ whose collapsible part Γ_2 is defined by the 2-cells γ_u of $C_2(n)$ and the well-founded order $\preccurlyeq_{\text{deglex}}$. Thus, for every 2-cell $\gamma_{x_p\ldots x_1} : c_{x_p}\ldots c_{x_1} \Longrightarrow c_{x_p\ldots x_1}$ in $C_2(n)$, we eliminate the generator $c_{x_p\ldots x_1}$ together with the 2-cell $\gamma_{x_p\ldots x_1}$, from the bigger column to the smaller one with respect to the order $\preccurlyeq_{\text{deglex}}$.

4.4.6. Knuth coherent presentation. Using the Tietze transformations constructed in the previous sections, we consider the following composite of Tietze transformations

$$\mathcal{R} \;:=\; R_{\Gamma_2} \circ T^{-1}_{\eta,\varepsilon\leftarrow\alpha'} \circ T^{-1}_{\gamma\leftarrow\alpha'} \circ R_{\Gamma_3}$$

defined from $\overline{\mathrm{Col}}_3(n)^{\top}$ to $\mathrm{Knuth}^{cc}_3(n)^{\top}$ as follows. Firstly, the transformation $\mathcal{R}$ eliminates the 3-cells of $\overline{\mathrm{Col}}_3(n)$ of the form $A_{x,v,t}$, $B_{x,v,t}$ and $C_{x,v,t}$ which are not of the form $C'_{x,v,t}$ and reduces its set of 2-cells to $\mathrm{PreCol}_2(n)$. Secondly, this transformation coherently replaces the 2-cells $\gamma_{x_p\ldots x_1}$ by the 2-cells $\alpha'_{x_p,x_{p-1}\ldots x_1}$, for each column $x_p\ldots x_1$ such that $\ell(x_p\ldots x_1) > 2$, the 2-cells $\alpha'_{x,zy}$ by $\eta^{c}_{x,y,z}$ for $1 \leqslant x \leqslant y < z \leqslant n$ and the 2-cells $\alpha'_{y,zx}$ by $\varepsilon^{c}_{x,y,z}$ for $1 \leqslant x < y \leqslant z \leqslant n$. Finally, for each column $x_p\ldots x_1$, the transformation $\mathcal{R}$ eliminates the generator $c_{x_p\ldots x_1}$ together with the 2-cell $\gamma_{x_p\ldots x_1}$ with respect to the order $\preccurlyeq_{\text{deglex}}$.

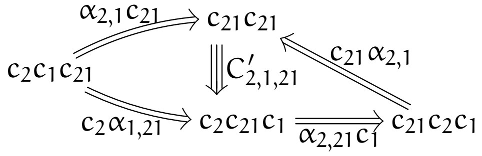
Let us denote by $\mathrm{Knuth}_3(n)$ the extended presentation of the monoid $\mathbf{P}_n$ obtained from $\mathrm{Knuth}_2(n)$ by adjunction of the following set of 3-cells

$$\{\, \mathcal{R}(C'_{x,v,t}) \quad \text{for } x^{\times 1}v^{\times 2}t \,\} \cup \{\, \mathcal{R}(D_{x,v,t}) \quad \text{for } x^{\times 2}v^{\times 2}t \,\}.$$

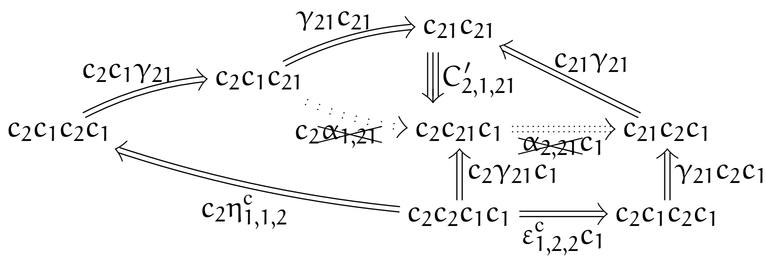
The transformation $\mathcal{R}$ being a composite of Tietze transformations, it follows the following result.

4.4.7. Theorem. *For $n > 0$, the $(3,1)$-polygraph $\mathrm{Knuth}_3(n)$ is a coherent presentation of the monoid $\mathbf{P}_n$.*

4.4.8. Example: Knuth's coherent presentation of the monoid $\mathbf{P}_2$. We have seen in Example 4.3.5 that the $(3,1)$-polygraphs $\mathrm{Col}_3(2)$, $\overline{\mathrm{Col}}_3(2)$ and $\mathrm{PreCol}_3(2)$ are equal. The coherent presentation $\mathrm{PreCol}_3(2)$ has three 2-cell $\alpha_{2,1}$, $\alpha_{1,21}$, $\alpha_{2,21}$ and the following 3-cell:

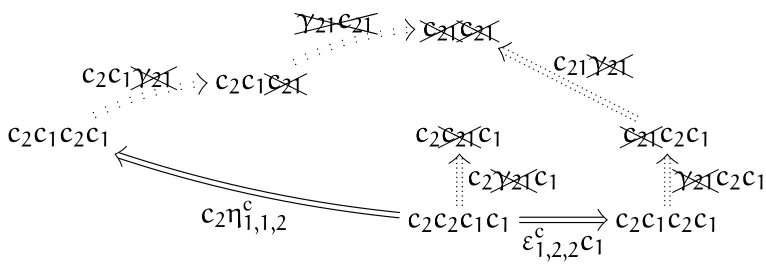


By definition of the 2-cells of $C_2(2)$, we have $\gamma_{21} := \alpha_{2,1}$. Thus we obtain that $T^{-1}_{\gamma\leftarrow\alpha'}(C'_{2,1,21}) = C'_{2,1,21}$ up to replace all the 2-cells $\alpha_{2,1}$ in $C'_{2,1,21}$ by γ_{21}. Hence, the coherent presentation $\mathrm{CPC}_3(2)$ is equal to $\mathrm{PreCol}_3(2)$. In order to compute the 3-cell $T^{-1}_{\eta,\varepsilon\leftarrow\alpha'}(T^{-1}_{\gamma\leftarrow\alpha'}(C'_{2,1,21}))$, the 2-cells $\alpha_{1,21}$ and $\alpha_{2,21}$ in $C'_{2,1,21}$ are respectively replaced by the 2-cells $\eta^{c}_{1,1,2}$ and $\varepsilon^{c}_{1,2,2}$ as in the following diagram

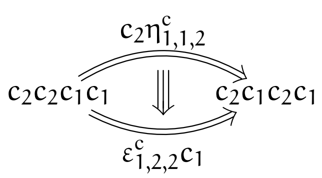


(14)

where the cancel symbol means that the corresponding 2-cell is removed. Hence the coherent presentation $\mathrm{Knuth}^{cc}_3(2)$ of $\mathbf{P}_2$ has for 1-cells c_1, c_2 and c_{21}, for 2-cells $\alpha_{2,1}$, $\alpha_{1,21}$ and $\alpha_{2,21}$ and the only 3-cell (14). Let us compute the Knuth coherent presentation $\mathrm{Knuth}_3(2)$. The 3-cell $R_{\Gamma_2}(T^{-1}_{\eta,\varepsilon\leftarrow\alpha'}(T^{-1}_{\gamma\leftarrow\alpha'}(C'_{2,1,21})))$ is obtained from (14) by removing the 2-cell γ_{21} together with the 1-cell c_{21}. Thus we obtain the following 3-cell, where the cancel symbol means that the corresponding element is removed,



Hence, the Knuth coherent presentation $\mathrm{Knuth}_3(2)$ of the monoid $\mathbf{P}_2$ has generators c_1 and c_2 subject to the Knuth relations $\eta^{c}_{1,1,2} : c_2c_1c_1 \Rightarrow c_1c_2c_1$ and $\varepsilon^{c}_{1,2,2} : c_2c_2c_1 \Rightarrow c_2c_1c_2$ and the following 3-cell



In this way, we obtain the Knuth coherent presentation of the monoid $\mathbf{P}_2$ that we obtain in Example 4.3.5 as a consequence of the fact that the 2-polygraph $\mathrm{Knuth}_2(2)$ is convergent.

4.4.9. Procedure to compute the 3-cells of $\mathrm{Knuth}_3(n)$**.** We present a procedure that computes the 2-sources and the 2-targets of the 3-cells of the Knuth coherent presentation $\mathrm{Knuth}_3(n)$, using the constructions given in Sections 3 and 4. The first step consists to define a procedure, called ReduceG3($\alpha_{u,v}$), that replaces a 2-cell $\alpha_{u,v}$ of $\mathrm{Col}_2(n)$ by a 2-cell of the 2-category $\mathrm{PreCol}_2(n)^*$ using a reduction defined in 4.3.1 with respect to the 3-cells $A_{x,v,t}$, $B_{x,v,t}$ and $C_{x,v,t}$, where x is in $[n]$ and v and t are in $\mathrm{col}(n)$. Given u in $\mathrm{col}(n)$ such that $\ell(u) \geqslant 2$ and $u = x_p x_{p-1} \dots x_2 x_1$, we will denote x_p (resp. x_1) by $\mathrm{first}(u)$ (resp. $\mathrm{last}(u)$) and the column $x_{p-1} \dots x_1$ (resp. $x_p \dots x_2$) by $\mathrm{rem}^f(u)$ (resp. $\mathrm{rem}^l(u)$). If $\ell(u) = 1$, we set $\mathrm{first}(u) = \mathrm{last}(u) = u$ and $\mathrm{rem}^f(u)$ and $\mathrm{rem}^l(u)$ are the empty columns.

ReduceG3($\alpha_{u,v}$):
Input: $\alpha_{u,v}$ in $\mathrm{Col}_2(n)$.
$\alpha = \alpha_{u,v}$;
case $u \overset{\times 1}{} v$ **do**
  if $\ell(u) \geqslant 2$ **then**
  $x = \mathrm{first}(u)$; $u_2 = \mathrm{rem}^f(u)$;
  $\beta =$ ReduceG3($\alpha_{u_2,v}$) ;
  $\alpha = \alpha^-_{x,u_2} c_v \star_1 c_x \beta \star_1 \alpha_{x,u_2 v}$; **else return** α;
case $u \overset{\times 2}{} v$ **do**
  if $\ell(u) \geqslant 2$ *and* $\ell(v) \geqslant 2$ **then**
  $x = \mathrm{first}(u)$; $u_2 = \mathrm{rem}^f(u)$;
  $w = C_l(u_2 v)$; $w' = C_r(u_2 v)$; $a = C_l(xw)$; $a' = C_r(xw)$;
  $\beta =$ ReduceG3($\alpha_{u_2,v}$) ;
  $\alpha = \alpha^-_{x,u_2} c_v \star_1 c_x \beta \star_1 \alpha_{x,w} c_{w'} \star_1 c_a \alpha_{a',w'}$;
  if $\ell(u) = 1$ *and* $\ell(v) \geqslant 2$ **then**
  $v_1 = \mathrm{rem}^l(v)$; $y = \mathrm{last}(v)$;
  $e = C_l(u v_1)$; $e' = C_r(u v_1)$;
  η_1 = ReduceG3($\alpha_{v_1,y}$); η_2 = ReduceG3(α_{u,v_1}); η_3 = ReduceG3($\alpha_{e,e'y}$) ;
  $\alpha = c_u \eta_1^- \star_1 \eta_2 c_y \star_1 c_e \alpha_{e',y} \star_1 \eta_3$;
  if $\ell(u) = 1$ *and* $\ell(v) = 2$ **then**
  return α ;

We define the procedure ElimAlpha($\alpha_{x,v}$) that replaces a 2-cell $\alpha_{x,v}$ of $\mathrm{PreCol}_2(n)$ by a 2-cell of the 2-category $\mathrm{Knuth}_2^{cc}(n)^*$, using the Tietze transformations given in 4.4.1 and 4.4.3. In the sequel, we will represent every 1-composite $f_1 \star_1 \dots \star_1 f_k$ of 2-cells by a list $[f_1, \dots, f_k]$ of 2-cells. If $L = [L[0], \dots, L[k-1]]$ is a list of length k and u and v are in $[n]^*$, we will denote by uLv the list $[uL[0]v, \dots, uL[k-1]v]$.

ElimAlpha($\alpha_{x,v}$):
Input: $\alpha_{x,v}$ in $\mathrm{PreCol}_2(n)$.
case $x \overset{\times 1}{} v$ **do**
  if $\ell(v) > 1$ **then**
  return $[c_x \gamma^-_v, \gamma_{xv}]$ *;* **else return** $[\gamma_{xv}]$;
case $x \overset{\times 2}{} v$ **do**
  $z = \mathrm{first}(v)$; $y = \mathrm{last}(v)$;
  if $x \leqslant y < z$ **then**
  return $[(\eta^c_{x,y,z})^-, c_x \gamma^-_{zy}, \gamma_{zx} c_y]$;
  if $y < x \leqslant z$ **then**
  return $[c_u \gamma^-_{zy}, \varepsilon^c_{x,z,y}, \gamma_{xy} c_z]$;

We define the procedure ElimAG(f) that replaces in a 2-cell f of the 2-category $\mathrm{PreCol}_2(n)^*$, every $\alpha_{x,v}$ in $\mathrm{PreCol}_2(n)$ by ElimAlpha$(\alpha_{x,v})$. In a second step, it replaces every γ_u in $C_2(n)$ by 1_u, with respect to the reduction R_{Γ_2} defined in 4.4.5.

ElimAG(f):
Input: $f = f_1 \star_1 \ldots \star_1 f_k$, where for $i = 1, \ldots, k$, $f_i = u_i \alpha_i v_i$,
with $u_i, v_i \in [n]^*$ and $\alpha_i \in \mathrm{PreCol}_2(n)$.
$L = [\,]$;
for $i = 0$ *to* $k - 1$ **do**
  $L[i] = u_{i+1}$ElimAlpha$(\alpha_{i+1})v_{i+1}$;
end
for $i = 0$ *to* $k - 1$ **do**
  for $j = 0$ *to* $\ell(L[i]) - 1$ **do**
    if $L[i][j] = u_j \beta_j v_j$, *with* $u_j, v_j \in [n]^*$ *and* β_j *or* β_j^- *are in* $C_2(n)$ **then**
    $L[i][j] = 1_{u_j v_j}$;
  end
end
return L.

We define the procedure Compute$C'(n)$ that computes the 2-sources and the 2-targets of the 3-cells $\mathcal{R}(C'_{x,v,t})$ of the Knuth coherent prensentation, where $\mathcal{R}$ is the Tietze transformation defined in 4.4.6.

Compute$C'(n)$:
Input: $n > 0$.
$K = \emptyset$;
for x *in* $[n]$ *and* v *and* t *in* $\mathrm{col}(n)$ *such that* $x^{\times 1} v^{\times 2} t$ **do**
  $w = C_l(vt)$; $w' = C_r(vt)$; $s = C_r(xw)$;
  α = ElimAG$(\alpha_{x,v})c_t$;
  α_1 = ElimAG(ReduceG3$(\alpha_{v,t})$) ; α_2 = ElimAG$(\alpha_{x,w})$;
  α_3 = ElimAG(ReduceG3$(\alpha_{s,w'})$) ;
  $\alpha' = [c_x\alpha_1, \alpha_2 c_{w'}, c_{xv}\alpha_3]$;
  $K = K \cup \{(\alpha, \alpha')\}$;
end
return K.

We define a procedure, called Compute$D(n)$, that computes the 2-sources and the 2-targets of the 3-cells $\mathcal{R}(D_{x,v,t})$ of the Knuth coherent prensentation, where $\mathcal{R}$ is the Tietze transformation defined in 4.4.6.

ComputeD(n):
Input: $n > 0$.
$K = \emptyset$;
for x *in* $[n]$ *and* v *and* t *in* col(n) *such that* $x^{\times 2} v^{\times 2} t$ **do**
$e = C_l(xv)$; $e' = C_r(xv)$; $b = C_l(P(e't))$; $b' = C_r(e't)$;
$w = C_l(vt)$; $w' = C_r(vt)$; $a = C_l(xw)$; $a' = C_r(xw)$;
α_1 = ElimAG(ReduceG3($\alpha_{x,v}$)) ; α_2 = ElimAG(ReduceG3($\alpha_{e',t}$)) ;
α_3 = ElimAG(ReduceG3($\alpha_{e,b}$)) ;
$\alpha = [\alpha_1 c_t, c_e \alpha_2, \alpha_3 c_{b'}]$;
α'_1 = ElimAG(ReduceG3($\alpha_{v,t}$)) ; α'_2 = ElimAG(ReduceG3($\alpha_{x,w}$)) ;
α'_3 = ElimAG(ReduceG3($\alpha_{a',w'}$)) ;
$\alpha' = [c_x \alpha'_1, \alpha'_2 c_{w'}, c_a \alpha'_3]$;
$K = K \cup \{(\alpha, \alpha')\}$;
end
return K.

Finally, a way to compute the 2-sources and the 2-targets of the 3-cells of the Knuth coherent presentation $\text{Knuth}_3(n)$ is to apply at the same time the procedures ComputeC$'(n)$ and ComputeD(n).

4.4.10. Coherent presentations in small ranks. Let us denote by $\text{Knuth}_2^{KB}(n)$ the convergent 2-polygraph obtained from $\text{Knuth}_2(n)$ by the Knuth-Bendix completion using the lexicographic order. For $n = 3$, the polygraph $\text{Knuth}_2^{KB}(3)$ is finite, but $\text{Knuth}_2^{KB}(n)$ is infinite for $n \geqslant 4$, [13]. Let us denote by $\text{Knuth}_3^{KB}(n)$ the Squier completion of $\text{Knuth}_2^{KB}(n)$. For $n \geqslant 4$, the polygraph $\text{Knuth}_2^{KB}(n)$ having an infinite set of critical branching, the set of 3-cells of $\text{Knuth}_3^{KB}(n)$ is infinite. However, the $(3,1)$-polygraph $\text{Knuth}_3(n)$ is a finite coherent convergent presentation of $\mathbf{P}_n$. Table 1 presents the number of cells of the coherent presentations $\text{Knuth}_3(n)$, $\overline{\text{Col}}_3(n)$ and $\text{Col}_3(n)$ of the monoid $\mathbf{P}_n$.

n	$\text{Col}_1(n)$	$\text{Knuth}_2(n)$	$\text{Knuth}_2^{KB}(n)$	$\text{Col}_2(n)$	$\text{Knuth}_3^{KB}(n)$	$\text{Knuth}_3(n)$	$\overline{\text{Col}}_3(n)$	$\text{Col}_3(n)$
1	1	0	0	0	0	0	0	0
2	3	2	2	3	1	1	1	1
3	7	8	11	22	27	24	34	42
4	15	20	∞	115	∞	242	330	621
5	31	40	∞	531	∞	1726	2225	6893
6	63	70	∞	2317	∞	10273	12635	67635
7	127	112	∞	9822	∞	55016	65282	623010
8	255	168	∞	40971	∞	275868	318708	5534197
9	511	240	∞	169255	∞	1324970	1500465	48052953
10	1023	330	∞	694837	∞	6178939	6892325	410881483

Table 1: Number of cells of $(3,1)$-polygraphs $\text{Knuth}_3(n)$, $\overline{\text{Col}}_3(n)$ and $\text{Col}_3(n)$, for $1 \leqslant n \leqslant 10$.

4.4.11. Actions of plactic monoids on categories. In [6], the authors give a description of the category of actions of a monoid on categories in terms of coherent presentations. Using this description, Theorem 4.4.7 allows to present actions of plactic monoids on categories as follows. The category $\text{Act}(\mathbf{P}_n)$ of actions of the monoid $\mathbf{P}_n$ on categories is equivalent to the category of 2-functors from the $(2,1)$-category $\text{Knuth}_2(n)^\top$ to the category Cat of categories, that sends the 3-cells of $\text{Knuth}_3(n)$ to commutative diagrams in Cat.

REFERENCES

[1] A. Berele. A Schensted-type correspondence for the symplectic group. *J. Combin. Theory Ser. A*, 43(2):320–328, 1986.

[2] L. Bokut, Y. Chen, W. Chen, and J. Li. New approaches to plactic monoid via Gröbner–Shirshov bases. *J. Algebra*, 423:301–317, 2015.

[3] A. J. Cain, R. D. Gray, and A. Malheiro. Crystal monoids & crystal bases: rewriting systems and biautomatic structures for plactic monoids of types A_n, B_n, C_n, D_n, and G_2 . arXiv:1412.7040, 2015.

[4] A. J. Cain, R. D. Gray, and A. Malheiro. Finite Gröbner–Shirshov bases for Plactic algebras and biautomatic structures for Plactic monoids. *J. Algebra*, 423:37–53, 2015.

[5] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[6] S. Gaussent, Y. Guiraud, and P. Malbos. Coherent presentations of Artin monoids. *Compos. Math.*, 151(5):957–998, 2015.

[7] Y. Guiraud and P. Malbos. Polygraphs of finite derivation type. *Mathematical Structures in Computer Science*, pages 1–47, 2017. to appear.

[8] Y. Guiraud, P. Malbos, and S. Mimram. A homotopical completion procedure with applications to coherence of monoids. In *24th International Conference on Rewriting Techniques and Applications*, volume 21 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 223–238. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2013.

[9] N. Hage. Finite convergent presentation of plactic monoid for type C. *Internat. J. Algebra Comput.*, 25(8):1239–1263, 2015.

[10] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the q-analogue of classical Lie algebras. *J. Algebra*, 165(2):295–345, 1994.

[11] D. Knuth and P. Bendix. Simple word problems in universal algebras. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pages 263–297. Pergamon, Oxford, 1970.

[12] D. E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.*, 34:709–727, 1970.

[13] Ł. Kubat and J. Okniński. Gröbner-Shirshov bases for plactic algebras. *Algebra Colloq.*, 21(4):591–596, 2014.

[14] A. Lascoux, B. Leclerc, and J.-Y. Thibon. Crystal graphs and q-analogues of weight multiplicities for the root system A_n. *Lett. Math. Phys.*, 35(4):359–374, 1995.

[15] A. Lascoux and M.-P. Schützenberger. Sur une conjecture de H. O. Foulkes. *C. R. Acad. Sci. Paris Sér. A-B*, 286(7):A323–A324, 1978.

[16] A. Lascoux and M.-P. Schützenberger. Le monoïde plaxique. In *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*, volume 109 of *Quad. "Ricerca Sci."*, pages 129–156. CNR, Rome, 1981.

[17] C. Lecouvey. Private communication. 2016.

[18] C. Lenart. On the combinatorics of crystal graphs. I. Lusztig's involution. *Adv. Math.*, 211(1):204–243, 2007.

[19] P. Littelmann. A plactic algebra for semisimple Lie algebras. *Adv. Math.*, 124(2):312–331, 1996.

[20] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.

[21] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.

[22] M.-P. Schützenberger. La correspondance de Robinson. In *Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976)*, pages 59–113. Lecture Notes in Math., Vol. 579. Springer, Berlin, 1977.

[23] J. T. Sheats. A symplectic jeu de taquin bijection between the tableaux of King and of De Concini. *Trans. Amer. Math. Soc.*, 351(9):3569–3607, 1999.

[24] C. C. Squier, F. Otto, and Y. Kobayashi. A finiteness condition for rewriting systems. *Theoret. Comput. Sci.*, 131(2):271–294, 1994.

[25] S. Sundaram. Orthogonal tableaux and an insertion algorithm for $\mathrm{SO}(2n+1)$. *J. Combin. Theory Ser. A*, 53(2):239–256, 1990.

NOHRA HAGE
nohra.hage@univ-st-etienne.fr
Univ Lyon, Université Jean Monnet
CNRS UMR 5208, Institut Camille Jordan
Maison de l'Université, 10 rue Tréfilerie, CS 82301
F-42023 Saint-Étienne Cedex 2, France

PHILIPPE MALBOS
malbos@math.univ-lyon1.fr
Univ Lyon, Université Claude Bernard Lyon 1
CNRS UMR 5208, Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne cedex, France