Algebraic Confluences

Part II. Two-dimensional Homotopy and Rewriting

Philippe Malbos

Motivation

Fact. The homological finiteness condition left-FP_3 is not sufficient for a finitely presented decidable monoid to admit a finite convergent presentation.

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Example. (Squier, 1994) The monoid

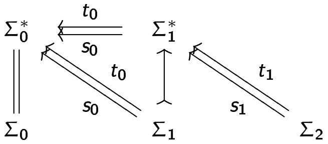
$$\mathbf{S_1} = \langle\ a, b, t, x, y \mid at^n b \Rightarrow 1,\ xa \Rightarrow atx,\ xt \Rightarrow tx,\ xb \Rightarrow bx,\ xy \Rightarrow 1\ \rangle.$$

- has a decidable word problem,
- is of homological type left-FP_3,
- does not have a finite convergent presentation,
- does not have **finite derivation type**.

Homotopical Squier Theorem

Homotopical Squier Theorem

- Σ a 2-polygraph.



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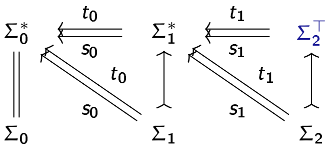
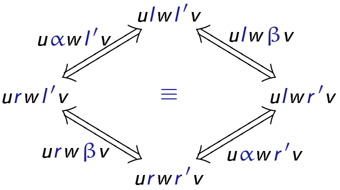


- $\Sigma_2^\top$ free **category enriched in groupoid** on Σ:
  - 0-cells : Σ_0,
  - 1-cells strings in Σ_1^*,
  - 2-cells : reductions and their inverses $\Leftrightarrow$,
  - plus **Peiffer elements**:



$l \overset{\alpha}{\Rightarrow} r$, $l' \overset{\beta}{\Rightarrow} r'$.

Homotopical Squier Theorem

Definition. A **homotopy relation** on $\Sigma_2^\top$ is an equivalence relation $\equiv$ on parallel 2-cells stable under context and composition:

- $f \equiv g$ implies $ufv \equiv ugv$,
- $f \equiv g$ implies $k \star_1 f \star_1 h \equiv k \star_1 g \star_1 h$.

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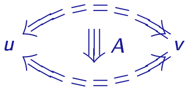
Definition. A **homotopy basis** is a cellular extension Σ_3 made of 3-cells



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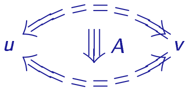

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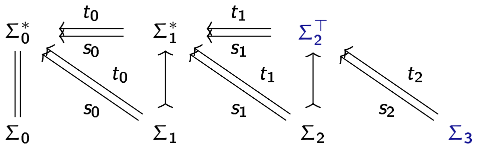
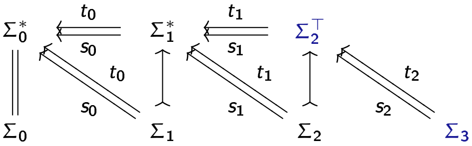


Homotopical Squier Theorem

Definition. Σ has **finite derivation type** (**FDT**) if

- **i)** Σ is finite,
- **ii)** $\Sigma_2^\top$ has a finite homotopy basis Σ_3.



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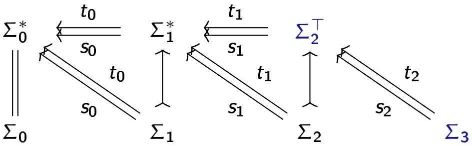

Theorem. (Squier, 1994)

i) Property FDT is Tietze invariant for finite rewriting systems.

ii) A monoid having a finite convergent rewriting system has FDT.

Example. (Squier, 1994) The monoid

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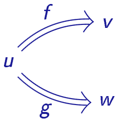
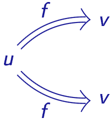
Homotopical Squier Theorem : proof

Branchings

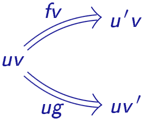
- Let Σ be a 2-polygraph.
- A **branching** of Σ is a pair (f, g) of 2-cells of Σ_2^* with a common source:



- A branching (f, g) is **local** when f and g are rewriting steps.
- Local branchings are
  - **aspherical**



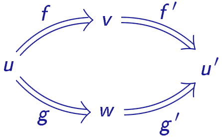
  - **Peiffer**



  - or **overlapping**.

Generating confluences

- A branching $(f, g) : u \Rightarrow (v, w)$ is **confluent** when there exist 2-cells $f' : v \Rightarrow u'$ and $g' : w \Rightarrow u'$ in Σ_2^* such that



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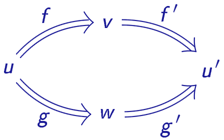
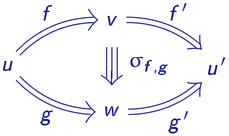


• A family of **generating confluences** of Σ is a cellular extension of $\Sigma_2^{\top}$ that contains exactly one 3-cell



for every critical branching (f, g) of Σ.

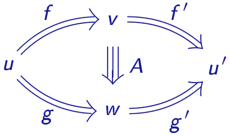
- If Σ is confluent, it always admit a family of generating confluences.
- Such a family is not necessarily unique.

Generating confluences

Let Σ be a convergent 2-polygraph. Let Γ be a family of generating confluences of Σ.

Lemma 1.

For every local branching $(f, g) : u \Rightarrow (v, w)$ of Σ, there exist 2-cells f' and g' in Σ_2^* and a 3-cell A in $\Gamma^\top$, as in the following diagram:



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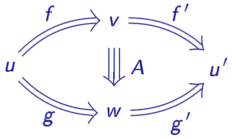
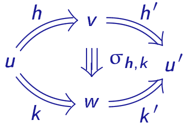


Proof.

For aspherical or Peiffer branching, choose f' and g'such that $f \star_1 f' = g \star_1 g'$ and A is identity.

An overlapping branching (f, g) that is not critical is of the form $(f, g) = (uhv, ukv)$ with (h, k) critical.

Consider



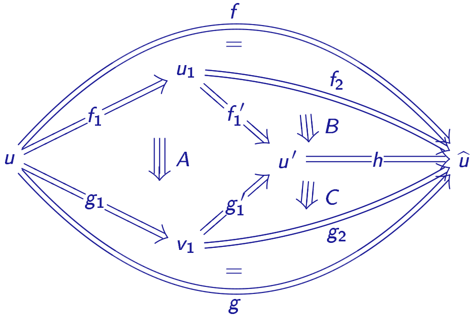
in Γ.

Set $A = u\sigma_{h,k}v$, $f' = uh'v$ and $g' = kuk'v$.

Generating confluences

Lemma 2.
For every parallel 2-cells f and g of Σ_2^* whose common target is a normal form, there exists a 3-cell from f to g in Γ.

Proof. By Noetherian induction on the common source of f and g.



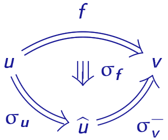
Homotopical Squier Theorem

Proposition. Let Σ be a convergent 2-polygraph. Every family Γ of generating confluences of Σ is a homotopy basis of $\Sigma^{\top}$.

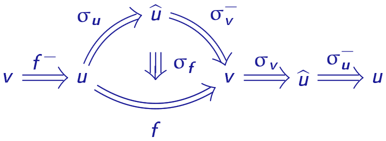
Proof. Consider a 2-cell $f : u \Rightarrow v$ in Σ_2^*. Using the confluence, choose 2-cells

$$\sigma_u : u \Rightarrow \widehat{u} \qquad \text{and} \qquad \sigma_v : v \Rightarrow \widehat{v} = \widehat{u} \qquad \text{in} \qquad \Sigma_2^*.$$

By construction, the 2-cells $f \star_1 \sigma_v$ and σ_u are parallel and their common target $\widehat{u}$ is a normal form. By Lemma 2, there exists a 3-cell $\sigma_f : f \star_1 \sigma_v \Rrightarrow \sigma_u$ in $\Gamma^{\top}$ or, equivalently, a 3-cell $\sigma_f : f \Rrightarrow \sigma_u \star_1 \sigma_v^-$ in $\Gamma^{\top}$:



Moreover, the $(3,1)$-category $\Gamma^{\top}$ contains a 3-cell $\sigma_{f^-} : f^- \Rrightarrow \sigma_v \star_1 \sigma_u^-$, given as the composite:



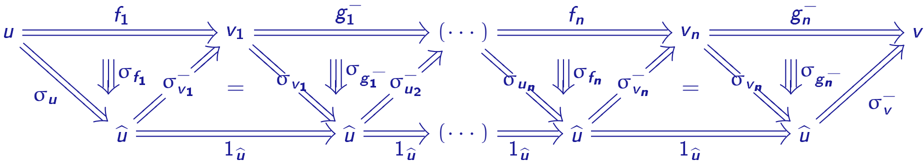
Homotopical Squier Theorem

Proposition. Let Σ be a convergent 2-polygraph. Every family Γ of generating confluences of Σ is a homotopy basis of $\Sigma^{\top}$.

Proof.
Consider a 2-cell $f : u \Rightarrow v$ in $\Sigma_2^{\top}$. It can be decomposed into a "zig-zag"

$$u \overset{f_1}{\Longrightarrow} v_1 \overset{g_1^-}{\Longrightarrow} u_2 \overset{f_2}{\Longrightarrow} (\cdots) \overset{g_{n-1}^-}{\Longrightarrow} u_n \overset{f_n}{\Longrightarrow} v_n \overset{g_n^-}{\Longrightarrow} v$$

where each f_i and g_i is a 2-cell of Σ_2^*. We define σ_f as the following composite 3-cell of $\Gamma^{\top}$, with source f and target $\sigma_u \star_1 \sigma_v^-$:



We proceed similarly for any 2-cell $g : u \Rightarrow v$ of $\Sigma_2^{\top}$, to get a 3-cell σ_g from g to $\sigma_u \star_1 \sigma_v^-$ in $\Gamma^{\top}$. Thus, the composite $\sigma_f \star_2 \sigma_g^-$ is a 3-cell of $\Gamma^{\top}$ from f to g.

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References

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