Identities among relations for polygraphic rewriting

July 6, 2010

References:

- (P. M., Y. Guiraud) *Identities among relations for higher-dimensional rewriting systems*, arXiv:0910.4538, to appear;
- (P. M., Y. Guiraud) Coherence in monoidal track categories, arXiv:1004.1055, submitted;
- (P. M., Y. Guiraud) *Higher-dimensional categories with finite derivation type*, Theory and Applications of Categories, 2009;

Contents

Two-dimensional homotopy for polygraphs

Track n-categories and homotopy bases

Definitions.

- A **track n-category** is an $(n-1)$-category enriched in groupoids.

- A **homotopy basis** of an n-category $\mathcal{C}$ is a cellular extension Γ such that the track n-category $\mathcal{C}/\Gamma$ is aspherical

i.e., for every n-spheres $\cdot \underset{g}{\overset{f}{\rightrightarrows}} \cdot$ in $\mathcal{C}$, there exists an $(n+1)$-cell $\cdot \underset{g}{\overset{f}{\rightrightarrows}} \!\Downarrow\! \cdot$ in $\mathcal{C}(\Gamma)$,

i.e., $\overline{f} = \overline{g}$ in $\mathcal{C}/\Gamma$.

Definition. An n-polygraph Σ has **finite derivation type** (**FDT**) if

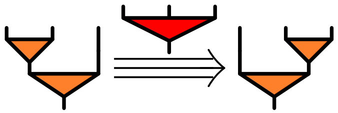
i) Σ is finite,

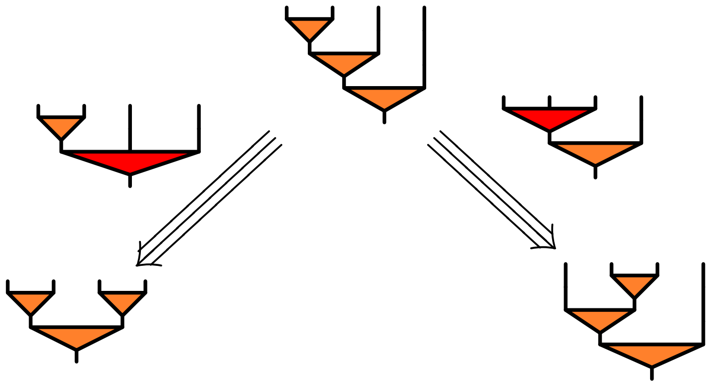
ii) the free track n-category $\Sigma^{\top} = \Sigma^{*}_{n-1}(\Sigma_n)$ admits a finite homotopy basis.

Proposition. The property FDT is Tietze invariant for finite polygraphs.

Critical branchings

- A branching is **critical** when it is a "minimal overlapping" of n-cells, such as:



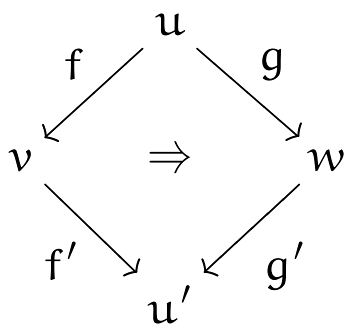


- Newman's diamond lemma ('42) :

$$\text{Termination} + \text{confluence of critical branchings} \Rightarrow \text{Convergence}$$

The homotopy basis of generating confluences

- A **generating confluence** of an n-polygraph Σ is an $(n+1)$-cell



with (f, g) critical branching.

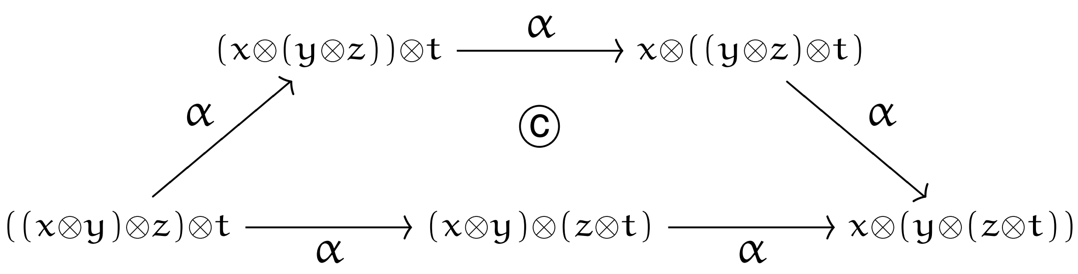
Theorem. Let Σ be a convergent n-polygraph.
A cellular extension of Σ^* made of one generating confluence for each critical branching of Σ is a homotopy basis of $\Sigma^\top$.

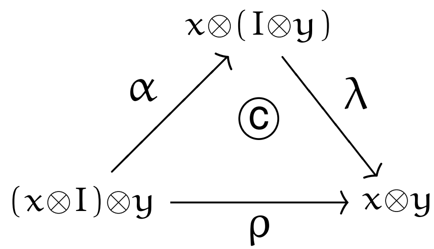
Example I : Mac Lane's coherence theorem

- Monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ with natural isomorphisms

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z) \qquad \lambda_x : I \otimes x \xrightarrow{\sim} x \qquad \rho_x : x \otimes I \xrightarrow{\sim} x$$

such that the following diagrams commute:



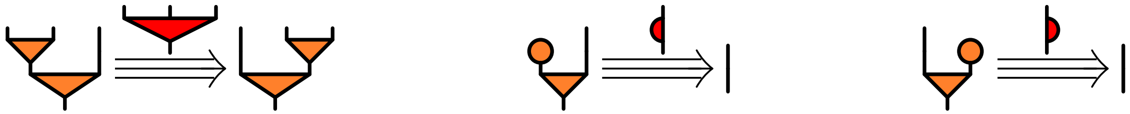


Theorem. (Mac Lane's coherence theorem '63)

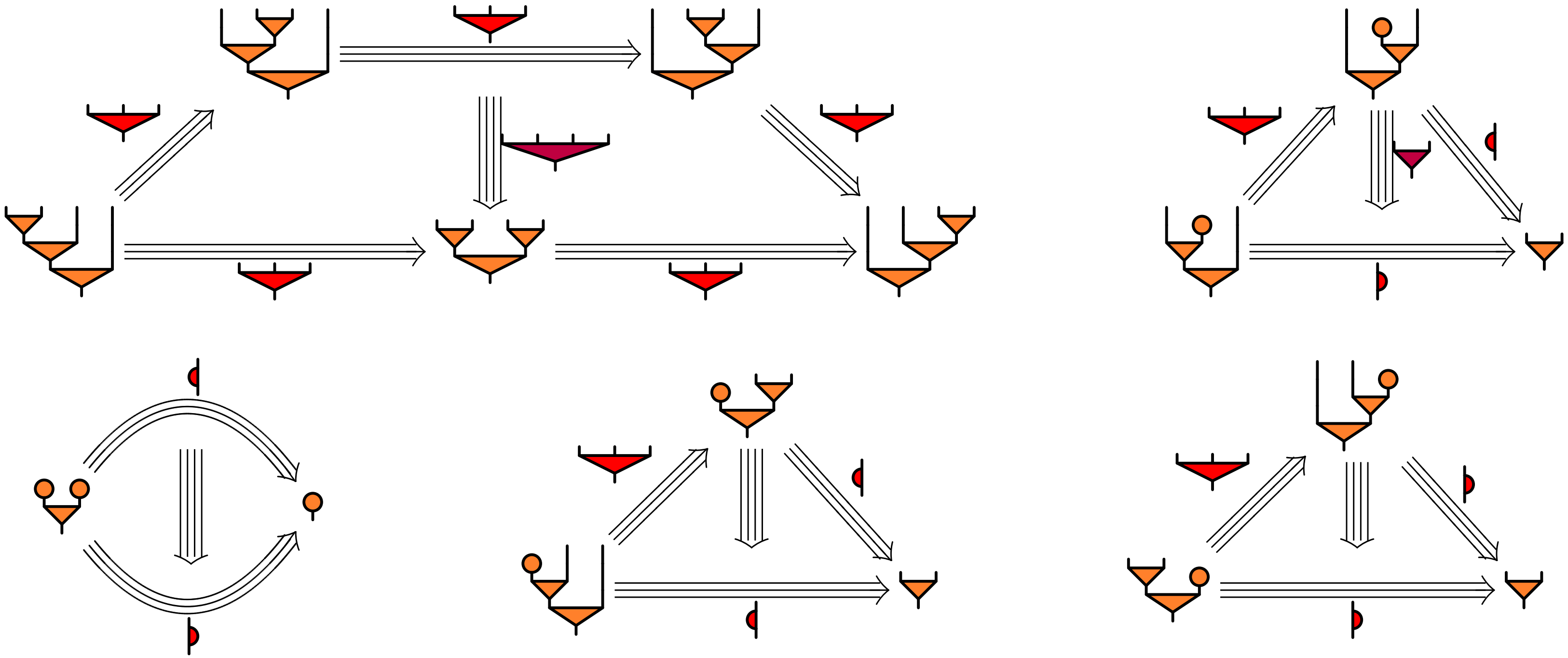
In a monoidal category, all the diagrams built from $\mathcal{C}$, $\otimes$, I, α, λ and ρ are commutative.

Example I : Mac Lane's coherence theorem

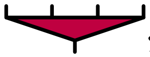
- Let Mon be the 3-polygraph:
  - one 0-cell, one 1-cell, two 2-cell : ,
  - three 3-cells :

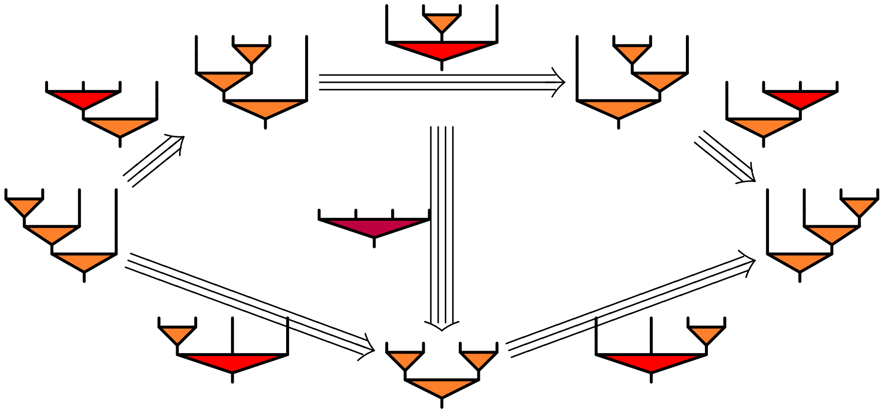


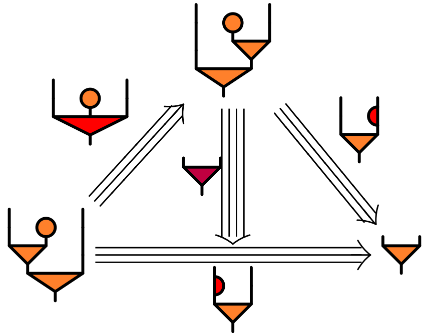
Lemma. Mon terminates and is locally confluent, with the following five generating confluences:



Example I : Mac Lane's coherence theorem

- Let $\Gamma = \{$, $\}$ be the cellular extension of Mon^* with :





- The track 3-category $\mathrm{Mon}^\top/\Gamma$ is the theory of monoidal categories, *i.e.*, there is an equivalence:

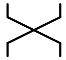
$$\text{Monoidal categories} \quad \leftrightarrow \quad 3\text{-functors } \mathrm{Mon}^\top/\Gamma \longrightarrow \mathbf{Cat}$$

Theorem. The 3-category $\mathrm{Mon}^\top/\Gamma$ is aspherical.

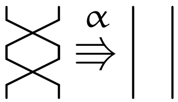
Corollary : Mac Lane's coherence theorem.

Example II : polygraph of permutations

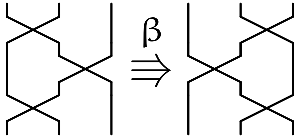
- Perm 3-polygraph of permutations :
  - one 0-cell, one 1-cell, one 2-cell :



  - two 3-cells :

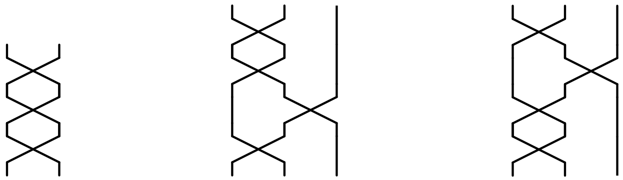


and

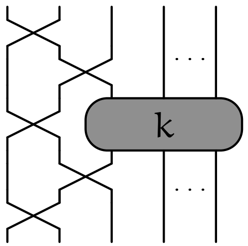


Theorem. Perm is finite, convergent and has FDT.

- Perm has an infinite set of critical branchings.
  - three **regular** critical branchings

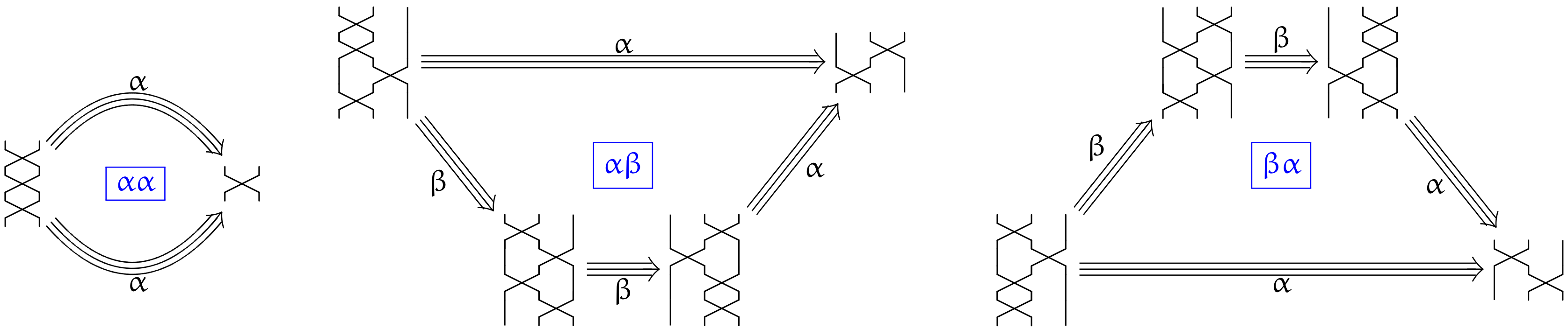


  - a **right-indexed** critical branching

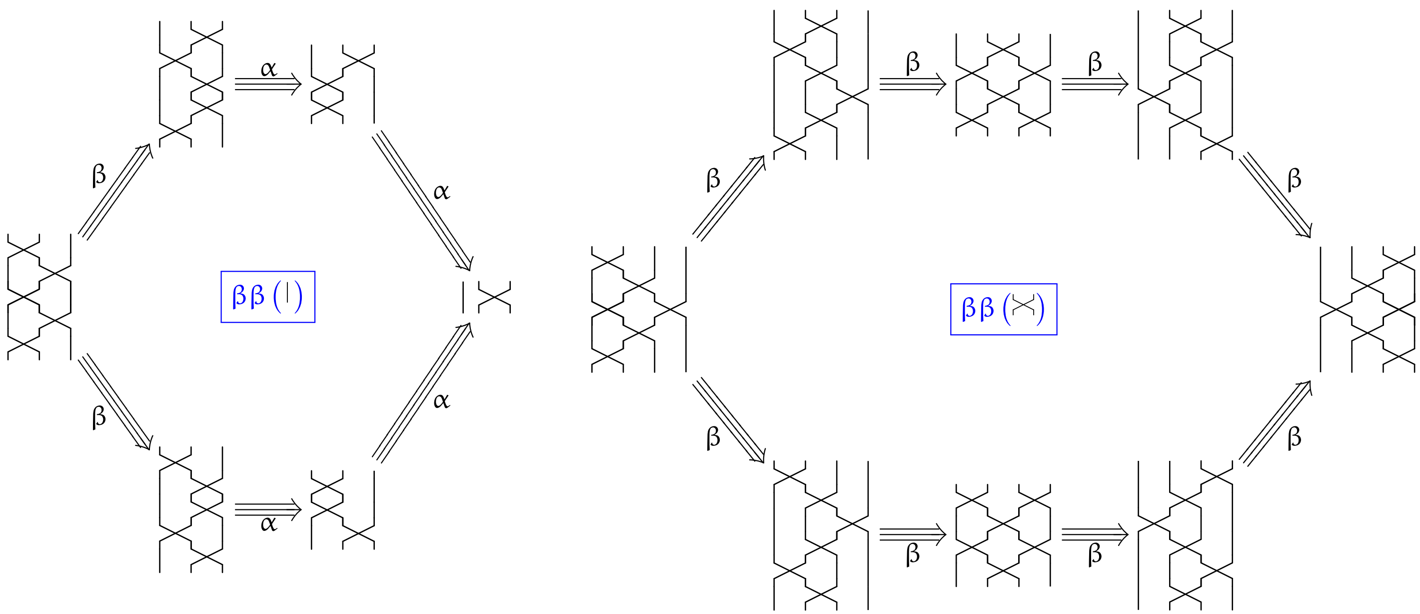


Example II : polygraph of permutations

- regular critical branchings are confluent



- right-indexed have two **normal instances**



- $\{\alpha\alpha,\ \alpha\beta,\ \beta\alpha,\ \beta\beta\,(|),\ \beta\beta\,(\times)\}$ is a homotopy base.

Generating confluences and finite derivation type

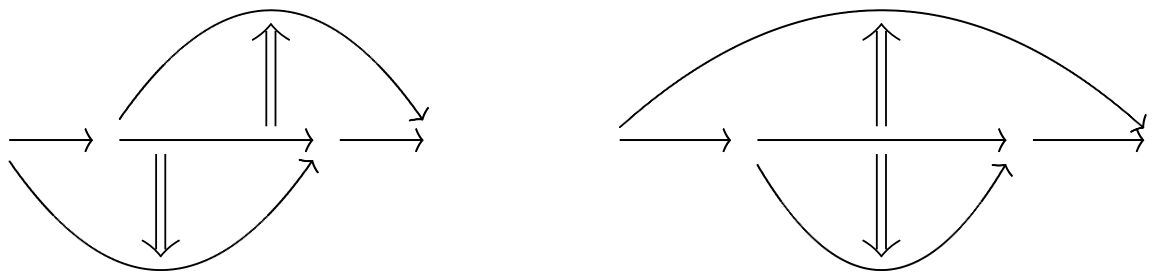
- Generating confluences of a convergent n-polygraph Σ form a homotopy basis of $\Sigma^\top$.

Corollary. A finite convergent n-polygraph with a finite number of critical branchings has FDT.

Corollary. (Squier '94) A finite convergent 2-polygraph has FDT.

- Two shapes of branching in a 2-polygraph :

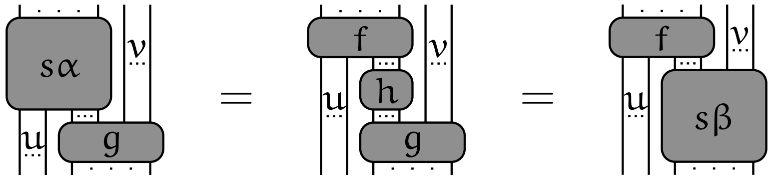
Regular critical branching — **Inclusion** critical branching



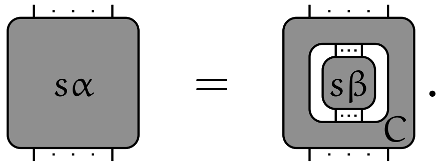
Theorem. For $n \geq 3$, there exist finite convergent n-polygraphs without FDT.

Critical branchings in 3-polygraphs

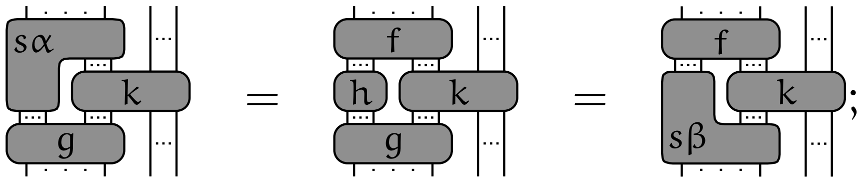
- **Regular** critical branching



- **Inclusion** critical branching

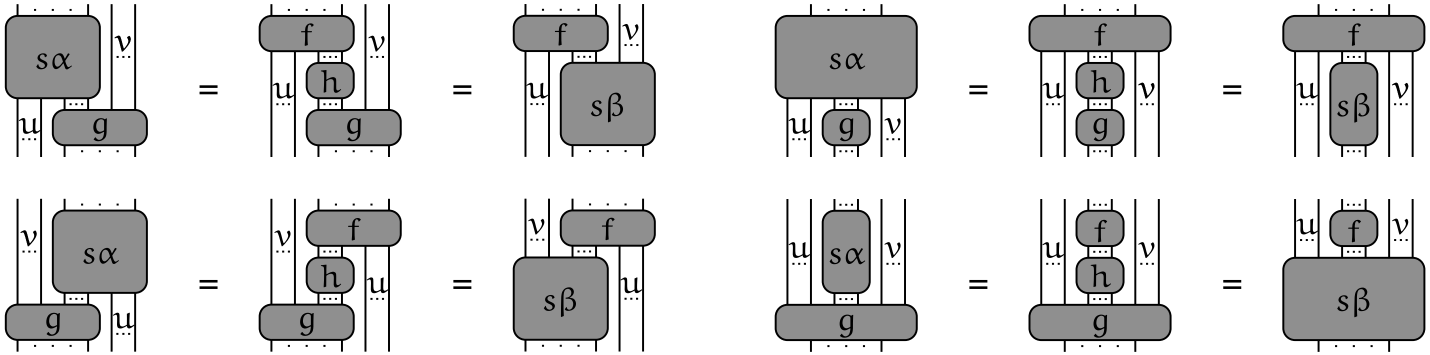


- **Right-indexed** critical branching

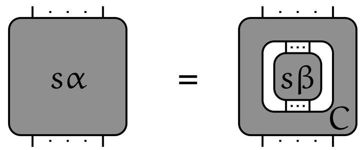


Critical branchings in 3-polygraphs

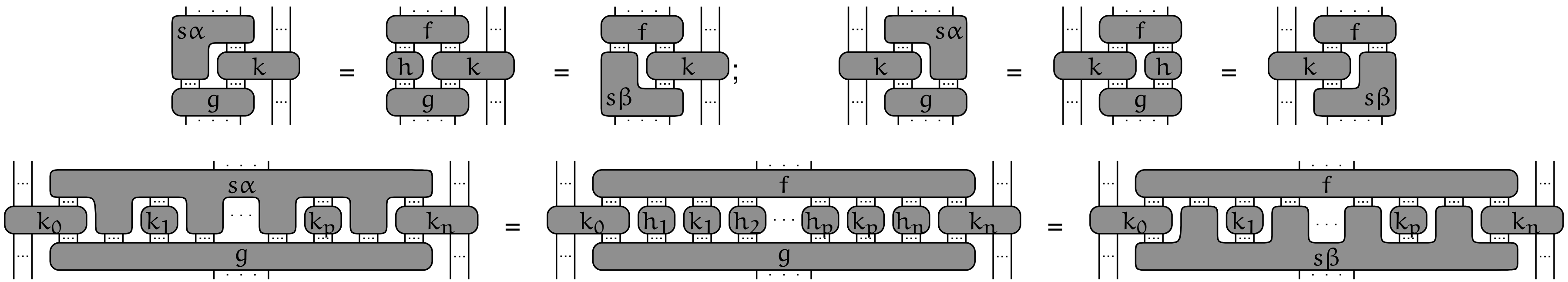
- **Regular**



- **Inclusion**



- **Indexed**

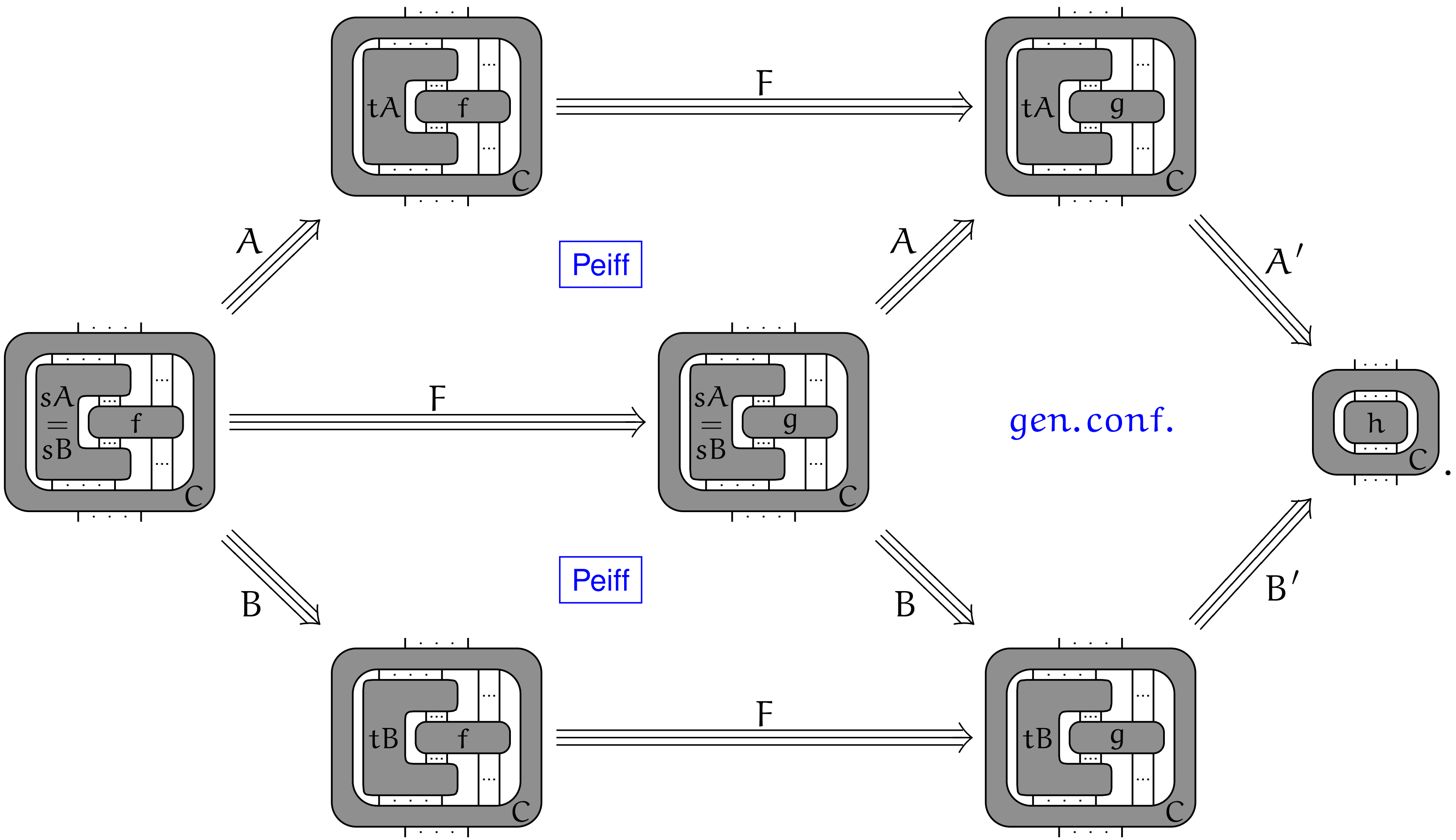


Critical branchings in 3-polygraphs

Theorem. Let Σ be a finite convergent 3-polygraph.

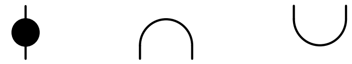
i) If Σ does not have indexed critical branching it has FDT.

ii) If Σ has indexed critical branchings with finitely many normal instances it has FDT.

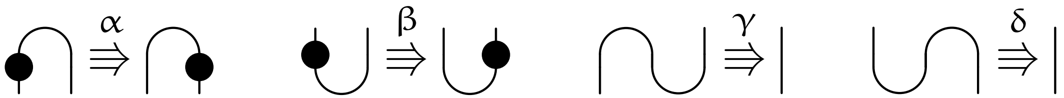


Example III : pearl necklaces

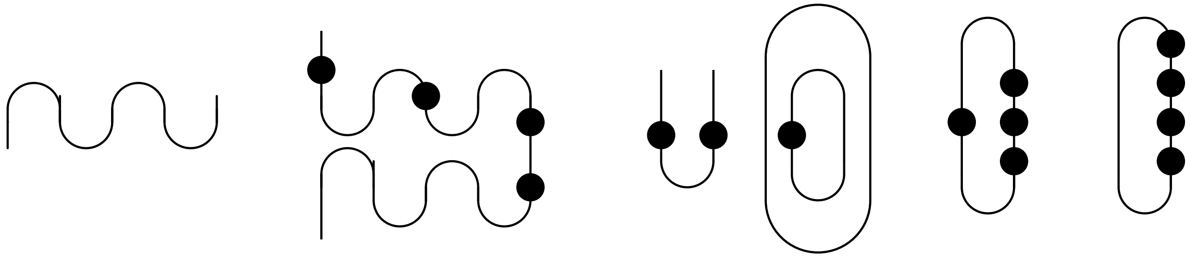
- Pearl 3-polygraph presenting the 2-category of **pearl necklaces**:
  - one 0-cell, one 1-cell, three 2-cells :



  - four 3-cells :



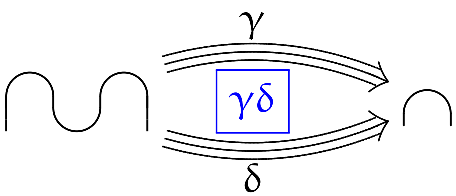
- Examples of 2-cells

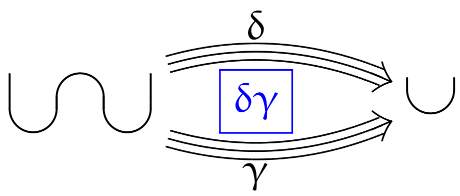


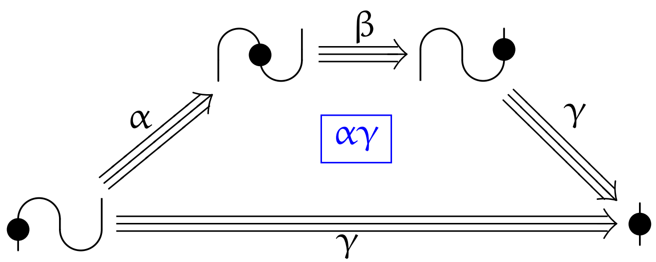
Proposition. Pearl is finite and convergent but does not have FDT.

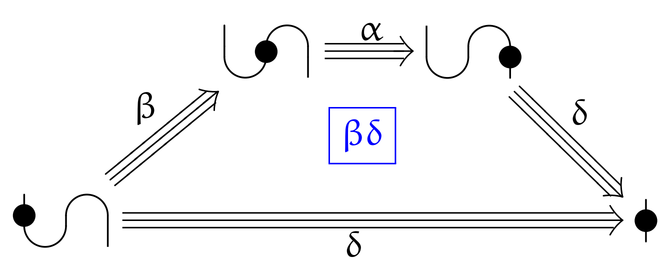
Example III : pearl necklaces

- Four regular critical branching

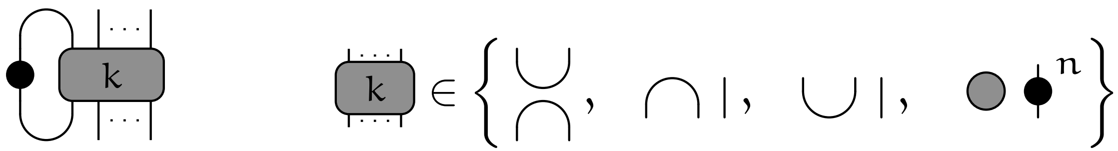


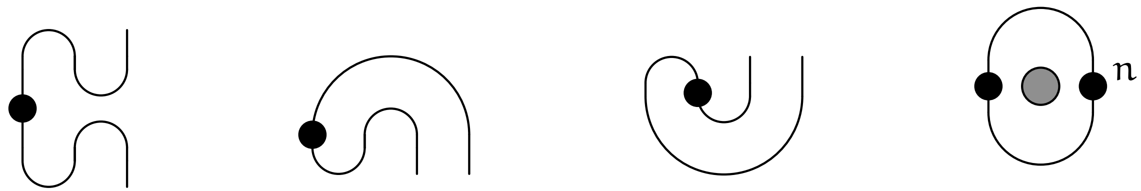




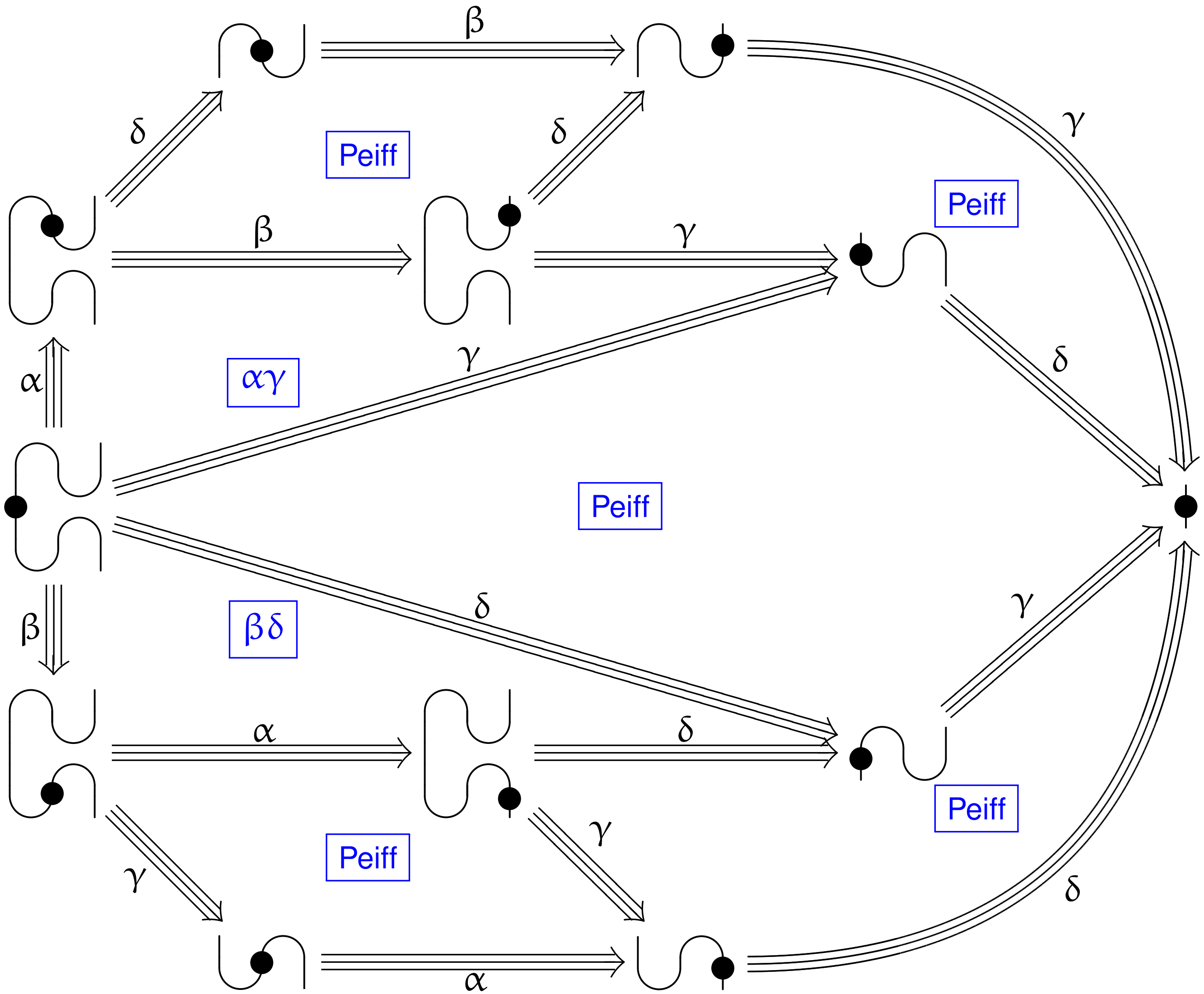


- One right-indexed critical branching

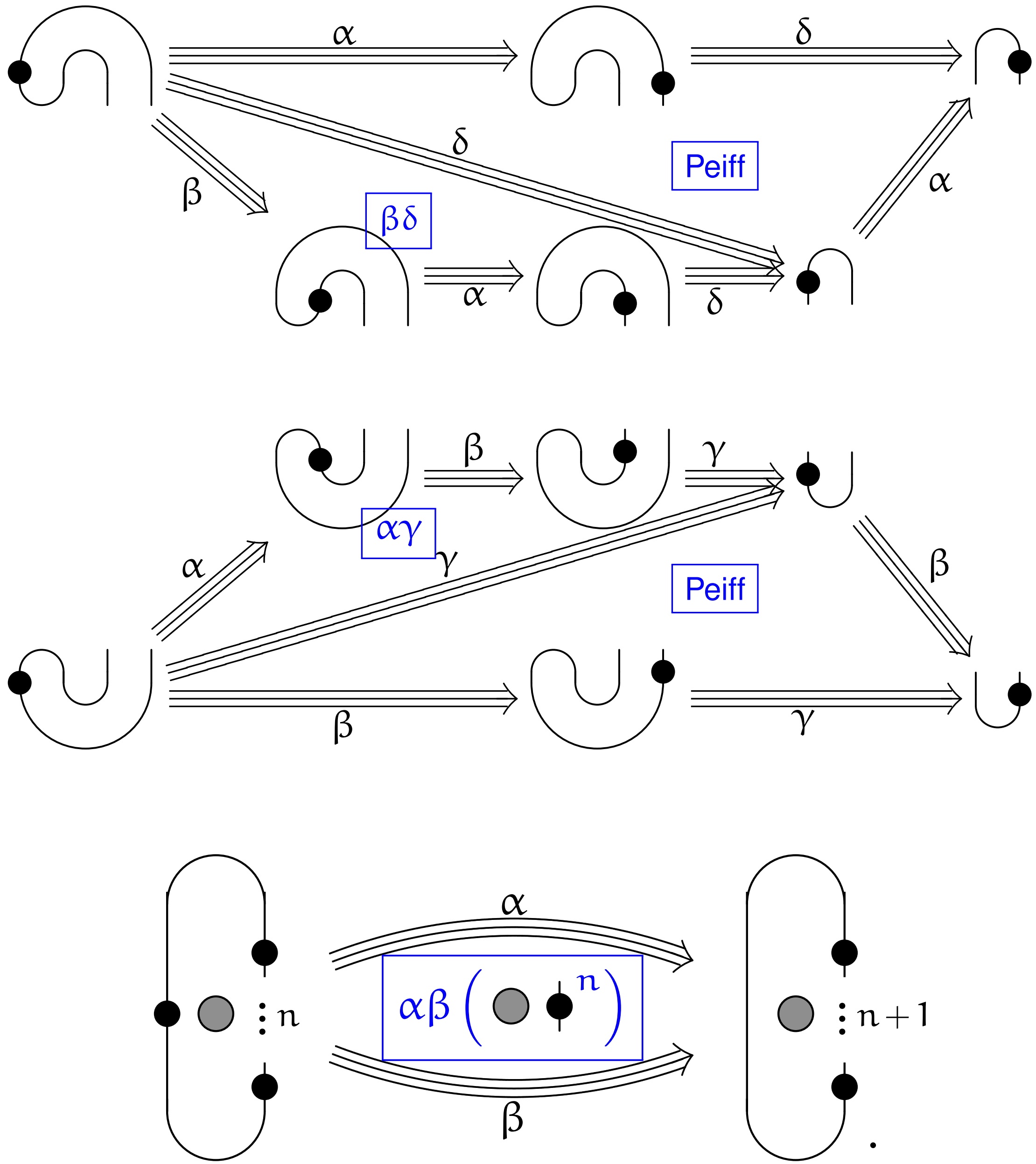




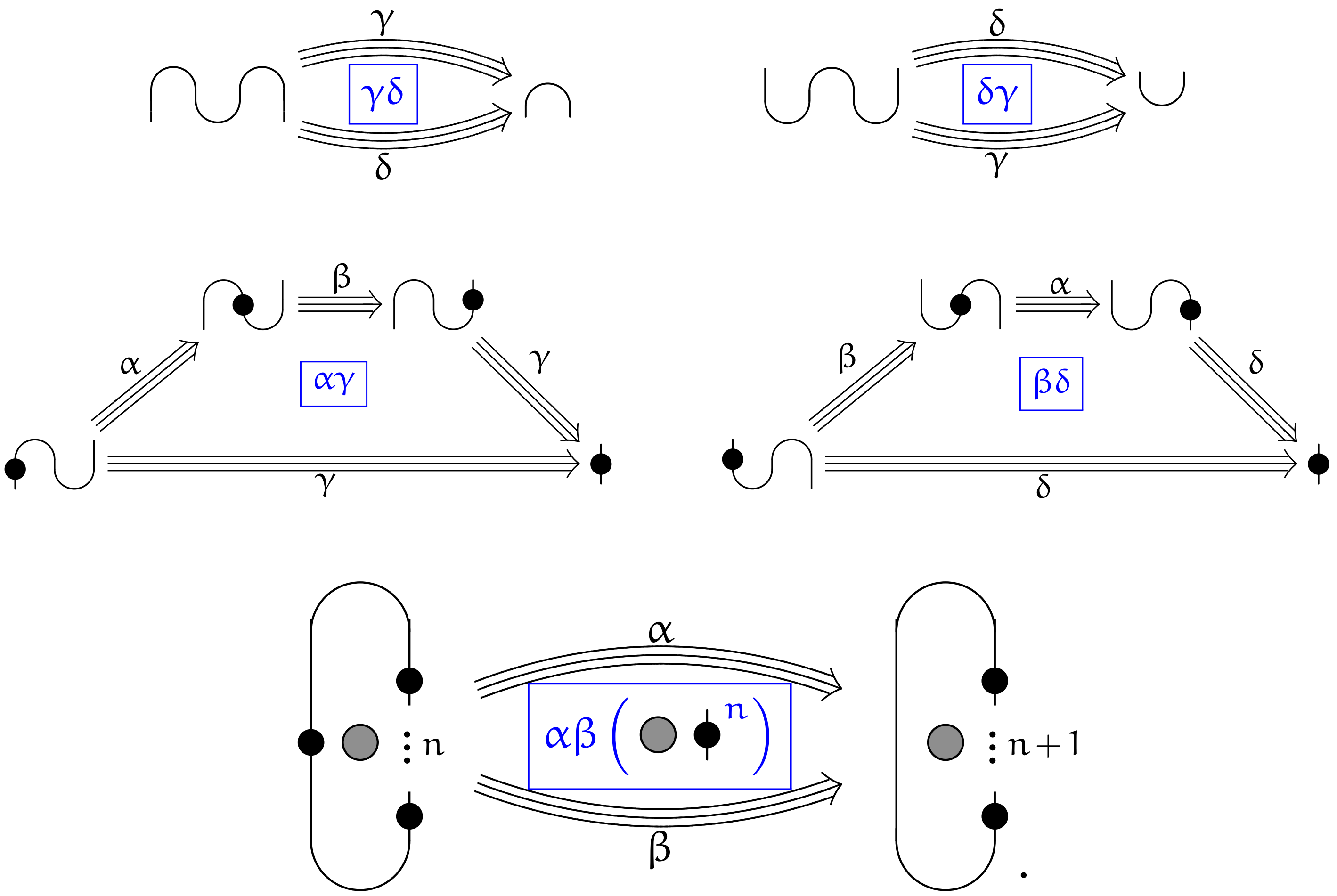
Example III : pearl necklaces



Example III : pearl necklaces



Example III : pearl necklaces



- An infinite homotopy base : $\{\, \gamma\delta,\ \delta\gamma,\ \alpha\gamma,\ \beta\delta,\ \alpha\beta\left(\bigcirc\ \bullet^{n}\right),\ n \in \mathbb{N} \,\}$.
- The 3-polygraph Pearl is finite and convergent but does not have finite derivation type

Identities among relations for polygraphs

The special case of groups

- (Peiffer-Reidemester-Whitehead '49) the **free crossed module**

$$\partial : C(\mathcal{P}) \longrightarrow F(X)$$

on a presentation $\mathcal{P} = (X, R)$ of a group $\mathbf{G}$.

- Whitehead's lemma ('49) : the following diagram commutes

$$\begin{array}{ccccccc} \ker \partial & \longrightarrow & C(\mathcal{P}) & \xrightarrow{\partial} & F(X) & \longrightarrow & \mathbf{G} \\ & & \downarrow{\scriptstyle (\ )_{ab}} & & \downarrow{\scriptstyle [\ ]} & & \\ \ker J & \longrightarrow & \mathbb{Z}\mathbf{G}[R] & \xrightarrow{J} & \mathbb{Z}\mathbf{G}[X] & \longrightarrow & \mathbb{Z}\mathbf{G} \end{array}$$

- Brown-Huebschmann ('82) : the Abelianisation map induces an isomorphism of $\mathbf{G}$-modules

$$\ker \partial \simeq \ker J$$

Theorem. (Cremanns-Otto '96) If $\mathcal{P}$ is finite, then $\mathbf{G}$ has FDT if and only if $\ker \partial$ is finitely generated.

Corollary. A finitely presented group is of type FP_3 if and only if it has FDT.

Identities among relations

- Let $\mathcal{C}$ be a n-category presented by a $(n+1)$-polygraph Σ.
- Let $\Sigma^{\top ab}$ be the Abelianised free $(n+1)$-track category on Σ.

$$\Sigma_{n+1}^{\top ab} \overset{s}{\underset{t}{\rightrightarrows}} \Sigma_n^* \twoheadrightarrow \mathcal{C}$$

Theorem A track category is Abelian if and only if it is linear.

- There exists a $\mathcal{C}$-module $\Pi(\Sigma)$ of **identities among relations** of Σ and an isomorphism of Σ_{n-1}^*-modules

$$\widetilde{\Pi}(\Sigma) \simeq \mathrm{Aut}^{\Sigma^{\top ab}}$$

- $\Pi(\Sigma)$ is defined as follows:
  - for an n-cell u of $\mathcal{C}$, then $\Pi(\Sigma)(u)$ is the quotient of

$$\mathbb{Z}\left\{ \lfloor f \rfloor \mid v \circlearrowleft f \text{ in } \Sigma^{\top ab},\ \overline{v} = u \right\}$$

by :

– $\lfloor f \star_n g \rfloor = \lfloor f \rfloor + \lfloor g \rfloor$ for every $f \circlearrowright v \circlearrowleft g$ with $\overline{v} = u$

– $\lfloor f \star_n g \rfloor = \lfloor g \star_n f \rfloor$ for every $v \underset{g}{\overset{f}{\rightleftarrows}} w$ with $\overline{v} = \overline{w} = u$

Generating identities among relations

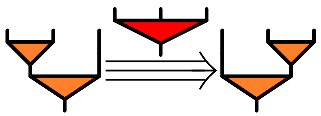
Theorem. Any homotopy basis of the free track category $\Sigma^{\top}$ forms a generating set for the module $\Pi(\Sigma)$.

Corollary. If a polygraph Σ has FDT, then the module $\Pi(\Sigma)$ is finitely generated.

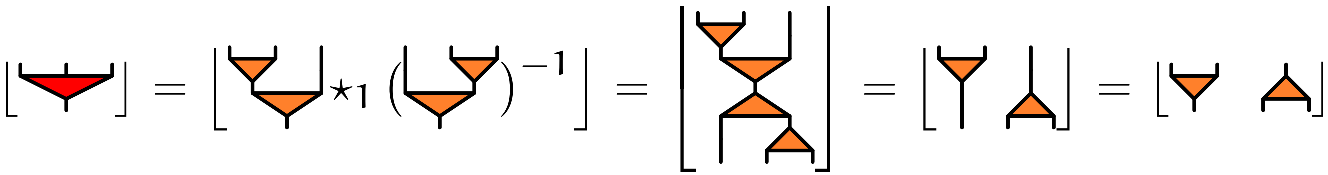
Corollary. If Σ is convergent, the module $\Pi(\Sigma)$ is generated by the generating confluences of Σ.

Example. Let As be the 2-polygraph $(*, |, \triangledown)$.

- As is a finite convergent presentation of the monoid $\langle a \mid aa = a \rangle$.
- As has one generating confluence:



- Hence the following element generates $\Pi(\Sigma)$:



Finite Abelian derivation type

Definition. An n-polygraph Σ has **finite Abelian derivation type** (**TDF$_{\mathrm{ab}}$**) if

i) Σ is finite,

ii) $\Sigma^{\top_{\mathrm{ab}}}$ admits a finite homotopy basis.

- TDF implies TDF$_{\mathrm{ab}}$,

Proposition. Let $\mathcal{C}$ be an n-category presented a polygraph Σ, then Σ has TDF$_{\mathrm{ab}}$ if and only if the $\mathcal{C}$-module $\Pi(\Sigma)$ is finitely generated.

Corollary. A 1-category has TDF$_{\mathrm{ab}}$ if and only if it is of homological type FP_3.