Coherence of quasi-terminating decreasing 2-polygraphs

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Motivation: compute syzygies

- A **syzygy** is a relation between generators (from Greek $\sigma\upsilon\zeta\upsilon\gamma\iota\alpha$, a pair).

- Syzygies problem in linear algebra.

  ▷ Given a finitely generated module M on a commutative ring R and a set of generators:

$$\{\mathbf{y_1}, \ldots, \mathbf{y_k}\},$$

  ▷ a **syzygy** of M is an element $(\lambda_1, ..., \lambda_k)$ in R^k for which

$$\lambda_1 \mathbf{y_1} + \ldots + \lambda_k \mathbf{y_k} = 0.$$

  ▷ The set of all syzygies with respect to a given generating set is a submodule of R^n called the **module of syzygies**.

- Schreyer, 1980 : computation of linear syzygies by means of the **division algorithm**.

  ▷ Buchberger's completion algorithm for computing Gröbner bases allows the computation of the first syzygy module.

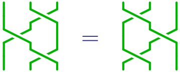
  ▷ The reduction to zero of the S-polynomial of two polynomials in a Gröbner basis gives a syzygy.

Motivation: compute syzygies for presentations of monoids

- Syzygy problem for a monoid **M**
  - presented by **generators** and **relations**.
  - We would like build a (small !) **cofibrant approximation** of **M** in the category of $(\infty, 1)$-categories,
    - that is, a free $(\infty, 1)$-category homotopically equivalent to **M**.
- In low dimensions : **coherent presentations**
  - generators, rules, **syzygies**.
- Applications:
  - Explicit description of actions of a monoid on categories in representation theory.
  - Coherence theorems for monoids.
  - Algorithms in homological algebra.

Examples

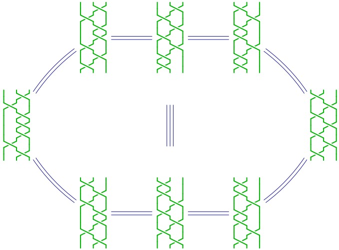
- The **Artin monoid** $\mathbf{B}_3^+$ of braids on 3 strands.

$s =$  $t =$  

- The **Artin presentation**:

$$\mathrm{Art}_2(\mathbf{B}_3^+) = \langle\, s, t \quad \big| \quad tst = sts \rangle$$

- We will prove that there is no syzygy between relations induced by $tst = sts$.



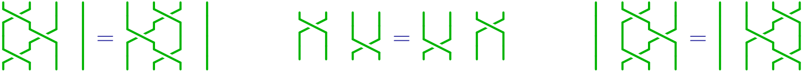
With presentation $\mathrm{Art}_2(\mathbf{B}_3^+)$ two proofs of the same equality in $\mathbf{B}_3^+$ are equal.

Motivation

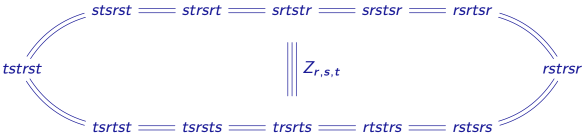
▶ The **Artin monoid** $\mathbf{B}_4^+$ of braids on 4 strands.

▶ The **Artin presentation**

$$\mathrm{Art}_2(\mathbf{B}_4^+) = \langle\, r, s, t \;\mid\; rsr = srs,\;\; rt = tr,\;\; tst = sts \,\rangle$$



▶ The relations amongst the braid relations on 4 strands are generated by the following **Zamolodchikov relation** (Deligne, 1997).



Motivation

▶ Computation of finite coherent presentations with **homotopical completion-reduction procedure** (Guiraud-M.-Mimram, RTA 2013).

- ▷ Knuth-Bendix's completion procedure.
- ▷ Squier's homotopical theorem for convergent rewriting systems.
- ▷ Homotopically reduce generators, rules, syzygies.

▶ The **Knuth-Bendix** procedure does not terminate for

- ▷ $\mathbf{B}_3^+ = \langle\, s, t \mid sts = tst \rangle$ on the two generators s and t, (Kapur-Narendran, 1985)
- ▷ Plactic monoid $\mathbf{P_4}$ on the generators 1, 2, 3, 4, (Kubat-Okniński, 2014).

▶ Computation of coherent presentations with convergent presentations using new generators.

- ▷ The Artin monoid $\mathbf{B}^+(\mathbf{W})$ with Garside's presentation, (Gaussent-Guiraud-M., 2015)
- ▷ Plactic monoid $\mathbf{P_n}$ with column presentation, (Hage-M., 2016).

Coherent presentations

Other possibility: weaken the termination hypothesis.

Plan

I. Polygraphs and coherent presentations of monoids

- Polygraphs as higher-dimensional rewriting systems
- Coherent presentations of monoids
- Homotopical completion-reduction procedure

II. Decreasing two-dimensional polygraphs

- Labelled polygraphs
- Decreasing two-dimensional polygraphs
- Decreasingness of Peiffer branchings

III. Coherence by decreasingness

- Decreasing Squier's completion
- Main result
- Example

Part I. Coherent presentations of monoids

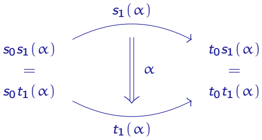
Polygraphs

- A 1-**polygraph** is an directed graph (Σ_0, Σ_1)

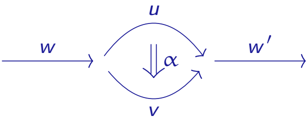
$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1$$

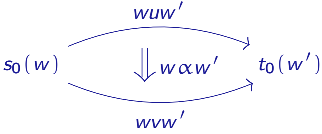
- A 2-**polygraph** is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ where
  - ▷ (Σ_0, Σ_1) is a 1-polygraph,
  - ▷ Σ_2 is a **globular extension** of the free 1-category Σ_1^*.

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2$$



- A **rewriting step** is a 2-cell of the free 2-category Σ_2^* over Σ with shape





where $u \overset{\alpha}{\Longrightarrow} v$ is a 2-cell of Σ_2 and w, w' are 1-cells of Σ_1^*.

Termination

- A 2-polygraph Σ **terminates** if it does not generate any infinite reduction sequence

$$u_1 \Longrightarrow u_2 \Longrightarrow \cdots \Longrightarrow u_n \Longrightarrow \cdots$$

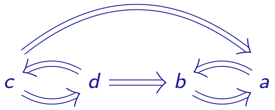
- A 2-polygraph Σ is **quasi-terminating** if every infinite reduction sequence

$$u_1 \Longrightarrow u_2 \Longrightarrow \cdots \Longrightarrow u_n \Longrightarrow \cdots$$

cycles, that is the sequence contains an infinite number of occurrences of the same 1-cell.

- A 1-cell u of Σ_1^* is called a **semi-normal form** if for any rewriting step with source u leading to a 1-cell v, there exists a rewriting sequence from v to u.

- If Σ is quasi-terminating, any 1-cell u of Σ_1^* admits a semi-normal form.
  - Note that, this semi-normal form is neither irreducible nor unique in general.



Example

▶ The 2-polygraph

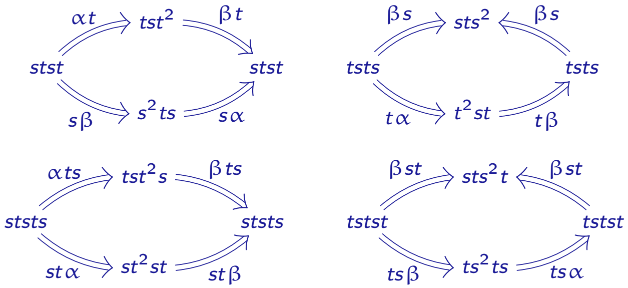
$$\Sigma(\mathbf{B}_3^+) = \langle s, t \mid sts \overset{\alpha}{\Longrightarrow} tst,\ tst \overset{\beta}{\Longrightarrow} sts \rangle$$

presents the monoid $\mathbf{B}_3^+$.

- ▷ It is not terminating but it is quasi-terminating.
- ▷ It has four critical branchings:

$$(\alpha t, s\beta), \quad (\beta s, t\alpha), \quad (\alpha ts, st\alpha) \quad \text{and} \quad (\beta st, ts\beta).$$

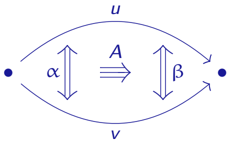
These four branchings are confluent as follows



Polygraphs

- A $(3,1)$-**polygraph** is a data made of
  - a 2-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2)$,
  - a globular extension Σ_3 of the free $(2,1)$-category $\Sigma_2^\top$.

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2^\top \underset{t_2}{\overset{s_2}{\leftleftarrows}} \Sigma_3$$



- The $(2,1)$-category $\Sigma_2^\top$ corresponds to the 2-category of congruences generated by Σ_2.

Coherent presentations of categories

▶ A **coherent presentation** of $\mathbf{M}$ is a $(3,1)$-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$ such that

▷ $(\Sigma_0, \Sigma_1, \Sigma_2)$ is a presentation of $\mathbf{M}$:

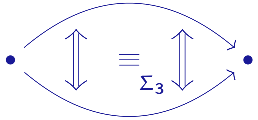
$$\Sigma_0 = \{\bullet\} \qquad \text{and} \qquad \mathbf{M} \simeq \Sigma_1^*/\Sigma_2,$$

▷ the cellular extension Σ_3 is a **homotopy basis**.

In other words:

▷ the quotient $(2,1)$-category $\Sigma_2^\top/\Sigma_3$ is aspherical,

▷ the congruence generated by Σ_3 on the $(2,1)$-category $\Sigma_2^\top$ contains every pair of parallel 2-cells.



▷ 3-cells of Σ_3 generate a tiling of $\Sigma_2^\top$.

Coherent presentations

Problems.

1. How to compute a coherent presentation ?
2. How to reduce a coherent presentation ?

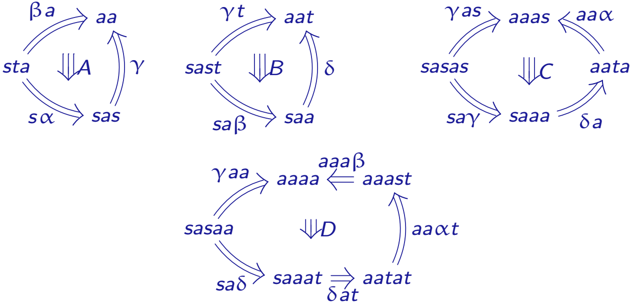
Homotopical completion-reduction procedure

Example. The **Kapur-Narendran's presentation** of $\mathbf{B}_3^+$, obtained from Artin's presentation by coherent adjunction of the Coxeter element st

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \ \mid\ ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

The deglex order generated by $t > s > a$ proves the termination of Σ_2^{KN}.

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \ \mid\ ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\Longrightarrow} aa,\ saa \overset{\delta}{\Longrightarrow} aat \ \mid\ A,\ B,\ C,\ D \,\rangle$$



However. The coherent presentation $\mathcal{S}(\Sigma_2^{\mathrm{KN}})$ obtained is bigger than necessary.

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a \,\rangle$$

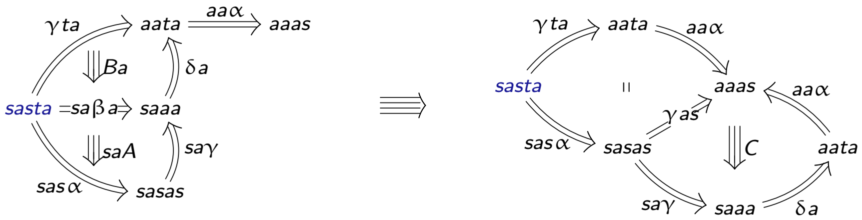
$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as\,, st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, \cancel{C}, D \,\rangle$$

- There are four critical triple branchings, overlapping on

$$sasta,\; sasast,\; sasasas,\; sasasaa.$$

  - Critical triple branching on *sasta* proves that C is redundant:



$$C = sas\alpha^{-1} \star_1 (Ba \star_1 aa\alpha) \star_2 (saA \star_1 \delta a \star_1 aa\alpha)$$

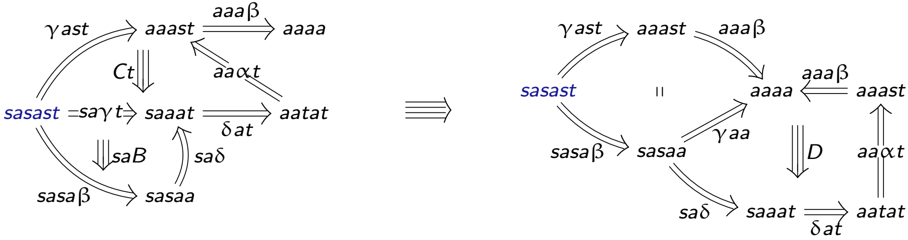
Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\Longrightarrow} aa,\ saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\Longrightarrow} aa,\ saa \overset{\delta}{\Longrightarrow} aat \mid A, B, \cancel{C}, \cancel{D} \,\rangle$$

- ▷ Critical triple branching on $sasast$ proves that D is redundant:



$$D = sasa\beta^{-1} \star_1 \big((Ct \star_1 aaa\beta) \star_2 (saB \star_1 \delta at \star_1 aa\alpha t \star_1 aaa\beta) \big)$$

The homotopical completion-reduction procedure

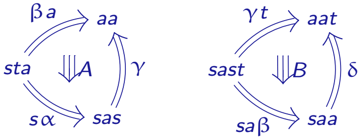
Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a,\, sas \overset{\gamma}{\Longrightarrow} aa,\, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as\,,\, st \overset{\beta}{\Longrightarrow} a,\; \cancel{sas \overset{\gamma}{\Longrightarrow} aa},\; \cancel{,\, saa \overset{\delta}{\Longrightarrow} aat} \mid \cancel{A}, \cancel{B}, \cancel{C}, \cancel{D} \rangle$$

▷ The 3-cells A and B are collapsible and the rules γ and δ are redundant.



The homotopical completion-reduction procedure

Example.

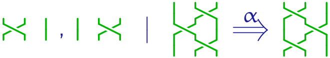
$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a,\; sas \overset{\gamma}{\Longrightarrow} aa,\; saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, \cancel{a} \;\mid\; tst \overset{\alpha}{\Longrightarrow} sts,\; \cancel{st \overset{\beta}{\Longrightarrow} a},\; \cancel{sas \overset{\gamma}{\Longrightarrow} aa},\; \cancel{saa \overset{\delta}{\Longrightarrow} aat} \mid \cancel{A}, \cancel{B}, \cancel{C}, \cancel{D} \,\rangle$$

- ▷ The rule $st \overset{\beta}{\Longrightarrow} a$ is collapsible and the generator a is redundant.

$$\mathcal{R}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t \;\mid\; tst \overset{\alpha}{\Longrightarrow} sts \;\mid\; \emptyset \,\rangle$$

$= \langle$  $\mid \; \emptyset \,\rangle$

Coherent presentations

Problems.

1. How to compute a coherent presentation without adding generators ?
2. How to weaken the terminating hypothesis ?

Part II. Decreasing two-dimensional polygraphs

Labelled two-dimensional polygraphs

▶ A **well-founded labelling** for a 2-polygraph Σ is a data $(W, \prec, \psi)$ made of a set W, a well-founded order $\prec$ on W and a map

$$\psi : \Sigma_{stp} \longrightarrow W$$

that associates to a rewriting step f a **label** $\psi(f)$.

▶ Given a rewriting sequence $f = f_1 \cdot \ldots \cdot f_k$, we denote by

$$L^W(f) = \{\psi(f_1), \ldots, \psi(f_k)\}$$

the set of labels of rewriting steps in f.

Labelling to the semi-normal form

- Let Σ be a confluent and quasi-terminating 2-polygraph.
  - By quasi-termination, any 1-cell u admits a (non-unique) semi-normal form.
  - Given a 1-cell u in Σ_1^*, we fix a semi-normal form $\widetilde{u}$.
  - By confluence, any two congruent 1-cells of Σ_1^* have the same semi-normal form.
- The **labelling to the semi-normal form** associated is the map

$$\psi^{\mathrm{SNF}} : \Sigma_{stp} \longrightarrow \mathbb{N}$$

defined, for any rewriting step f of Σ.

$$\psi^{\mathrm{SNF}}(f) = d(t_1(f), \widetilde{t_1(f)}),$$

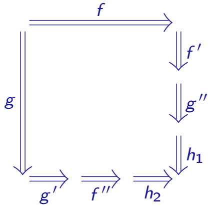
the length of the shortest rewriting sequence from $t_1(f)$ to its semi-normal form.

Decreasing two-dimensional polygraphs

▶ Decreasingness from ARS, (van Oostrom, 1994).

▷ Let Σ be a 2-polygraph with a well-founded labelling $(W, \psi, \prec)$.

▶ A local branching (f, g) of Σ is **decreasing** if there is a **decreasing confluence diagram**:



with

i) for each $k \in L^W(f')$, we have $k \prec \psi(f)$,

ii) for each $k \in L^W(g')$, we have $k \prec \psi(g)$,

iii) f'' (resp. g'') is an identity or a rewriting step labelled by $\psi(f)$ (resp. $\psi(g)$),

iv) for each $k \in L^W(h_1) \cup L^W(h_2)$, we have $k \prec \psi(f)$ or $k \prec \psi(g)$.

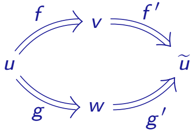
▶ A 2-polygraph Σ is **decreasing** if there exists a well-founded labelling $(W, \prec, \psi)$ of Σ making all its local branching decreasing.

Theorem. Any decreasing 2-polygraph is confluent.

Decreasingness from quasi-termination

▶ Any confluent and quasi-terminating 2-polygraph Σ is decreasing with respect to any semi-normal form labelling ψ^{SNF}.

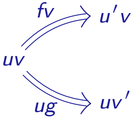
▷ For any local branching $u \Rightarrow (v, w)$ there is a semi-normal form $\widetilde{u}$ giving a confluence diagram as follows:



▷ We choose the rewriting sequences f' and g' of minimal length, thus making this confluence diagram decreasing with respect ψ^{SNF}.

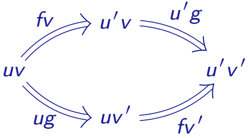
Decreasingness of Peiffer branchings

- Given a Peiffer branching



of a 2-polygraph Σ.

- We will call **Peiffer confluence** the following confluence diagram



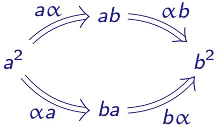
- If Σ is decreasing,
  - all its Peiffer branchings can be completed into a decreasing confluence diagram.
  - However, the Peiffer confluence for this branching is not necessarily decreasing.
  - it is the case for a labelling SNF when the source uv is already the chosen semi-normal form.

Decreasingness of Peiffer branchings

Example. Consider the 2-polygraph $\Sigma = \langle a, b \mid a \overset{\alpha}{\Longrightarrow} b, b \overset{\beta}{\Longrightarrow} a \rangle$.

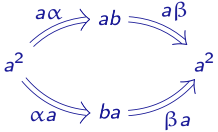
- ▷ Σ is confluent and quasi-terminating.
- ▷ For each 1-cell u of Σ_1^*, we set $\widetilde{u} = a^{\ell(u)}$.
- ▷ Σ is decreasing for the labelling ψ^{SNF} associated.

▶ The following Peiffer confluence:



is not decreasing. We have $\psi^{\mathrm{SNF}}(\alpha a) = \psi^{\mathrm{SNF}}(a\alpha) = 1$ and $\psi^{\mathrm{SNF}}(\alpha b) = \psi^{\mathrm{SNF}}(b\alpha) = 2$.

▶ This Peiffer branching is decreasing by using the diagram



Indeed, $\psi^{\mathrm{SNF}}(a\beta) = \psi^{\mathrm{SNF}}(\beta a) = 0$.

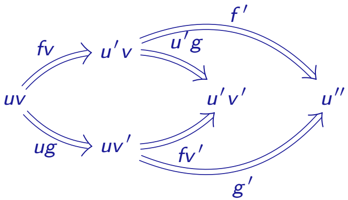
Peiffer decreasingness

▶ Let Σ be a 2-polygraph and let Σ_3 be a globular extension of the $(2,1)$-category $\Sigma_2^\top$.

▶ The 2-polygraph Σ is **Peiffer decreasing with respect to** Σ_3 if there exists a well-founded labelling $(W, \prec, \psi)$ such that the following conditions hold

▷ Σ is decreasing with respect to $(W, \prec, \psi)$,

▷ for any Peiffer branching $(fv, ug) : uv \Rightarrow (u'v, uv')$, there exists a decreasing confluence diagram $(fv \cdot f', ug \cdot g')$:



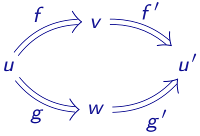
such that

$$u'g \star_1 (fv')^- \equiv_{\Sigma_3} f' \star_1 (g')^-.$$

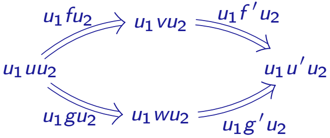
Whisker compatibility

- Let Σ be a 2-polygraph with a well-founded labelling $(W, \prec, \psi)$.

- The labelling is **whisker compatible** if for any decreasing confluence diagram



where (f, g) is a local branching, and for any 1-cells u_1 and u_2 in Σ_1^*, then the following confluence diagram is decreasing:



- Note that a labelling SNF is not whisker compatible in general.

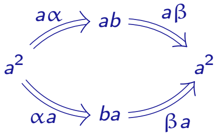
Example

Example. Consider the 2-polygraph $\Sigma = \langle a, b \mid a \overset{\alpha}{\Longrightarrow} b,\ b \overset{\beta}{\Longrightarrow} a \rangle$.

- ▷ For each 1-cell u of Σ_1^*, we set $\widetilde{u} = a^{\ell(u)}$.
- ▷ The labelling ψ^{SNF} associated is whisker compatible.
- ▷ $\psi^{\mathrm{SNF}}(u_1 f u_2) = \psi^{\mathrm{SNF}}(u_1) + \psi^{\mathrm{SNF}}(f) + \psi^{\mathrm{SNF}}(u_2)$, for f in Σ_{stp} and 1-cells u_1, u_2.

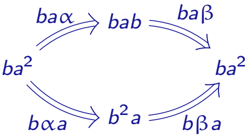
▶ If the labelling ψ^{SNF} is associated to semi-normal forms of the form

- ▷ $\widetilde{u} = a^{\ell(u)}$, for any 1-cell u such that $\ell(u) \neq 3$, and $\widetilde{a^3} = a^3$ and $\widetilde{b^3} = b^3$.
- ▷ The diagram



is decreasing with $\psi^{\mathrm{SNF}}(a\alpha) = \psi^{\mathrm{SNF}}(\alpha a) = 1$ and $\psi^{\mathrm{SNF}}(a\beta) = \psi^{\mathrm{SNF}}(\beta a) = 0$.

- ▷ However, the diagram



is not decreasing with $\psi^{\mathrm{SNF}}(ba\alpha) = \psi^{\mathrm{SNF}}(b\alpha a) = 1$ and $\psi^{\mathrm{SNF}}(ba\beta) = \psi^{\mathrm{SNF}}(b\beta a) = 2$.

Example

- Consider the 2-polygraph

$$\Sigma(\mathbf{B}_3^+) = \langle s, t \mid sts \overset{\alpha}{\Longrightarrow} tst,\ tst \overset{\beta}{\Longrightarrow} sts \rangle$$

- We define the labelling SNF ψ^{SNF} associated to semi-normal forms given for each 1-cell u of $\Sigma(\mathbf{B}_3^+)_1^*$ by

$$\widetilde{u} = (sts)^{N_u} v,$$

where v is a 1-cell of $\Sigma(\mathbf{B}_3^+)_1^*$ and

$$N_u = \max\{n \mid u = (sts)^n v \text{ holds in } \mathbf{B}_3^+\}.$$

- The labelling ψ^{SNF} is whisker compatible.
  - Indeed, for any rewriting steps f and g, have

$$\psi^{\mathrm{SNF}}(g) < \psi^{\mathrm{SNF}}(f) \qquad \text{implies} \qquad \psi^{\mathrm{SNF}}(u_1 f u_2) < \psi^{\mathrm{SNF}}(u_1 g u_2)$$

for any 1-cells u_1 and u_2.

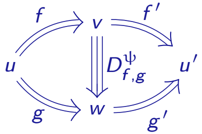
Part III. Coherence by decreasingness

Decreasing Squier's completion

- Let Σ be a decreasing 2-polygraph for a well-founded labelling $(W, \prec, \psi)$.

- A **family of generating decreasing confluences** of Σ with respect to ψ is a globular extension of the $(2,1)$-category $\Sigma_2^\top$ that contains,

  - for every critical branching (f, g) of Σ, one 3-cell of the form



  - where the confluence diagram $(f \cdot f', g \cdot g')$ is decreasing with respect to ψ.

- Any decreasing 2-polygraph admits such a family of generating decreasing confluences.

- Such a family is not unique in general.

Decreasing Squier's completion

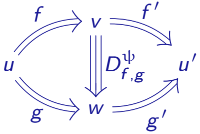
- Let Σ be a decreasing 2-polygraph for a well-founded labelling $(W, \prec, \psi)$.

- A **decreasing Squier's completion** of Σ with respect to ψ is a $(3,1)$-polygraph $\mathcal{D}(\Sigma, \psi)$
  - that extends the 2-polygraph Σ,
  - by a globular extension

$$\mathcal{O}(\Sigma, \psi) \cup \mathcal{L}(\Sigma)$$

where

- $\mathcal{O}(\Sigma, \psi)$ is a chosen family of generating decreasing confluences with respect to ψ,



- $\mathcal{L}(\Sigma)$ is a loop extension of Σ, containing exactly one loop for each equivalence classes of elementary loops of Σ_2^*.

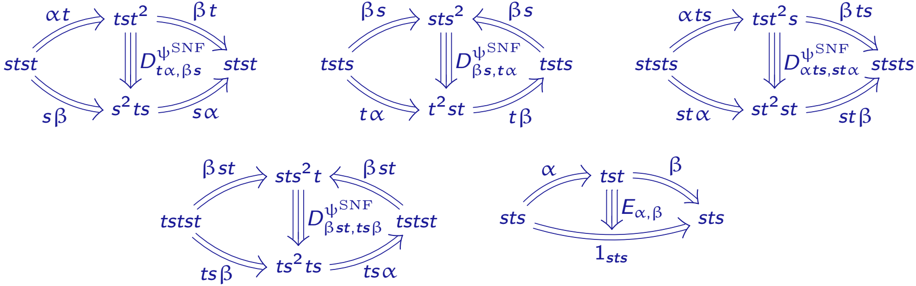
Decreasing Squier's completion

▶ **Example**. The 2-polygraph

$$\Sigma(\mathbf{B}_3^+) = \langle s, t \mid sts \overset{\alpha}{\Longrightarrow} tst,\ tst \overset{\beta}{\Longrightarrow} sts \rangle$$

is decreasing for the labelling SNF ψ^{SNF} defined with the semi-normal form of the $(sts)^N v$.

▶ A decreasing Squier's completion of the 2-polygraph $\Sigma(\mathbf{B}_3^+)$ is given by



The confluences diagrams are decreasing:

$$\psi^{\mathrm{SNF}}(\alpha t) = \psi^{\mathrm{SNF}}(s\beta) = 1 \quad \text{and} \quad \psi^{\mathrm{SNF}}(\beta t) = \psi^{\mathrm{SNF}}(s\alpha) = 0.$$

$$\psi^{\mathrm{SNF}}(\beta s) = 0,\ \psi^{\mathrm{SNF}}(t\alpha) = 2 \quad \text{and} \quad \psi^{\mathrm{SNF}}(t\beta) = 1, \psi^{\mathrm{SNF}}(\beta s) = 0.$$

$$\psi^{\mathrm{SNF}}(\alpha ts) = \psi^{\mathrm{SNF}}(st\alpha) = 1 \quad \text{and} \quad \psi^{\mathrm{SNF}}(\beta ts) = \psi^{\mathrm{SNF}}(st\beta) = 0.$$

$$\psi^{\mathrm{SNF}}(\beta st) = 0,\ \psi^{\mathrm{SNF}}(ts\beta) = 2 \quad \text{and} \quad \psi^{\mathrm{SNF}}(ts\alpha) = 1, \psi^{\mathrm{SNF}}(\beta st) = 0.$$

Decreasing Squier's completion

Theorem. (Alleaume-M., 2016)

Let Σ be a 2-polygraph and let ψ^{SNF} be a SNF labelling of Σ.

Let $\mathcal{D}(\Sigma, \psi^{\mathrm{SNF}})$ be a decreasing Squier's completion of Σ.

If the three following conditions hold

- Σ is quasi-terminating,
- ψ^{SNF} is whisker compatible,
- Σ is Peiffer decreasing with respect to ψ^{SNF} and with respect to $\mathcal{D}(\Sigma, \psi^{\mathrm{SNF}})$.

Then $\mathcal{D}(\Sigma, \psi^{\mathrm{SNF}})$ is a coherent presentation of the monoid presented by Σ.

Corollary (Squier, 1994)

Let Σ be a convergent 2-polygraph. Any Squier's completion of Σ is a coherent presentation of the monoid presented by Σ.

Decreasing Squier's completion

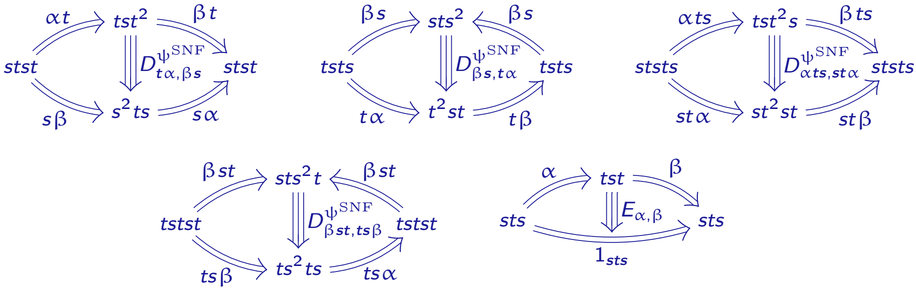
Example. Consider the 2-polygraph

$$\Sigma(\mathbf{B}_3^+) = \langle s, t \mid sts \overset{\alpha}{\Longrightarrow} tst,\ tst \overset{\beta}{\Longrightarrow} sts \rangle$$

with the labelling ψ^{SNF} defined using the semi-normal forms $(sts)^N v$.

- ▷ $\Sigma(\mathbf{B}_3^+)$ is quasi-terminating,
- ▷ ψ^{SNF} is whisker compatible
- ▷ $\Sigma(\mathbf{B}_3^+)$ is Peiffer decreasing with respect to ψ^{SNF} and with respect to $\mathcal{L}(\Sigma)$.

▶ Thus the following 3-cells extend ψ^{SNF} into a coherent presentation of $\mathbf{B}_3^+$:



▶ This is another proof that Artin's presentation of $\mathbf{B}_3^+$ has no syzygy.