Boundary value problems for the infinity Laplacian: regularity and geometric results

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Initial motivation

Study the overdetermined boundary value problems

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = c & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta_\infty^N u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = c & \text{on } \partial\Omega . \end{cases}$$

Δ_∞ = infinity Laplacian

Δ_∞^N = normalized infinity Laplacian

Symmetry results

The overdetermined boundary value problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \\ |\nabla u| = c & \text{on } \partial\Omega\,, \end{cases}$$

admits a solution $\iff$ Ω is a ball.

[Serrin 1971]

Serrin's result extends to the case of the p-Laplacian operator, and of more general elliptic operators in divergence form

[Garofalo-Lewis 1989, Damascelli-Pacella 2000, Brock-Henrot 2002, F.-Gazzola-Kawohl 2006]

What happens for $p = +\infty$?

Symmetry breaking may occur!

This intriguing discovery leads to study a number of

geometric and regularity matters

Outline

- I. Background: overview on infinity Laplacian and viscosity solutions
- II. Overdetermined problem: a simple case (web functions)
- III. Geometric intermezzo
- IV. Regularity results for the Dirichlet problem
- V. Overdetermined problem: the general case

The infinity Laplace operator

$$\Delta_\infty u := \langle \nabla^2 u \cdot \nabla u, \nabla u \rangle \qquad \text{for all } u \in C^2(\Omega)$$

Where the name comes from:

Formally, it is the limit as $p \to +\infty$ of the p-Laplacian $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2} \nabla u)$

$$\Delta_p u = |\nabla u|^{p-2} \Delta u + (p-2) |\nabla u|^{p-4} \Delta_\infty u$$

If divide the equation $\Delta_p u = 0$ by $(p-2)|\nabla u|^{p-4}$, we obtain

$$0 = \frac{|\nabla u|^2}{p-2} \Delta u + \Delta_\infty u \, .$$

As $p \to +\infty$, we formally get $\Delta_\infty u = 0$.

A quick overview

- *Origin:* [Aronsson 1967] discovered the operator and found the "singular" solution

$$u(x,y) = x^{4/3} - y^{4/3}\ , \qquad \Delta_\infty u = 0 \text{ in } \mathbb{R}^2 \setminus \{axes\}\,.$$

- *Viscosity solutions*: [Bhattacharya, DiBenedetto, Manfredi 1989], [Jensen 1998] proved the existence and uniqueness of a *viscosity* solution to

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega\,. \end{cases}$$

  Optimization of Lipschitz extension of functions: $u \in AML(g)$, i.e.
  $u = g$ on $\partial\Omega$ and $\forall A \subset\subset \Omega$, $\forall v = u$ on ∂A, $\|\nabla u\|_{L^\infty(A)} \leq \|\nabla v\|_{L^\infty(A)}$

- *Calculus of Variations in L^∞* [Juutinen 1998, Barron 1999, Crandall-Evans-Gariepy 2001, Crandall 2005, Barron-Jensen-Wang 2001]

- *Regularity of ∞-harmonic functions*

  – $C^{1,\alpha}$ for $n = 2$ [Savin 2005, Evans-Savin 2008]

  – differentiability in any space dimension [Evans-Smart 2011]

  Remark: C^1 regularity in dimension $n > 2$ is a major open problem!

- *Inhomogeneous problems*

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

  – existence and uniqueness of a viscosity solution u [Lu-Wang 2008]

  – u is everywhere differentiable [Lindgren 2014]

▷ Recent trend: study problems involving the *normalized infinity Laplacian*, in connection with *"Tug-of-War differential games"*

$$\Delta_\infty^N u := \begin{cases} \langle \nabla^2 u \cdot \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \rangle & \text{if } \nabla u \neq 0 \\ \left[\lambda_{min}(\nabla^2 u), \lambda_{max}(\nabla^2 u)\right] & \text{if } \nabla u = 0 \end{cases} \qquad \text{for all } u \in C^2(\Omega).$$

Existence and uniqueness of a viscosity solution have been proved for

$$\begin{cases} -\Delta_\infty^N u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

[Peres-Schramm-Sheffield-Wilson 2009, Lu-Wang 2010, Armstrong-Smart 2012]

Viscosity solutions

- A viscosity solution to $-\Delta_\infty u = 1$ in Ω is a function $u \in C(\Omega)$ which is both a viscosity sub-solution and a viscosity super-solution, meaning that, for all $x \in \Omega$ and for all smooth functions φ:

$$-\Delta_\infty \varphi(x) \leq 1 \quad \text{if } u \prec_x \varphi\,, \qquad -\Delta_\infty \varphi(x) \geq 1 \quad \text{if } \varphi \prec_x u$$

- For solutions to $-\Delta_\infty^N u = 1$ the above inequalities must be replaced by

$$\begin{cases} -\dfrac{\Delta_\infty \varphi(x)}{|\nabla \varphi(x)|^2} \leq 1 & \text{if } \nabla \varphi(x) \neq 0 \\ -\lambda_{\max}(\nabla^2 \varphi(x)) \leq 1 & \text{if } \nabla \varphi(x) = 0 \end{cases} \qquad \begin{cases} -\dfrac{\Delta_\infty \varphi(x)}{|\nabla \varphi(x)|^2} \geq 1 & \text{if } \nabla \varphi(x) \neq 0 \\ -\lambda_{\min}(\nabla^2 \varphi(x)) \geq 1 & \text{if } \nabla \varphi(x) = 0\,. \end{cases}$$

[Crandall-Ishii-Lions 1992]

Simplified version of the overdetermined problem

Q. For which domains Ω is it true that the unique solution u to

$$(D) \quad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is of the form

$$u(x) = \varphi(d_\Omega(x)) \quad \text{in } \Omega \quad ?$$

We call such a function u a *web-function*.

Remark: u web $\Rightarrow |\nabla u| = |\varphi'(0)| = c \quad$ on $\partial\Omega$.

Basic example: web solution on the ball

Look for a radial solution to problem (D) in a ball $B_R(0)$:

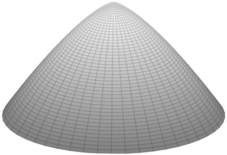
$$\begin{cases} -\Delta_\infty u = 1 & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases}$$

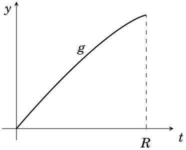
If $u(x) = \varphi(R - |x|)$, we have to solve the $1D$ problem

$$-\varphi''(R-|x|)\,[\varphi'(R-|x|)]^2 = 1, \qquad \varphi(0) = 0, \qquad \varphi'(R) = 0.$$

The solution is

$$f(t) = c_0[R^{4/3} - (R-t)^{4/3}], \quad c_0 = 3^{4/3}/4 \qquad (\Rightarrow u \in C^{1,1/3}(B_R))$$





Similar computations in the normalized case, with profile

$$g(t) = \frac{1}{2}[R^2 - (R-t)^2] \qquad (\Rightarrow u \in C^{1,1}(B_R))$$

Heuristics

Assume that u is a C^2 solution to problem (D) in a domain Ω.

Gradient flow (characteristics) $\begin{cases} \dot{\gamma}(t) = \nabla u\big((\gamma(t))\big) \\ \gamma(0) = x \end{cases}$

P-function $P(x) := \dfrac{|\nabla u(x)|^4}{4} + u(x)$

$$\frac{d}{dt} P(\gamma(t)) = |\nabla u|^2 \langle \nabla^2 u \cdot \nabla u, \nabla u \rangle + |\nabla u|^2 = |\nabla u|^2 (\Delta_\infty u + 1) = 0 \Rightarrow$$

$\Rightarrow P(\gamma(t)) = \lambda$ (P is constant along characteristics)

$\Rightarrow u(\gamma(t))$ can be explicitly determined by solving an ODE

Unfortunately from this information we cannot reconstruct u because we do not know the geometry of characteristics! ... BUT, if $u = \varphi(d_\Omega)$:

- ∇u is parallel to $\nabla d_\Omega \Rightarrow$ characteristics are line segments normal to $\partial\Omega$
- By solving an ODE for φ as in the radial case, we get:

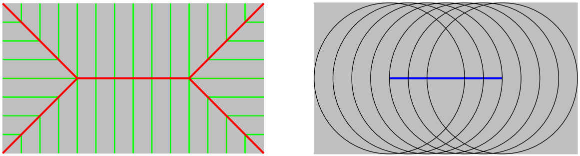
  $$\varphi(t) = f(t) := c_0 \left[R^{4/3} - (R-t)^{4/3} \right] \qquad (R = \text{length of the characteristic})$$

- If we ask u to be differentiable, all characteristics must have the same length equal to the inradius ρ_Ω and u is given by

  $$u(x) = \Phi_\Omega(x) := c_0 \left[\rho_\Omega^{4/3} - (\rho_\Omega - d_\Omega(x))^{4/3} \right].$$

When do characteristics have the same length?

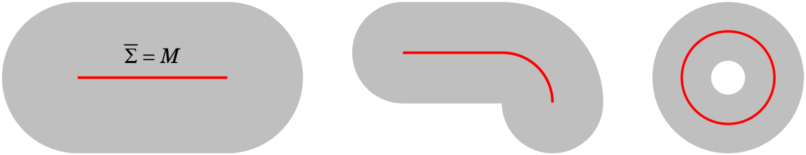
- False in general



- True $\iff \overline{\Sigma}(\Omega) = M(\Omega)$, where

  Cut locus $\overline{\Sigma}(\Omega) :=$ the closure of the singular set $\Sigma(\Omega)$ of d_Ω

  High ridge $M(\Omega) :=$ the set where $d_\Omega(x) = \rho_\Omega$



Theorem (web-viscosity solutions)
The unique viscosity solution to problem

$$(D) \qquad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is a web-function if and only if $M(\Omega) = \overline{\Sigma}(\Omega)$. In this case,

$$u(x) = \Phi_\Omega(x) := c_0 \left[\rho_\Omega^{4/3} - (\rho_\Omega - d_\Omega(x))^{4/3} \right].$$

- For the *normalized operator* Δ_∞^N, an analogous result holds true, with Φ_Ω replaced by $\Psi_\Omega(x) := \frac{1}{2}[\rho_\Omega^2 - (\rho_\Omega - d_\Omega(x))^2]$.
- In the *regular case* (C^1 solutions, C^2 domains) the result was previously obtained by Buttazzo-Kawohl 2011.
- *Proof:* we use viscosity methods + non-smooth analysis results (in particular, *a new estimate of d_Ω near singular points*).

Singular sets of d_Ω

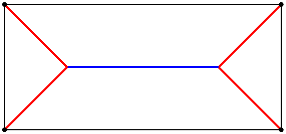
Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain.

$$M(\Omega) \subseteq \Sigma(\Omega) \subseteq C(\Omega) \subseteq \overline{\Sigma}(\Omega)\,.$$

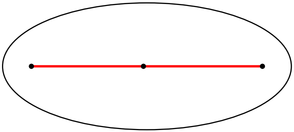
- *$M(\Omega)$:= the high ridge of Ω*
  is the set where d_Ω attains its maximum over $\overline{\Omega}$;
- *$\Sigma(\Omega)$:= the skeleton of Ω*
  is the set of points with multiple projections on $\partial\Omega$;
- *$C(\Omega)$:= the central set of Ω*
  is the set of the centers of all maximal balls contained into Ω;
- *$\overline{\Sigma}(\Omega)$:= the cut locus of Ω*
  is the closure of $\Sigma(\Omega)$ in $\overline{\Omega}$.

In general the inclusions are strict

- when $\Omega = R$ is a rectangle, one has $M(R) \subsetneq \Sigma(R) = C(R) \subsetneq \overline{\Sigma}(R)$;



- when $\Omega = E$ is an ellipse, one has $M(E) \subsetneq \Sigma(E) \subsetneq C(E) = \overline{\Sigma}(E)$;



- more pathological examples:

  $\Sigma(\Omega)$ is always C^2-rectifiable [Alberti 1994]
  $\overline{\Sigma}(\Omega)$ may have positive Lebesgue measure [Mantegazza-Mennucci 2003]
  $C(\Omega)$ may fail to be $\mathscr{H}^1$-rectifiable [Fremlin 1997]
  and may have Hausdorff dimension 2 [Bishop-Hakobyan 2008]

Which is the geometry of an open set Ω when $\overline{\Sigma}(\Omega) = M(\Omega)$?

Remark: If $\overline{\Sigma}(\Omega) = M(\Omega) =: S$, then

S is a closed set with empty interior and *positive reach*

Definition [Federer 1959]:

S has *positive reach* if, for every x in an open tubular neighborhood outside S, there is a unique minimizer of the distance function from x to S

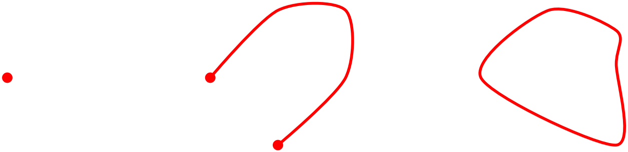
$\Leftrightarrow$ S is *proximally C^1*, namely $\exists r_S > 0 \ : \ d_S$ is C^1 on $\{0 < d_S(x) < r_S\}$

Similar definition for *proximally C^2* sets.

Which is the geometry of a closed set S with empty interior and positive reach?

$\Rightarrow$ The set Ω will be a tubular neighborhood of S of radius ρ_Ω.

Theorem (Characterization of proximally C^1 sets with empty interior in $\mathbb{R}^2$)
Let $S \subset \mathbb{R}^2$ be closed, with empty interior, proximally C^1, and connected.
Then S is either a singleton, or a 1-dimensional manifold of class $C^{1,1}$.



Proof: purely geometrical, hard to extend to higher dimensions...

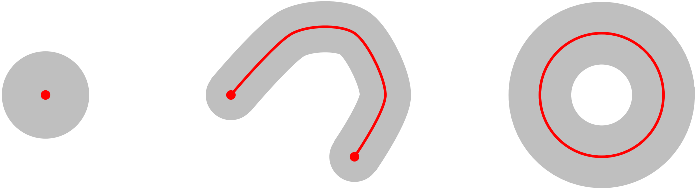
Theorem (Characterization of proximally C^2 sets with empty interior in $\mathbb{R}^2$)

Let $S \subset \mathbb{R}^2$ be closed, with empty interior, proximally C^2, and connected. Then S is either a singleton, or a 1-dimensional manifold of class C^2 without boundary.

Theorem (Characterization of planar domains with $M(\Omega) = \overline{\Sigma}(\Omega)$)

Let $\Omega \subset \mathbb{R}^2$ be an open bounded connected set with $M(\Omega) = \overline{\Sigma}(\Omega)$.
Then:

- Ω is either a disk or a parallel neighborhood of a 1-dim. $C^{1,1}$ manifold.
- If Ω is $C^2 \Rightarrow$ the case of manifold with boundary cannot occur.
- If Ω is also simply connected $\Rightarrow$ Ω is a disk.



Theorem (Extension to higher dimensions)

Let $\Omega \subset \mathbb{R}^n$ be an open bounded *convex* set of class C^2.

If $M(\Omega) = \overline{\Sigma}(\Omega)$, then Ω is a ball.

In the web case:

We now know for which domains a web solution to the Dirichlet pb. exists.

In the general (non-web) case:

- The geometry of characteristics is unknown.
- Even worse, we do not know if the gradient flow is well posed!
  (∇u is in $L^\infty_{loc}(\Omega)$, NOT in $\mathrm{Lip}_{loc}(\Omega)$.)

However:

To have local forward uniqueness for the gradient flow, it is enough that u is *locally semiconcave* [Cannarsa-Yu 2009], i.e. $\exists C \geq 0$ s.t.

$$u(x+h) + u(x-h) - 2u(x) \leq C|h|^2 \qquad \forall [x-h, x+h] \subset \Omega.$$

We need a *regularity result*!

Theorem (power-concavity of solutions)
Assume that Ω is convex, and let u be the unique viscosity solution to problem

$$(D) \qquad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u^{3/4}$ is concave in Ω.

- ▷ Counterpart of a well-known result for the *p-Laplacian* [Sakaguchi 1987]
- ▷ For the *normalized operator* Δ_∞^N, an analogous result holds true, with concavity exponent equal to $1/2$.

Proof:

We adapt the convex envelope method [Alvarez-Lasry-Lions 1997].

The function $w := -u^{3/4}$ solves

$$\begin{cases} -\Delta_\infty w - \frac{1}{w}\left[\frac{1}{3}|\nabla w|^4 + \left(\frac{3}{4}\right)^3\right] = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega\,. \end{cases}$$

We show that w^{**} is a supersolution to the same problem.

By applying a comparison principle, we get $w^{**} \geq w$.

Hence $w = w^{**}$, i.e. w is convex. □

Corollary (local semiconcavity and C^1-regularity of solutions)

Assume that Ω is convex, and let u be the unique viscosity solution to problem

$$(D) \qquad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then u is locally semiconcave and continuously differentiable in Ω.

- Same result for the *normalized operator* Δ_∞^N.
- The *optimal expected regularity* is of type $C^{1,\alpha}$.
  In the normalized case, we can prove that u is $C^{1,1} \Leftrightarrow M(\Omega) = \overline{\Sigma}(\Omega)$.

Assuming Ω convex, characteristics are now back at our disposal!

Heuristics - continued

$$P(x) := \frac{|\nabla u|^4}{4} + u\,, \quad \text{with } u \text{ solution to } (D)$$

- Along characteristics: $\frac{d}{dt}\big(P(\gamma(t))\big) = 0 \quad \Rightarrow \quad P(\gamma(t))$ is constant

- Assuming $u = 0$ and $|\nabla u| = c$ on $\partial\Omega \;\Rightarrow P$ is constant on Ω.

- If P is constant on $\Omega \;\Rightarrow u$ solves a first order HJ equation

  $\Rightarrow$ by uniqueness [Barles 1990]
  $u(x) = \Phi_\Omega(x) := c_0 \left[\rho_\Omega^{4/3} - (\rho_\Omega - d_\Omega(x))^{4/3}\right]$

  $\Rightarrow$ by the results in the web-case $\quad M(\Omega) = \overline{\Sigma}(\Omega)$.

Lemma 1 (*P*-function inequalities)
Assume Ω is convex. Then

$$\min_{\partial\Omega} \frac{|\nabla u|^4}{4} \le P(x) \le \max_{\overline{\Omega}} u \qquad \forall x \in \overline{\Omega}.$$

Proof:

The *supremal convolutions*

$$u^\varepsilon(x) = \sup_y \left\{ u(y) - \frac{|x-y|^2}{2\varepsilon} \right\}$$

are of class $C^{1,1}$ and are *sub-solutions* of the PDE

$\Rightarrow P_\varepsilon := \frac{|\nabla u^\varepsilon|^4}{4} + u^\varepsilon$ is increasing along the gradient flow of u^ε

$\Rightarrow$ in the limit as $\varepsilon \to 0$ we obtain the required inequalities. $\square$

Lemma 2 (matching of upper and lower bounds)

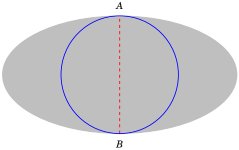
Assume Ω convex. If u satisfies the overdetermined condition $|\nabla u| = c$ on $\partial\Omega$, then

$$\frac{c^4}{4} = \min_{\partial\Omega} \frac{|\nabla u|^4}{4} = \max_{\overline{\Omega}} u\,.$$

Proof: *Key remark*: the web-function Φ_Ω is a *super-solution* to $-\Delta_\infty u = 1$

$\Longrightarrow \Phi_B \leq u \leq \Phi_\Omega$ on B = inner ball of radius ρ_Ω

$\Longrightarrow \Phi_B = u = \Phi_\Omega$ on $\gamma = [x, y]$, with $x \in M(\Omega)$, $y \in \partial\Omega$



$$\Longrightarrow u = c_0 \left[\rho_\Omega^{4/3} - (\rho_\Omega - d_\Omega(x))^{4/3}\right] \qquad \text{on } \gamma$$

$$\Longrightarrow \max_{\overline{\Omega}} u = u(x) = c_0 \rho_\Omega^{4/3} = \frac{|\nabla u(y)|^4}{4} = \min_{\partial\Omega} \frac{|\nabla u|^4}{4}.$$

□

Theorem (Serrin-type theorem for Δ_∞ and Δ_∞^N)

Assume that Ω is convex. Then each of the overdetermined problems

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = c & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta_\infty^N u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = c & \text{on } \partial\Omega \end{cases}$$

admits a solution $\iff M(\Omega) = \overline{\Sigma}(\Omega)$.

By the previous geometric results + convexity assumption:

- If $n = 2 \iff \Omega$ is a *stadium*.
- If $n = 2$ and Ω is $C^2 \iff \Omega$ is a *ball*.

Link between symmetry breaking and boundary regularity!

Open problems

- Prove Serrin-type theorem for Δ_∞ or Δ_∞^N without the convexity restriction.
- Characterize domains with $M(\Omega) = \overline{\Sigma}(\Omega)$ in higher dimensions.
- Study the regularity preserving properties of the parabolic flow governed by Δ_∞ or Δ_∞^N.

Many thanks for your attention

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