Equilibrium states for non hőlderian Random Dynamical Systems.

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Abstract

We study random dynamical systems which are assumed to have summable continuity module. Existence of a unique equilibrium state in the relativized variationnal principle is obtained. Moreover, in the particular case of small random perturbations, we obtain an estimation of the rate of convergence to equilibrium and strong stability properties.

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Introduction

A random dynamical system (RDS) describes the successive application of different transformations chosen at random (see [12], [13] for a background on RDS). Random perturbations appear as particular sorts of RDS ; transformations are chosen randomly in some neighborhood of a given transformation.
The concepts of equilibrium states and pressure arose from statistical mechanics and are fundamental for the "thermodynamic formalism" on compact spaces (in the sense of Bowen [7] and Ruelle [20]). These concepts were extended to random dynamical systems using the "relativized" variationnal principle of Ledrappier and Walters [16]. As in the deterministic case, existence and uniqueness of such an equilibrium state is related to existence and uniqueness of generalized eigenvalues, eigenfunctions and conformal measures for the random transfer operators $\mathcal{L}_\omega$ (i.e., positive numbers c_ω, functions h_ω and measures ν_ω verifying $\mathcal{L}_\omega h_\omega = c_\omega h_{S\omega}$, $\mathcal{L}_\omega^\star \nu_{S\omega} = c_\omega \nu_\omega$), see [10], [14], [4].
Many results have been obtained for expanding RDS with Hőlder derivative (see [2], [5], [10], [14]).

The present work is devoted to the non hőlderian case. We will be concerned with interactions having summable continuity module (see section 1 for precise definitions). This corresponds to dynamics whose derivative have summable continuity module (this will be denoted by $C^{1+\mathrm{sum}}$). For example, we may consider a piecewise C^1 dynamic of the interval, whose derivative has continuity module in $(1+|\log t|)^{-1-\alpha}$, $\alpha>0$ ([8]). In this situation, even in the deterministic case, exponential rates of convergence should not be expected (see [11]). We will only be concerned with symbolic dynamics on a finite state space but the reader should have in mind that this setting covers different cases. In appendix A, we explain how to code small perturbations of a C^1 Axiom A on a subshift of finite type.
We consider RDS with summable continuity module and some integrability conditions. We show existence and uniqueness of generalized eigenvalues, eigenfunctions, conformal measures and convergence of the compositions $\mathcal{L}_{S^n\omega}\circ\cdots\circ\mathcal{L}_\omega$ to a rank one operator.
In the particular case of small random perturbations, we also obtain a uniform bound for the rate of convergence on a dense subspace of $C(X)$, strong stability properties (theorem 1.4) and the differentiability of the pressure function $p(t)=\pi(tF)$, $t\in\mathbb{R}$ (proposition 1.5).
In order to prove these results, Birkoff cones techniques are used. These techniques were introduced by Ferrero and Schmitt [10] and more recently extensively applied by Liverani [17]. They were also employed in a random framework in [2]. Here, we use a sequence of convex cones rather than a universal one ; this allows us to capture non exponential rate of convergence. This modification of the Hilbert metric technique was introduced in [11].

Section 1 contains precise definitions, statement of results and recalls facts on Birkoff cones. Generalized eigenfunctions, eigenvalues and conformal measures are constructed in section 2. Section 3 concerns small perturbations, stability properties and the differentiability of the pressure.

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1 Setting, statement of results and basic definitions and results on cones.

Let (X, σ) be an aperiodic subshift of finite type on a finite alphabet A i.e., let B be a finite aperiodic (by aperiodic, we mean that $B^M > 0$ for some integer M) matrix with $b_{i,j} \in \{0, 1\}$, $i, j \in A$, X is the set of B-admissible sequences of elements of A :

$$X = \{(x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}} \ / \ b_{x_i,x_{i+1}} = 1, \quad i \in \mathbb{N}\}$$

This space X is endowed with the metric : $d(x, y) = \rho^n$, whenever $x_j = y_j$ for $j = 0, \cdots, n-1$ and $x_n \neq y_n$ with $0 < \rho < 1$. We shall write $x \overset{n}{\sim} y$ whenever $d(x, y) \leq \rho^n$.
For any $f \in C(X) = C(X, \mathbb{R})$, the continuity module of f is the sequence $v(f) = (v_n(f))_{n\geq 0}$ with :

$$v_n(f) = \sup_{x \overset{n}{\sim} y} \mid f(x) - f(y) \mid \quad v_0(f) = \sup f - \inf f \ .$$

Given a weight function $\phi \in C(X)$, we consider the transfer operator associated to ϕ and acting on $C(X)$:

$$\mathcal{L}_\phi f(x) = \sum_{\alpha\in A \ / \ \alpha x\in X} e^{\phi(\alpha x)} f(\alpha x) \text{ for } \ f \in C(X)$$

where ax is the concatenation of a and x, for $a \in A^n$ and $x \in X$; its dual $\mathcal{L}^\star_\phi$ is defined by :

$$\int hd(\mathcal{L}^\star_\phi \mu) = \int \mathcal{L}_\phi h d\mu \ \text{ for } h \in C(X) \text{ and } \mu \text{ a Borel measure on } X.$$

Transfer operators acting on spaces of functions having a given continuity module will be studied.
To be precise, given any positive and decreasing toward zero sequence $\chi(n)$, the metric $d'(x, y) = \chi(n)$ whenever $d(x, y) = \rho^n$ and the space B_χ of functions which are Lipschitz with respect to d' may be considered. That is :

$$B_\chi = \{f \in C(X) \ / \ \exists K \geq 0 \ / \ \forall n \geq 1, \ v_n(f) \leq K\chi(n)\}.$$

For any $f \in B_\chi$, set :

$$K_\chi(f) = \inf\{K \ / \ \forall n \geq 0, \ v_n(f) \leq K\chi(n)\} \text{ and}$$

$$\| f \|_{B_\chi} = \max(\| f \|_\infty, K_\chi(f)).$$

Clearly, $\| \ \|_{B_\chi}$ is a norm on B_χ. Endowed with this norm, B_χ is a Banach space which is dense in $C(X)$ by the Stone-Weierstrass theorem.
Let also $\mathcal{M}(X)$ be the space of Borel measures on X and $\mathcal{P}(X)$ the subspace of probabilities.

In what follows, we consider random weights whose continuity module are comparable to a reference sequence $\chi(n)$.

1.1 Statement of results.

Let $\chi(n)$ be a positive and decreasing toward zero sequence, $(\Omega, \mathcal{F}, P, S)$ an abstract invertible ergodic dynamical system and $F : \Omega \longrightarrow C(X)$ a measurable application. For $\omega \in \Omega$ and $n \in \mathbb{N}$, we write $v_n(\omega)$ instead of $v_n(F_\omega)$ and $\chi(0) = V$. Let $D > 1$ be fixed.

We state the following hypothesis on F (recall that M is the smallest integer such that $B^M > 0$) :

- $(H1)$ $\int \|F_\omega\|_\infty dP < \infty$.
- $(H2)$ $\sup_{p\in\mathbb{N}} \frac{\sum_{i=p+1}^{\infty} v_i(S^{-i}\omega)}{\chi(p)} = C(\omega) \in L^1(\Omega)$.
- $(H3)$ $\exp\left(2V[\frac{(D^2-1)}{D}C(\omega) + C(S^{-M}\omega)] + 2\sum_{k=1}^{M-1} v_0(S^{(k-M)}\omega)\right) \in L^2(\Omega)$ if M is greater than 1.
  $\exp[2V(1+D)C(\omega)] \in L^2(\Omega)$ if $M = 1$.

Remark 1.1 *Assumption $(H1)$ will be used to prove that $\log c_\omega$ is in $L^1(\Omega)$ where c_ω, $\omega \in \Omega$ are generalized eigenvalues for $\mathcal{L}_\omega$; assumption $(H3)$ will be used in lemma 2.5 to prove that $(\mathcal{L}_{n,S^{-n}\omega}\mathbf{1})_{n\in\mathbb{N}}$.*

Since $C(\omega)$ belongs to $L^1(\Omega)$, $\sum_{i=1}^{\infty} v_i(S^{-i}\omega)$ is P-integrable so, for almost every ω, $\sum_{i=1}^{\infty} v_{p+i}(S^{-i}\omega)$ tends to zero as p tends to infinity. So, we may consider the sequence $\eta^\omega(p) = \sum_{i=1}^{\infty} v_{p+i}(S^{-i}\omega)$ and the associated space B^ω. Let us write $\mathcal{L}_{n,\omega} = \mathcal{L}_{F_{S^{n-1}\omega}} \circ \cdots \circ \mathcal{L}_{F_\omega}$ $\omega \in \Omega$ the compositions of transfer operators. It may easily be verified that $\mathcal{L}_{n,\omega}$ maps B^ω into $B^{S^n\omega}$ and that $\mathcal{L}_{n,\omega}$ verifies the cocycle relation

$$\mathcal{L}_{n+m,\omega} = \mathcal{L}_{m,S^n\omega} \circ \mathcal{L}_{n,\omega}.$$

Let us write $\mathcal{B} = \bigcup_{\omega\in\Omega} B^\omega$. We will obtain the following results.

Theorem 1.2 *Under hypothesis (H1), (H2) and (H3), there exists unique measurable maps :*

$$\begin{array}{llll} H: & \Omega \longrightarrow \mathcal{B} & C: \ \Omega \longrightarrow \mathbb{R}^+/\{0\} & \nu: \ \Omega \longrightarrow \mathcal{P}(X) \\ & \omega \longrightarrow h_\omega \in B^\omega & \quad\ \omega \longrightarrow c_\omega & \quad\ \ \omega \longrightarrow \nu_\omega \end{array}$$

such that :

1. $\forall \omega \in \Omega$, $\mathcal{L}_{F\omega} h_\omega = c_\omega h_{S\omega}$ *and* $\mathcal{L}^\star_{F\omega} \nu_{S\omega} = c_\omega \nu_\omega$,

2. $\nu_\omega(h_\omega) = 1$, $\log c_\omega$ *and* $\| \log h_\omega \|$ *are integrable,*

3.
$$\forall f \in C(X), \ \left\| \frac{\mathcal{L}_{n,\omega} f}{c_{S^{n-1}\omega} \times \ldots \times c_\omega} - h_{S^n\omega} \int f d\nu_\omega \right\|_\infty \overset{n\to\infty}{\longrightarrow} 0. \tag{1.1}$$

4. *The probability* μ *defined on* $\Omega \times X$ *by* $\mu(G) = \int_\Omega \int_X G d\mu_\omega dP$ *for* G *in* $L(\Omega \times X)$ *is the unique equilibrium state for* F *in the relativized variationnal principle of Ledrappier and Walters ([17], [16], see also [4]).*

Remark 1.3 *Recall that for* G *in* $L(\Omega \times X)$, *the pressure may be defined by using the following variationnal principle (see [4] for details) :*

$$\pi(F) := \sup_{\mu \in \mathcal{M}(P,\sigma)} \{h_\mu(\sigma) + \int F d\mu\},$$

where $h_\mu(\sigma)$ *is the relativized entropy*[1] *and* $\mathcal{M}(P,\sigma)$ *is the space of families of Borel probabilities* μ_ω *such that for any* f *in* $C(X)$ *and any* ω *in* Ω, $\int_X f \circ \sigma d\mu_\omega = \int_X f d\mu_{S\omega}$ $\forall \omega \in \Omega$.
Using standard arguments (see for example, [15], [14] or [4]), existence and uniqueness[2] *of* $(h_\omega, c_\omega, \nu_\omega)_{\omega\in\Omega}$ *given by theorem 1.2 and integrability of* $\| \log h_\omega \|$ *and* $\log c_\omega$ *imply that* $\mu = h\nu$ *is the unique equilibrium state for* F *and* $\pi(F) = \int \log c_\omega dP(\omega)$.

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[1]Let $p : \Omega \times X \longrightarrow \Omega$ be the canonical projection and Θ the product map : $\Theta(\omega, x) = (S\omega, \sigma x)$, then the relativized entropy $h_\mu(\sigma)$ is the entropy of Θ relative to S : $h_\mu(\sigma) = h_\mu(\Theta | p^{-1}\mathcal{F})$.

[2]It may be easily seen that under properties *1.* and *2.* of theorem 1.2, convergence (1.1) implies uniqueness of $(h_\omega, c_\omega, \nu_\omega)_{\omega\in\Omega}$.

1.2 Random perturbations.

Now we will emphasize the improvements obtained in the particular case of Random Perturbations.

Let $\phi \in C(X)$ be a given weight function which is assumed to have summable continuity module : $\sum_{n\geq 1} v_n(\phi) = V < \infty$. It will be randomly perturbed by functions having close continuity module.

We will especially consider the following two spaces :

$$B = B_\chi \text{ with } \chi(n) = \sum_{p=n+1}^{\infty} v_p(\phi) \text{ and } E = B_{v(\phi)}.$$

For small ε, we consider an ε-neighborhood B_ε of ϕ in the space E. We assume that $(\Omega_\varepsilon, \mathcal{F}_\varepsilon, P_\varepsilon, S_\varepsilon)$ - abstract invertible ergodic dynamical systems - and $F_\varepsilon : \Omega_\varepsilon \longrightarrow B_\varepsilon$ - measurable applications - are given. Let us write $\mathcal{L}_{n,\omega,\varepsilon} = \mathcal{L}_{F_\varepsilon S_\varepsilon^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{F_\varepsilon \omega}$ $\omega \in \Omega_\varepsilon$.

We have, $v_n(\omega) \leq v_n(\phi)(1+\varepsilon)$ and $\|F_\varepsilon\omega\|_\infty \leq \|\phi\|_\infty + \varepsilon$. So $(H1)$, $(H2)$ and $(H3)$ are verified and we have theorem 1.2. It will be clear from the proofs that the rate of convergence in (1.1) is bounded by $L\gamma^{l(n)}\|f\|_B$, for $f \in B$, where $(l(n))_{n\in\mathbb{N}}$ is a strictly increasing sequence of integers, $L > 0$ and $0 < \gamma < 1$, L and γ depend neither on ω, nor on ε.

Let (h_0, c, ν) be the triple associated to ϕ, i.e. $h_0\nu$ is the unique equilibrium state associated to ϕ and $\mathcal{L}_\phi h_0 = ch_0$ (see [11]). It is then natural to wonder about the stability of h_ω, c_ω, ν_ω.

Theorem 1.4 *We have the following strong stability results.*

$$\lim_{\varepsilon\to 0} \sup_{\omega\in\Omega_\varepsilon} \|h_0 - h_\omega\|_\infty = 0,$$

$$\lim_{\varepsilon\to 0} \sup_{\omega\in\Omega_\varepsilon} \frac{c}{c_\omega} = 1,$$

$$\forall f \in C(X),\ \lim_{\varepsilon\to 0} \sup_{\omega\in\Omega_\varepsilon} |\nu(f) - \nu_\omega(f)| = 0.$$

Finally, we are concerned with the differentiability of the pressure which is useful to obtain large deviations theorems.

Let ε be fixed, we write F instead of F_ε and we consider the function $p(t) = \pi(tF)$, $t \in \mathbb{R}$. Using theorem 1.4, we obtain the following result.

Proposition 1.5 *The function p is differentiable with derivative $p'(t) = t\mu_{tF}(F)$.*

Theorem 1.2 is proved by using Birkoff cones techniques in a very similar way as in [11] (deterministic case). We refer the reader to this article for the proofs of lemmas that go verbatim along the deterministic lines.

1.3 Basic definitions and results on cones.

Let Λ be a closed cone of positive functions and $\dot{\Lambda}$ its projective space :

$$\dot{\Lambda}=\left\{f\in\Lambda\ /\ \int fdm=1\right\}$$

for a finite measure m on X whose support is X.
For any f and g in $\dot{\Lambda}$, there exists a largest $\lambda(f,g)$ (maybe zero) and a smallest $\mu(f,g)$ (maybe, $\mu(f,g)=\infty$) such that :

$$\lambda(f,g)f\leq g\leq\mu(f,g)f \text{ and } g-\lambda(f,g)f\in\dot{\Lambda}\ \ \mu(f,g)f-g\in\dot{\Lambda}\ .$$

The Hilbert pseudo-metric θ_Λ on $\dot{\Lambda}$ is defined by :

$$\theta_\Lambda(g,f)=\log\frac{\mu(f,g)}{\lambda(f,g)}.$$

The importance of this metric is due to the following three propositions from Birkoff which we state in the particular case of cones of continuous functions.

Proposition 1.6 *[3] Let θ_+ denote the projective metric on the cone $C^+(X)$ and $f\in C^+(X)$. The set $\pi_f=\{g\in\Lambda\ /\ \theta_+(g,f)<\infty\}$ is a complete metric space for the metric θ_+.*

Remark 1.7 *It will be useful to remark that for $f\in C^+(X)$, $\theta_+(f,\mathbf{1})=\log\dfrac{\sup f}{\inf f}$.*

On the other hand, let P be a positive operator on $C^+(X)$ and Λ, Λ' two cones such that $P\Lambda\subset\Lambda'$. We set :

$$diam_{\theta_{\Lambda'}}(P\Lambda)=\sup_{f,g\in\Lambda}\theta_{\Lambda'}(Pf,Pg).$$

We have the following fundamental result of contraction.

Proposition 1.8 *[3] Let P be a positive operator on $C^+(X)$ and Λ, Λ' two cones such that $P\Lambda\subset\Lambda'$ then for any f and $g\in\Lambda$, we have :*

$$\theta_{\Lambda'}(Pf,Pg)\leq\tanh\left[\frac{1}{4}diam_{\theta_{\Lambda'}}(P\Lambda)\right]\theta_\Lambda(f,g).$$

Finally, the following proposition gives a comparison between θ_Λ and $\|\ \|_\infty$.

Proposition 1.9 *[3] For any f and g in Λ such that $\int fdm=1$ and $\int gdm=1$, we have :*

$$\|f-g\|_\infty\leq(e^{\theta_\Lambda(f,g)}-1)\|g\|_\infty.$$

2 Proof of theorem 1.2.

In order to prove theorem 1.2, we construct h_ω as the projective limit of $\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}$, this also gives existence of c_ω. Then $\nu_\omega(f)$ is obtained as the limit of μ_l and λ_l where μ_l and λ_l are the real numbers used to compute the projective distance between $\mathcal{L}_{m(l),\omega}h_\omega$ and $\mathcal{L}_{m(l),\omega}f$. The method is adapted from [11], we will emphasize the arguments due to the random situation and refer the reader to [11] for the other computations.

For any $D > 1$, we construct inductively a sequence of metrics $(\eta_l(p))_{l\in\mathbb{N}}$ and of positive integers $(n_l)_{l\in\mathbb{N}^\star}$:

- $\eta_0(p) = \chi(p)$
- $n_1 = \inf\{n \geq M \ / \ D\eta_0(n) \leq V\}$ where $V = \sup \chi(n) = \chi(0)$.
- $\eta_1(p) = D(\eta_0(p) + \eta_0(n_1 + p))$
- $n_2 = \inf\{n \geq M \ / \ D\eta_1(n) \leq V\}$
- $\eta_2(p) = D(\eta_0(p) + \eta_0(n_2 + p))$

  $\vdots$

- $\eta_l(p) = D(\eta_0(p) + \eta_{l-1}(n_l + p))$ with $n_l = \inf\{n \geq M \ / \ D\eta_{l-1}(n) \leq V\}$.

Remark 2.1 *By construction, the sequences η_l verify for any $l \geq 1$,*

$$\eta_l(0) = D(\eta_0(0) + \eta_{l-1}(n_l)) \leq V(D+1).$$

Remark 2.2 *For example, if the sequence $\chi(n)$ is geometric, we can take $n_l = M, \ \forall l \in \mathbb{N}$.*
If the sequence $\chi(n)$ is in $\frac{1}{n}$, we obtain n_l in $\log l$.

2.1 Construction of h_ω and c_ω.

For almost every $\omega \in \Omega$, let :

$$\Lambda_0^\omega = \left\{ g \in C^+(X) \ / \ g(x) \leq \exp(\eta_0^\omega(p))g(y) \text{ if } x \overset{p}{\sim} y \ p \geq 1 \right\}$$

with $\eta_0^\omega(p) = \sum_{i=1}^{\infty} v_{p+i}(S^{-i}\omega)$. It may be easily verified that $\Lambda_0^\omega \subset B^\omega$ and that $\mathcal{L}_{n,\omega}\Lambda_0^\omega \subset \Lambda_0^{S^n\omega}$ but the finiteness of the diameter is not assured by our

assumptions[3].
Moreover, for $f \in \Lambda_0^\omega$, $p \in \mathbb{N}$, $x \overset{p}{\sim} y$ and $\omega \in \Omega$ we have :

$$\mathcal{L}_{n_1,\omega} f(x) \ \leq \ \mathcal{L}_{n_1,\omega} f(y) \exp[\eta_0^\omega(n_1 + p) + \eta_0^{S^{n_1}\omega}(p)].$$

We are led to consider the following cones :

$$\Lambda_1^{S^{n_1}\omega} = \left\{ g \in C^+(X) \ / \ g(x) \leq \exp(\eta_1^{S^{n_1}\omega}(p)) g(y) \text{ if } x \overset{p}{\sim} y \ p \geq 0 \right\}$$

with $\eta_1^{S^{n_1}\omega}(p) = D(\eta_0^{S^{n_1}\omega}(p) + \eta_0^\omega(n_1 + p))$ and $D > 1$. We have $\mathcal{L}_{n_1,\omega}\Lambda_0^\omega \subset \Lambda_1^{S^{n_1}\omega}$. We need to estimate $\theta_{\Lambda_1^{S^{n_1}\omega}}(\mathcal{L}_{n_1,\omega} f, \mathcal{L}_{n_1,\omega} g)$ for f and g in Λ_0^ω. The following lemma is proved by using that by construction of n_1 and $(H2)$, $\eta_1^{S^{n_1}\omega}(0) \leq VC(S^{n_1}\omega)(D+1)$ and remarking that if $f \in \Lambda_0^\omega$, since $n_1 \geq M$,

$$\begin{aligned} \frac{\sup \mathcal{L}_{n_1,\omega} f}{\inf \mathcal{L}_{n_1,\omega} f} \ &\leq \ \exp\left(\eta_0^{S^{(n_1-M)}\omega}(0)\right) \\ &\quad (\#A)^M \exp(\sum_{k=1}^{M-1} v_0(S^{(n_1-M+k)}\omega)) := L(S^{n_1}\omega). \qquad (2.2) \end{aligned}$$

Lemma 2.3 *For any $\omega \in \Omega$, we have*

$$\begin{aligned} diam\theta_{\Lambda_1^{S^{n_1}\omega}} \mathcal{L}_{n_1,\omega}\Lambda_0^\omega \ &\leq \ 2\log\frac{D+1}{D-1} + 2V[C(S^{n_1}\omega)\frac{(D^2-1)}{D} + C(S^{(n_1-M)}\omega)] \\ &\quad +2M\log\#A + \sum_{k=1}^{M-1} v_0(S^{(n_1-M+k)}\omega) := M(S^{n_1}\omega). \end{aligned}$$

We then construct inductively a sequence of positive cones.

Proposition 2.4 *There exists a family Λ_l^ω of sub cones in $C^+(X)$ such that :*

- $\mathcal{L}_{n_l,\omega}\Lambda_{l-1}^\omega \subset \Lambda_l^{S^{n_l}\omega}$,
- $\forall f,\ g \ \in \Lambda_{l-1}^\omega$ *we have* $\theta_{\Lambda_{l-1}^\omega}(\mathcal{L}_{n_l,\omega} f, \mathcal{L}_{n_l,\omega} g) \leq M(S^{n_l}\omega)$.

Now, for any $n \in \mathbb{N}$, there is a unique integer $l(n)$ such that :

$$n_1 + \cdots + n_{l(n)} \leq n < n_1 + \cdots + n_{l(n)+1}.$$

Set $\gamma(\omega) = (1 - e^{-M(\omega)}) \geq \tanh\frac{M(\omega)}{4}$. Using proposition 1.8, for f and g in Λ_0^ω and $\omega \in \Omega$, we obtain the following inequality :

$$\theta_{\Lambda_l^{S^{n_1+\cdots n_l}\omega}}(\mathcal{L}_{n_l+\ldots+n_1,\omega} f, \mathcal{L}_{n_l+\ldots+n_1,\omega} g) \leq \prod_{i=2}^{l} \gamma(S^{m(i)}\omega) \ M(S^{n_1}\omega) := P_l(\omega).$$

[3]In fact, if $\chi(n)$ is a geometric sequence then $\mathrm{diam}_{\Lambda_0^\omega}\mathcal{L}_{n,\omega}\Lambda_0^\omega$ is finite but there is a large class of sequences for which we know that this diameter is not finite (this is the case for $\chi(n) = \frac{1}{n^\alpha}$ see [11]).

Now, $\Lambda_{l-1}^\omega \subset C^+(X)$, thus using proposition 1.8, we get $\theta_+ \leq \theta_{\Lambda_{l-1}^\omega}$ and for any $f, g \in \Lambda_0^\omega$, we have

$$\theta_+(\mathcal{L}_{n,\omega} f, \mathcal{L}_{n,\omega} g) \leq P_l(\omega). \tag{2.3}$$

Clearly, at this step, we have to prove that $P_l(\omega)$ goes to zero P-a.e. as l goes to infinity. This follows from hypothesis $(H3)$[4].

Lemma 2.5 *$P_l(\omega)$ converges a.e to zero as l goes to infinity.*

Proof : We just have to prove that $\prod_{i=2}^l \gamma(S^{m(i)}\omega)$ converges a.e. to zero. Let

$$\mathbf{A} = \bigcup_{C\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \bigcup_{l\geq n} \left\{ \omega \in \Omega \ / \ \sum_{i=2}^{l+1} -\log \gamma(S^{m(i)}\omega) < C \right\}$$

Following the strategy of the Borel-Cantelli lemma, to show that $P(\mathbf{A}) = 0$, it is enough to obtain :

$$\sum_{l\geq 1} P \left\{ \omega \in \Omega \ / \ \sum_{i=2}^{l+1} -\log \gamma(S^{m(i)}\omega) < C \right\} < \infty \text{ for any } C.$$

But,

$$P \left\{ \sum_{i=2}^{l+1} -\log \gamma(S^{m(i)}\omega) < C \right\} \leq (l-1) P \left\{ -\log \gamma(\omega) < \frac{C}{l-1} \right\},$$

now, $\sum_n nP\{-\log\gamma(\omega) < \frac{C}{n}\} < \infty$ if and only if $(-\log \gamma(\omega))^{-1} \in L^2(\Omega)$. Hypothesis $(H3)$ implies that $e^{M(\omega)} \in L^2$ and this gives the result since $\gamma(\omega) = (1 - e^{-M(\omega)})$. We used the fact that for a measurable real function f, $\int_\Omega f^2 dP = \int_{[0,\infty[} tP(f > t)dt$ and thus $f \in L^2(\Omega)$ if and only if $\sum_{n>0} nP(f > n) < \infty$. $\Box$

Now, we want to obtain h_ω as the θ_+-limit of $\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}$.
Let $p > n$, we write $n = n_1 + \cdots + n_l + r = m(l) + r$ and $p = n_1 + \cdots + n_l + \cdots + n_k + s = m(k) + s$, then we have :

$$\begin{aligned} &\theta_+(\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}, \mathcal{L}_{p,S^{-p}\omega}\mathbf{1}) = \\ &\theta_+ \left(\mathcal{L}_{m(l),S^{-m(l)}\omega} \circ \mathcal{L}_{r,S^{-n}\omega}\mathbf{1}, \mathcal{L}_{m(l),S^{-m(l)}\omega} \circ \mathcal{L}_{p-m(l),S^{p}\omega}\mathbf{1} \right). \end{aligned} \tag{2.4}$$

[4]If the sequence $\chi(n)$ is geometric, then $n_l = 1$ for any l thus, $P_l(\omega) = \prod_{i=2}^l \gamma(S^i\omega)\ M(S\omega)$; under the assumption that $\log \gamma \in L^1(\Omega)$, the ergodic theorem may be used to prove that $\prod_{i=2}^l \gamma(S^i\omega)$ goes to zero exponentially fast (see [10]).

Thus, using (2.3) and the fact that $\mathcal{L}_{r,S^{-n}\omega}\mathbf{1}$ and $\mathcal{L}_{n_l+\cdots+n_k+s,S^{-m(l)}\omega}\mathbf{1}$ belong to $\Lambda_0^{S^{-m(l)}\omega}$, we obtain

$$\theta_{C^+(X)}(\mathcal{L}_{n,S^{-n}\omega}\mathbf{1},\mathcal{L}_{p,S^{-p}\omega}\mathbf{1}) \leq P_l(S^{-m(l)}\omega).$$

But, since $P_l(\omega) \longrightarrow 0$ a.e. and is decreasing, it goes to zero in L^1. Besides, P is S-invariant, so $P_l(S^{-m(l)}\omega) \longrightarrow 0$ in L^1, thus there exists a subsequence $\rho(l)$ such that $P_{\rho(l)}(S^{-m(\rho(l))}\omega) \longrightarrow 0$ a.e. Now, for any $n \in \mathbb{N}$, there is a unique integer $k(n)$ such that :

$$n_1 + \cdots + n_{\rho(k(n))} \leq n < n_1 + \cdots + n_{\rho(k(n)+1)}.$$

Using the subsequence $m(\rho(k(n)))$ in (2.4) instead of $m(l(n))$, we obtain :

$$\theta_{C^+(X)}(\mathcal{L}_{n,S^{-n}\omega}\mathbf{1},\mathcal{L}_{p,S^{-p}\omega}\mathbf{1}) \leq P_{\rho(l)}(S^{-m(\rho(l))}\omega) \xrightarrow{\text{a.e.}} 0 \text{ as } l \text{ goes to infinity.}$$

Moreover, using (2.2), we get for any $n \in \mathbb{N}$ and $x, y \in X$, $n \geq M$,

$$L(\omega)^{-1}\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}(y) \leq \mathcal{L}_{n,S^{-n}\omega}\mathbf{1}(x) \leq L(\omega)\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}(y), \tag{2.5}$$

thus we obtain $\theta_{C^+(X)}(\mathcal{L}_{n,S^{-n}\omega}\mathbf{1},\mathbf{1}) < \infty$.
So, using proposition 1.6, we may consider h_ω the θ_+-limit of $\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}$ which verifies $\mathcal{L}_\omega h_\omega = c_\omega h_{S\omega}$.
Because of proposition 1.9, we get that h_ω is also the $\| \|_\infty$-limit of the sequence $\left(\frac{\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}}{\int \mathcal{L}_{n,S^{-n}\omega}\mathbf{1}dm}\right)_{n\in\mathbb{N}}$.
This implies that h_ω is in Λ_l^ω for any l ; also using (2.5),we get :

$$L(\omega)^{-2} \leq \inf h_\omega \leq \sup h_\omega \leq L(\omega)^2, \tag{2.6}$$

and thus, $\|\log h_\omega\|$ is integrable using $(H3)$.

We may now construct the measures ν_ω.

2.2 construction of the measures ν_ω.

For $f \in \Lambda_0^\omega$, let λ_l and μ_l be the real numbers used to define the projective $\Lambda_l^{S^{m(l)}\omega}$-distance between $\mathcal{L}_{m(l),\omega}f$ and $c_{m(l),\omega}h_{S^{m(l)}\omega}$, we get :

$$(\mu_{l+1} - \lambda_{l+1}) \leq (\mu_1 - \lambda_1)\prod_{i=2}^{l+1}(1 - e^{-M(S^{m(i)}\omega});$$

using lemma 2.5, this product goes to zero and we may consider $\lambda_{1,(\omega)} \leq \nu_\omega(f) = \lim \lambda_{l,(\omega)} = \lim \mu_{l(,\omega)} \leq \mu_{1,(\omega)}$.
We have :

$$\left\| \frac{\mathcal{L}_{m(l),\omega} f}{c_{m(l),\omega}} - h_{S^{m(l)}\omega} \nu_\omega(f) \right\|_\infty \leq (\mu_1 - \lambda_1) \prod_{i=2}^{l+1} (1 - e^{-M(S^{m(i)}\omega}) \| h_{S^{m(l)}\omega} \|_\infty.$$

We get the following estimates :

$$\begin{aligned} \mu_{1,(\omega)} &\leq \frac{D+1}{D-1} e^{\frac{V}{D}(D^2-1)C(S^{n_1}\omega)} L(S^{n_1}\omega) \|f\|_\infty \\ &:= K(\omega)\|f\|_\infty \\ \lambda_{1,(\omega)} &\geq K(\omega) \inf f \\ \frac{\|\mathcal{L}_{r,\omega}\mathbf{1}\|}{c_{r,\omega}} &\leq \frac{\sup h_\omega}{\inf h_\omega} \leq L(\omega)^4. \end{aligned}$$

To conclude, set

- $\overline{h}_\omega = h_\omega \nu_\omega(\mathbf{1})$
- $\overline{c}_\omega = \frac{c_\omega \nu_\omega(\mathbf{1})}{\nu_{S\omega}(\mathbf{1})}$
- $\overline{\nu}_\omega = \frac{\nu_\omega}{\nu_\omega(\mathbf{1})}$,

then, for $n = m(l) + r$,

$$\begin{aligned} \left\| \frac{\mathcal{L}_{n,\omega} f}{\overline{c}_{n,\omega}} - \overline{h}_{S^n\omega} \overline{\nu}_\omega(f) \right\|_\infty &\leq \frac{\|\mathcal{L}_{r,S^{m(l)}\omega}\mathbf{1}\|}{c_{r,S^{m(l)}\omega}} \left\| \frac{\mathcal{L}_{m(l),\omega} f}{\overline{c}_{m(l),\omega}} - \overline{h}_{S^{m(l)}\omega} \overline{\nu}_\omega(f) \right\|_\infty \\ &\leq \frac{\nu_{S^{m(l)}\omega}(\mathbf{1})}{\nu_\omega(\mathbf{1})} \prod_{i=2}^{l+1} (1 - e^{-M(S^{m(i)}\omega}) L(S^{m(l)}\omega)^6 \|f\|_\infty K(\omega) \\ &\leq \prod_{i=2}^{l+1} (1 - e^{-M(S^{m(i)}\omega}) L(S^{m(l)}\omega)^6 \|f\|_\infty K(S^{m(l)}\omega) \frac{1}{K(\omega)}. \end{aligned} \tag{2.7}$$

Now, we have that

$$\prod_{i=2}^{\rho(l)+1} (1 - e^{-M(S^{m(i)}\omega})) L(S^{m(\rho(l))}\omega)^6 K(S^{m(\rho(l))}\omega) \overset{l\to\infty}{\longrightarrow} 0 \text{ in } L^1,$$

thus we may find a subsequence $\rho'(l)$ such that it goes to zero a.e. But inequality (2.7) remains true for any subsequence $(k_l)_{l\in\mathbb{N}}$, thus we have :

$$\| \frac{\mathcal{L}_{n,\omega} f}{\overline{c}_{n,\omega}} - \overline{h}_{S^n\omega} \overline{\nu}_\omega(f) \|_\infty \overset{n\to\infty}{\longrightarrow} 0 \text{ for any } f \in \Lambda_0^\omega.$$

Finally, for any positive function $f \in B^\omega$, $f + \|f\|_{B^\omega} \in \Lambda_0^\omega$ and we get the convergence (1.1) for $f \in B^\omega$, thus for $f \in C(X)$ by density of B^ω in $C(X)$ (remark that $c_\omega = \nu_{S\omega}(\mathcal{L}_\omega \mathbf{1})$ and integrability of $\log c_\omega$ directly follows from $(H1)$). $\square$

3 Random Perturbations and Stability properties.

In this section, we use the setting and definitions of section 1.2 and prove theorem 1.4. The transfer operator associated to ϕ is written $\mathcal{L}$ and (h_0, c, ν) is the triple associated to. Let us recall that :

$$h = \lim_{n\to\infty} \frac{\mathcal{L}^n \mathbf{1}}{c^n} \qquad h_\omega = \lim_{n\to\infty} \frac{\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}}{c_{n,S^{-n}\omega}} \tag{3.8}$$

$$c = \int \mathcal{L}\mathbf{1} d\nu \qquad c_\omega = \int \mathcal{L}_\omega \mathbf{1} d\nu_{S\omega}. \tag{3.9}$$

We also have for any $n \in \mathbb{N}$, $\omega \in \Omega$, $x \in X$,

$$e^{-n\varepsilon} \le \frac{\mathcal{L}_{n,\omega}\mathbf{1}(x)}{\mathcal{L}^n \mathbf{1}(x)} \le e^{n\varepsilon}. \tag{3.10}$$

The stability result on c directly follows :

$$e^{-\varepsilon} \int\int \mathcal{L}_\omega \mathbf{1}\ d\nu d\nu_{S\omega} \le \int \mathcal{L}\mathbf{1} d\nu = \int\int \mathcal{L}\mathbf{1} d\nu d\nu_{S\omega} \le e^{\varepsilon} \int\int \mathcal{L}_\omega \mathbf{1}\ d\nu d\nu_{S\omega}$$

thus : $e^{-\varepsilon} c_\omega \le c \le e^{\varepsilon} c_\omega$. $\square$

3.1 Stability of h_ω.

In the particular case of random perturbations, we have : $C(\omega) \le 1 + \varepsilon$, $v_0(\omega) \le (\varepsilon + 1) v_0(\phi)$, thus $M(\omega)$ is bounded by some $M > 0$, let $\gamma = \tanh \frac{M}{4}$, we have that $P_l(\omega) \le \gamma^l$, $0 < \gamma < 1$. So, we get for any n and ω :

$$\begin{aligned} \left\| \frac{\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}}{c_{n,S^{-n}\omega}} - h_\omega \right\|_\infty &\le Cte\gamma^{l(n)-1}, \\ \left\| \frac{\mathcal{L}^n \mathbf{1}}{c^n} - h_0 \right\|_\infty &\le Cte\gamma^{l(n)-1} \end{aligned}$$

where the constant depends neither on ω, nor on ε. Moreover, we have

$$\begin{aligned} \left\| \frac{\mathcal{L}^n \mathbf{1}}{c^n} - \frac{\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}}{c_{n,S^{-n}}} \right\|_\infty &\le \left\| \frac{\mathcal{L}^n \mathbf{1}}{c^n} \right\|_\infty \left\| 1 - \frac{\mathcal{L}_{n,S^{-n}\omega}\mathbf{1}}{\mathcal{L}^n \mathbf{1}} \frac{c^n}{c_{n,S^{-n}\omega}} \right\|_\infty \\ &\le e^V (e^{2n\varepsilon} - 1). \end{aligned}$$

Then, for any n, ω we obtain : $\|h_0 - h_\omega\|_\infty \leq Cte\gamma^{l(n)} + e^V(e^{2n\varepsilon} - 1)$, where the constant depends neither on ω, nor on ε. So, for $\delta > 0$, choose n_0 such that $\gamma^{l(n_0)} < \delta$ and then, choose ε_0 such that for $\varepsilon < \varepsilon_0$, $(e^{2n_0\varepsilon} - 1) < \delta$ to obtain :

$$\lim_{\varepsilon\to 0} \sup_{\omega\in\Omega_\varepsilon} \|h_0 - h_\omega\|_\infty = 0. \square$$

3.2 Stability of ν_ω.

From section 2, we have that, for any $f \in C(X)$, $\left\|\frac{\mathcal{L}_{n,\omega} f}{c_{n,\omega}} - h_{S^n\omega}\nu_\omega(f)\right\|_\infty$ converges toward 0 uniformly in ω and ε. Let $f \in C(X)$, we write for any n

$$|\nu(f)-\nu_\omega(f)| \leq \left\|\nu(f) - \frac{\mathcal{L}^n f}{c^n h_0}\right\|_\infty + \left\|\frac{\mathcal{L}^n f}{c^n h_0} - \frac{\mathcal{L}_{n,\omega} f}{c_{n,\omega} h_{S^n\omega}}\right\|_\infty + \left\|\frac{\mathcal{L}_{n,\omega} f}{c_{n,\omega} h_{S^n\omega}} - \nu_\omega(f)\right\|_\infty.$$

On the other hand, we have $\left\|\frac{\mathcal{L}^n f}{c^n h_0} - \frac{\mathcal{L}_{n,\omega} f}{c_{n,\omega} h_{S^n\omega}}\right\|_\infty \leq \left\|\frac{\mathcal{L}^n \mathbf{1}}{c^n h_0} - \frac{\mathcal{L}_{n,\omega} \mathbf{1}}{c_{n,\omega} h_{S^n\omega}}\right\|_\infty \|f\|_\infty$ and the previous stability results imply :

$$\lim_{\varepsilon\to 0} \sup_{\omega\in\Omega} \left\|\frac{\mathcal{L}^n \mathbf{1}}{c^n h_0} - \frac{\mathcal{L}_{n,\omega} \mathbf{1}}{c_{n,\omega} h_{S^n\omega}}\right\|_\infty = 0.$$

At last, proceeding as before, we get :

$$\lim_{\varepsilon\to 0} \sup_{\omega\in\Omega} |\nu(f) - \nu_\omega(f)| = 0$$

This concludes the proof of theorem 1.4.

3.3 Differentiability of the pressure.

Let $F : \Omega \longrightarrow B_E(\phi, 1)$ where $B_E(\phi, 1)$ denotes the ball of radius 1, centered at ϕ in the space E. For any $t \in \mathbb{R}^\star$, tF belongs to $B_E(t\phi, t)$; replacing ε by t in section 2, we get that tF as a unique equilibrium state μ_t. We consider the function $p(t) = \pi(tF)$, $t \in \mathbb{R}$. Clearly, p is convex.
Let $t_0 \in \mathbb{R}$, $a \in \mathbb{R}$ is in the sub gradient $\partial p(t_0)$ of p at t_0 if and only if :

$$\forall t \in \mathbb{R}\ p(t) \geq p(t_0) + a(t - t_0).$$

Since p is convex, it is differentiable at t_0 if and only if $\partial p(t_0)$ reduces to one element. It is easily verified that $\mu_{t_0}(F) \in \partial p(t_0)$.
Now, let $c_{\omega,t}$, $h_{\omega,t}$ and $\nu_{\omega,t}$ denote the generalized eigenvalues, eigenfunctions and conformal measures associated to $\mathcal{L}_{\omega,t} = \mathcal{L}_{tF_\omega}$. We have for $f \in B$:

$$\left\|\frac{\mathcal{L}_{n,\omega,t} f}{c_{n,\omega,t}} - h_{S^n\omega,t}\nu_{\omega,t}(f)\right\|_\infty \leq C(t)\gamma_t^{l(n)}\|f\|_B,$$

and it follows from the constructions of section 2 that $C(t)$ and γ_t may be bounded by some constants $C > 0$ and $0 < \gamma < 1$ independent of t for t in a compact. Using this remark and proceeding as in the proof of the stability properties, we obtain that $t \longrightarrow \mu_t(F)$ is continuous.
The fact that μ_t is the only element of $\partial p(t_0)$ follows from the upper semi-continuity of the entropy map (see [4]) and the continuity of $t \longrightarrow \mu_t(F)$ at t_0. $\square$

A Coding construction for small random perturbations of Axiom A diffeomorphisms

Let f be a C^1 Axiom A diffeomorphism on a compact Riemannian manifold M, with hyperbolic set $\Lambda(f) = \bigcap_{n\in\mathbb{Z}} f^n U$, where U is a fundamental neighborhood of $\Lambda(f)$. We give a coding construction for small random perturbations of f. Let (Ω, S, P) be an abstract invertible ergodic dynamical system, B a small C^1-neighborhood of f. We consider a measurable application $g : \Omega \longrightarrow B$ and write $g(n,\omega) = g_{S^{n-1}\omega} \circ \cdots \circ g_\omega$ for $n \geq 0$ and $g(n,\omega) = g^{-1}_{S^n\omega} \circ \cdots g^{-1}_{S^{-1}\omega}$ for $n < 0$. Let $\Lambda_\omega = \bigcap_{n\in\mathbb{Z}} g(n,\omega)^{-1}U$. Clearly, g_ω maps Λ_ω into $\Lambda_{S\omega}$. Let (Σ, σ) be the subshift of finite type associated to f (see Bowen [7]), we will construct continuous and surjective maps π_ω such that the following diagram commutes :

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \pi_\omega \downarrow & & \downarrow \pi_{S\omega} \\ \Lambda_\omega & \xrightarrow{g_\omega} & \Lambda_{S\omega}. \end{array}$$

Bogenschütz and Gundlach [6] obtained such a coding for expanding random dynamical systems (not necesseraly perturbations) to a random subshift of finite type with bounded symbols. In [9], P. Collet construct random finite Markov partitions for an Axiom A perturbeted by independent noise.

Recall that there exists a continuous and surjective application π from Σ to $\Lambda(f)$ such that $f \circ \pi = \pi \circ \sigma$. Thus it is enough to construct a homeomorphism $\psi_\omega : \Lambda_\omega \mapsto \Lambda(f)$ such that $\psi_{S\omega} \circ g_\omega = f \circ \psi_\omega$. This construction is made by two "Random Shadowing lemmas", see Anosov ([1]), Bowen ([7]) and Newhouse ([18]) for the "classical" Shadowing Lemma. The proof of the folllowing lemma is directly adapted from Newhouse's proof of Theorem 3.7 (Stability) in [18].

Lemma A.1 *Let ε be an expansivity constant for f. If B is sufficently small, for any $z \in \Lambda_\omega$ there exists a unique $x = \psi_\omega x \in \Lambda(f)$ such that :*

$$\forall n \in \mathbb{Z}, d(f^n x, g(n,\omega)z) < \varepsilon/2.$$

Moreover, ψ_ω is continuous.

It remains to prove that ψ_ω is bijective.

Lemma A.2 *For small enough η and ball B, for $x \in \Lambda$, there exists a unique $z \in \Lambda_\omega$ such that $\forall n \in \mathbb{Z}$ $d(f^n x, g(n,\omega)z) < \eta$.*

In order to prove this lemma, we recall some results which follow straightforward from the theory of hyperbolic sets and Axiom A diffeomorphisms (see Newhouse [18]).
If B is small enough, there exists a fundamental neighborhood U of $\Lambda(f)$, $\lambda > 1$ such that :

1. The hyperbolic splitting $T_xM = E^u_x \oplus E^s_x, x \in \Lambda(f)$ may be extended to a continuous splitting $T_xM = E^1_x \oplus E^2_x, x \in U$,

2. for $g \in B$, $x \in U$, T_xg preserves the unstable cone field and is a λ-expansion on it i.e.

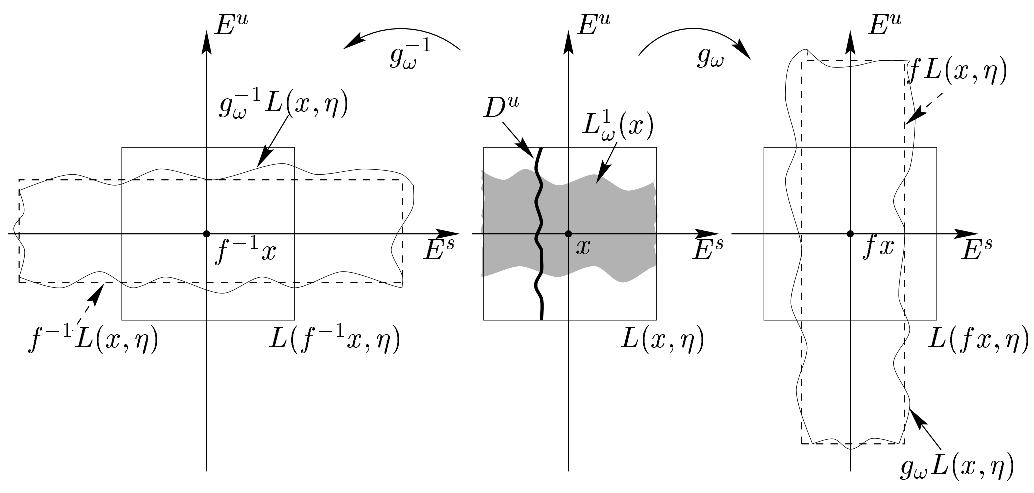
$$C^u_{x,\alpha} = \{v = v_1 + v_2 \in T_xM,\ v_1 \in E^1_x,\ v_2 \in E^2_x \ / \ |v_1| < \alpha|v_2|\}$$

   $T_xg(C^u_{x,\alpha}) \subset C^u_{gx,\alpha}$ and for $v \in C^u_{x,\alpha}$, $|T_xgv| > \lambda|v|$,

3. for $g \in B$, $x \in U$, T_xg^{-1} preserves the stable cone field and is a λ-expansion on it i.e.

$$C^s_{x,\alpha} = \{v = v_1 + v_2 \in T_xM,\ v_1 \in E^s_x,\ v_2 \in E^u_x \ / \ |v_1| \geq \alpha|v_2|\}$$

   $T_xg^{-1}(C^s_{x,\alpha}) \subset C^s_{g^{-1}x,\alpha}$ and for $v \in C^s_{x,\alpha}$, $|T_xg^{-1}v| > \lambda|v|$



Let $z \in \Lambda(f)$ and $L(z,\eta)$ be a small Liapunov neighborhood (see for example Pollicott [19]) of z, included in U, (i.e. $f(L(x,\eta))$ meet $L(fx,\eta)$ transversely

in the sense that their configuration is homeomorphic to the picture above.
We will call stable disk (resp. unstable disk) any C^1 disk D such that for any $x \in D \cap U$, $T_xD \subset C^s_{x,\alpha}$ (resp. $T_xD \subset C^u_{x,\alpha}$). For $A \subset M$, we will call u-width (resp s-width) $l^u(A) = \max\{\mathrm{diam} D^u \cap A, D^u \text{ unstable disk}\}$ (resp $l^s(A) = \max\{\mathrm{diam} D^s \cap A, D^s \text{ stable disk}\}$).
Clearly, $l^s(g_\omega L(f^{-1}x, \eta)) \leq \lambda^{-1} l^s L(f^{-1}x, \eta)$ and $l^u(g_\omega^{-1} L(fx, \eta)) \leq \lambda^{-1}\, l^u L(fx, \eta)$.
For $\omega \in \Omega$, $x \in \Lambda(f)$, let $L^1_\omega(x) = g_\omega^{-1}[L(fx, \eta)] \cap L(x, \eta)$ (see picture) and by induction, $L^n_\omega(x) = g_\omega^{-1}[L^{n-1}_{S^n\omega}(f^{n-1}x)] \cap L(x, \eta)$. We get a decreasing sequence of subset of $L(x, \eta)$ whose u-width go to zero ($l^u(L^n_\omega(x) \leq \lambda^{-n})$. Similarly, let $R^1_\omega(x) = g_{S^{-1}\omega}[L(f^{-1}x, \eta)] \cap L(x, \eta)$ and by induction, $R^n_\omega(x) = g_{S^{-1}\omega}[R^{n-1}_{S^{-n+1}\omega}(f^{-n+1}x)] \cap L(x, \eta)$. We get a decreasing sequence of subset of $L(x, \eta)$ whose s-width go to zero ($l^s(R^n_\omega(x) \leq \lambda^{-n})$. Now, $\bigcap_{n\in\mathbb{N}} L^n_\omega(x) \cap R^n_\omega(x)$ is non empty and has diameter zero. Moreover, any z satisfying the lemma needs to belong to $\bigcap_{n\in\mathbb{N}} L^n_\omega(x) \cap R^n_\omega(x)$ so we get the existence and uniqueness of z. $\square$

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