POLISH GROUPS AND BAIRE CATEGORY METHODS

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ABSTRACT. This article is a slightly modified version of the author's habilitation thesis, presenting his work on topics related to Polish groups, Baire category methods and metric model theory. Nearly all results presented are not new, though some arguments are. Among new results, we show that, for any countably infinite group Γ, all conjugacy classes in the space of actions of Γ on the Urysohn space are meager; and that the group of bounded isometries of the Urysohn space, endowed with the topology of uniform convergence, is path-connected.

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1. INTRODUCTION

The present article is a slightly modified version of the author's habilitation thesis; the first aim of such a thesis is to serve as an introduction to the author's work and domains of interest. In their current form, my hope is that these notes may be useful as an introduction to some of the uses of Baire category methods and ideas inspired by model theory.

When I first learned about the Baire category theorem, I thought it was remarkable that such a simple statement, with such a simple demonstration, could be used to establish the existence of apparently complicated mathematical objects. But that, to me, is not the main interest of Baire category notions; they are also particularly useful for instance as substitutes for measure-theoretic concepts in contexts where no natural measure is present. This phenomenon is particularly striking when one studies properties of Polish groups, which are the main subject of interest of this memoir. These groups appear in many places: infinite combinatorics, functional analysis, topological dynamics, ergodic theory... Isometry groups, homeomorphism groups, permutation groups can often be endowed with a Polish group structure and Baire category, or more generally descriptive-set-theoretic methods prove useful.

In the first section, we recall some definitions and concepts of Baire category theory, then present a panorama of Polish groups; we also discuss an interesting example of a group which (unfortunately?) cannot be endowed with a Polish group structure.

Next, we discuss the Urysohn space **U** and some of its siblings; this space, built by Urysohn in 1924, is characterized by the fact that it is both *universal* (it contains an isometric copy of any separable metric space) and *homogeneous* (any isometry between finite subspaces extends to an isometry of the whole space). These properties make the isometry group of **U** an interesting and rich object, for instance it contains an isomorphic copy of any Polish group. These notions of homogeneity and universality make sense in a variety of contexts and provide interesting problems. Before moving on to some of these, we discuss isometric embeddings of **U** into Banach spaces, which are surprisingly rigid: Holmes proved that there is essentially only one way of embedding **U** isometrically into a normed vector space, as soon as one has decided which point gets mapped to 0. We will investigate which spaces share this rigidity property.

Then we move on to actions of countable groups on some homogeneous structures, mainly the separable Hilbert space, the standard atomless probability algebra, and the Urysohn space. We study Baire category in the space of actions of some countable group Γ on one of these structures; this space has a natural Polish topology, and understanding generic properties of isometric actions, unitary representations and measure-preserving actions of countable groups also has some consequences on the structure of the ambient Polish group. This is a classic theme of research in ergodic theory, originally considered by Halmos in the case when the acting group is **Z** and much-studied since.

The last section, which is also the longest, bears the title "First order logic and Polish groups". This section actually contains little (or no?) logic, but the language and notions of first-order logic, and its metric avatar sometimes called "metric model theory", play a crucial role. Wittgenstein famously wrote that "the limits of my language are the limits of my world"; the limits of my language were pushed when I learned about metric model theory, and consequently the limits of my mathematical world were redefined by this new language. My hope is to convince the reader, whom I imagine to be somewhat skeptical, of the interest of considering Polish groups using a point of view influenced by model theory. It is certainly not new that this interaction is fruitful and natural in the context of automorphism groups of countable structures, which are exactly the closed subgroups of the permutation group of the integers; but it is only more recently that it appeared that model theory was relevant to the study of general Polish groups, and I believe some of the work presented here played a part in this realization.

I chose to use a relatively informal writing style, and to present few complete proofs; often a sketch of proof is proposed, sometimes a complete argument is given when it seems particularly enlightening to me or is not easily found in the literature. I tried to make this text accessible and interesting for a reader who is not a specialist of Polish groups; I hope that the experts will nevertheless find some food for thought.

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2. Baire category

2.1. **The basics of Baire category.** We begin by recalling the statement of the Baire category theorem.

Theorem 2.1 (Baire)**.** *Let X be a completely metrizable topological space, and $(O_n)_{n<\omega}$ be a countable family of dense open subsets of X. Then the intersection $\bigcap_{n<\omega} O_n$ is dense in X.*

Definition 2.2. Let X be a topological space. A subset of X is *meager* if it is contained in a countable union of closed sets, each of which has empty interior. A subset is *comeager*, or *generic*, if it contains a countable intersection of dense open sets.

Thus, the Baire category theorem is the assertion that comeager subsets of completely metrizable spaces are dense. Countable intersections of open sets are called G_δ sets; countable unions of closed sets are called F_σ sets. It is important to keep in mind that Baire's theorem is a *topological* theorem as opposed to a *metric* one: what matters is that there exists a compatible complete metric, even though the metric one "naturally" considers on X may not be complete. For instance, the usual distance on $]0,+\infty[$ is certainly not complete; however, the distance defined by $d(s,t) = |s-t| + |\frac{1}{s} - \frac{1}{t}|$ is, and it induces the usual topology on $]0,+\infty[$. This is part of a broader phenomenon.

Theorem 2.3 (Alexandrov)**.** *Let (X,d) be a metric space. There exists a complete metric compatible with the topology of X if, and only if, X is a G_δ subset of the completion of (X,d).*

This is easily seen to be equivalent to X being a G_δ subset of any metrizable space containing it. What matters most for us is that the Baire category theorem is true in any G_δ subset of a completely metrizable space.

Baire category notions are useful as notions of *largeness*: a comeager set may be thought of as being large, and a meager set as being small. The union of countably many small sets is still a small set; dually, the intersection of countably many large sets is still a large set. Of course, one is reminded of measure theory, where small sets are those which have measure zero, and large sets those with full measure. We will often work in contexts where there is no natural measure (something that will be made precise in the discussion at the beginning of the next section), so we have to content ourselves with category notions, cruder than measure-theoretic tools but which can be used in different contexts.

We say that a topological space X in which the Baire category theorem holds is a *Baire space*; any G_δ subset of a completely metrizable space is a Baire space, and any open subset of a Baire space is a Baire space. This gives rise to a local notion of largeness: given an open subset O of a topological space X and a subset $A \subseteq X$, we say that A is comeager in O if $A \cap O$ is a comeager subset of O. The local and global notions of largeness get along reasonably well: if A is globally large, then it is locally large everywhere; if A is large in an open set O, then $A \cap O$ is the intersection of O and a globally large subset of X.

We would want these local and global notions of largeness to get along even better; namely, a natural assumption would be that, if a set is not globally small, then it is locally large somewhere. This is not necessarily true (at least, not if one uses the usual axioms of Zermelo–Fraenkel set theory (ZF)[i]). The analogy with measure leads us to introduce a class of sets which behave well with regard to our notion of largeness; in measure theory, the measurable sets are those who differ from a Borel set by a negligible set. The same definition makes sense here.

Definition 2.4. Let X be a topological space. A subset A of X is *Baire-measurable* if there exists an open set O such that the symmetric difference $A \Delta O$ is meager.

It might be a bit surprising that the definition requires an open set rather than a Borel set; actually this does not matter: any Borel set is equal to an open set modulo a meager set, which is a consequence of the fact that the family of Baire-measurable subsets of a topological space X is a σ-algebra. Pursuing the analogy with measure, one could think of this as a strong form of regularity: if μ is an (outer) regular Borel measure on a topological space X, then any measurable set is equal to a G_δ set modulo a set of μ-measure 0.

Remark 2.5. The standard terminology for the above property is "A has the property of Baire". I always found it confusing, because my intuition is that a set with the property of Baire should satisfy the Baire category theorem, and this is obviously not always the case. For instance, the space of rational numbers has the property of Baire yet is a textbook example of a topological space failing to satisfy Baire's theorem. This is why I use the less standard, but to my mind more evocative, "Baire-measurable" terminology.

Note that from the definition of Baire-measurability it follows that if A is Baire-measurable and non-meager in a Polish space X then there exists a nonempty open subset O of X such that A is comeager in O. This is what we wanted: if a set is well-behaved (i.e. Baire measurable) and not small, then it is locally large somewhere.

Most of the time, we will not be working with general completely metrizable spaces, but merely with *separable* spaces.

Definition 2.6. A *Polish space* is a completely metrizable and separable topological space.

In particular, the topology of a Polish space is *second countable*, i.e. it admits a countable basis of open sets. We will often use the fact that any such space satisfies the *Lindelöff property*: from any open covering one can extract a countable subcovering. Note again that being a Polish space is a topological condition, not a metric one; often we will need to manipulate Polish spaces with noncomplete metrics. We use the terminology *Polish metric space* when we are concerned with complete separable metric spaces.

Remark 2.7. The term "Polish space" is often credited to Bourbaki, who were supposedly honoring the pioneering work of Polish topologists and set theorists during the first half of the twentieth century. In some papers, mostly from the fifties and sixties, one can find the term "polonais space" in articles written in English

[i]Throughout this text we work, as usual, in ZF + (Dependent Choice); the reader may safely assume that we work with the usual set theoretic axioms, and that we accept the axiom of choice.

(for instance [Eff65]). Using the French word for "Polish" was an interesting way to capture the influence of mathematicians of both countries on this notion, but it does not seem to have caught on.

We already saw that all Borel subsets of Polish spaces are Baire-measurable, which is useful but not sufficient for our purposes. The problem is that many naturally-defined subsets of Polish spaces turn out not to be Borel; an underlying issue is that the continuous image of a Borel set need not be Borel in general.

Definition 2.8. Let X be a Polish space. A subset $A \subseteq X$ is *analytic* if there exists a Polish space Y, a Borel mapping $f \colon Y \to X$ and a Borel subset B of Y such that $f(B) = A$. A subset A of X is *coanalytic* if its complement is analytic.

Actually, the condition above is equivalent to saying that there is a Polish space Y, and a *continuous* mapping from Y to X such that $f(Y) = A$.

Theorem 2.9 (Lusin-Sierpinski)**.** *Let X be a Polish space and A be an analytic subspace of X. Then A is Baire-measurable.*

Any Borel subset of a Polish space is analytic; one can use a diagonal argument to show that there exist analytic non Borel subsets of Polish spaces (any uncountable Polish space contains one). The following fundamental result may be considered as the starting point of descriptive set theory.

Theorem 2.10 (Lusin)**.** *Let X be a Polish space, and A be a subset of X. Then A is Borel if, and only if, A is both analytic and coanalytic.*

This result has the following spectacular consequence.

Theorem 2.11. *Let X, Y be Polish spaces, and $f \colon X \to Y$ be a function. Then f is Borel if, and only if, its graph is a Borel subset of $X \times Y$.*

I will not give proofs of these classical results here; we use [Kec95] as a general reference for descriptive-set-theoretic facts and theorems. The following fact is used in the classical proof of Theorem 2.9 and will be useful to us later.

Theorem 2.12. *Let X be a topological space, and let A be a subset of X. Denote by $U(A)$ the union of all open subsets of X in which A is comeager. Then $U(A) \setminus A$ is meager, and A is Baire-measurable if, and only if, $A \setminus U(A)$ is meager.*

The definition of $U(A)$ will play a role in the next section (in the proof of Pettis' lemma) as well as in the last section.

2.2. **Polish groups.**

Definition 2.13. A *topological group* is a group endowed with a topology for which the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous.
A *Polish group* is a topological group whose topology is Polish.

Polish groups are abundant in analysis but also, as we shall see, in ergodic theory and model theory. Below we will discuss important examples in some detail; for now, let us simply note that any countable discrete group is Polish, as is any locally compact metrizable group, any separable Banach space (the group operation being addition of vectors), etc. In locally compact groups, one can use the Haar measure to provide a notion of largeness which is well-behaved with respect to the group operations; while one cannot in general hope that the Haar measure is translation-invariant on both sides, it is invariant on one side (e.g. under the left translation action of the group on itself), and translates of subsets of measure zero always have measure zero.

The Haar measure was used by Weil in the thirties, generalizing a result of Steinhaus for $G = (\mathbf{R}, +)$, to prove that a discontinuous homomorphism defined on a locally compact group must be fairly wild.

Theorem 2.14 (Weil)**.** *Let G be a locally compact topological group, and A be a non-negligible Haar-measurable subset of G. Then AA^{-1} contains a neighborhood of the neutral element 1_G.*

Consequently, any Haar-measurable homomorphism from a locally compact group to a second-countable topological group must be continuous.

Sketch of proof. The first statement is proved using the regularity of the Haar measure; the second sentence is an easy exercise: let G be a locally compact group, H a second-countable topological group and $\varphi\colon G \to H$ a Haar-measurable homomorphism. Pick a nonempty open neighborhood V of 1_H; the Lindelöff property of $\varphi(G)$ implies that there exists a countable family h_n of elements of $\varphi(G)$ such that $\varphi(G) = \bigcup h_n(V \cap \varphi(G))$, from which one obtains a countable family (g_n) of elements of G such that $G = \bigcup g_n \varphi^{-1}(V)$.

Hence $\varphi^{-1}(V)$ is not negligible, so $\varphi^{-1}(V)(\varphi^{-1}(V))^{-1} \subseteq \varphi^{-1}(VV^{-1})$ contains a neighborhood of 1_G whenever V is an open neighborhood of 1_H. Given any open neighborhood W of 1_H, continuity of group operations implies that one can find an open neighborhood V of 1_H such that $VV^{-1} \subseteq W$. Hence $\varphi^{-1}(W)$ has nonempty interior; we just proved that φ is continuous at 1_G, hence continuous everywhere. □

Unfortunately, as soon as one gets out of the class of locally compact groups, one loses the Haar measure, in the worst way possible: by a result of Weil (see Appendix B of [GTW05] for a proof, coming from [Oxt46] and attributed to Ulam), a Polish group which admits a left-translation invariant measure class (i.e. a measure μ such that all its left translates are absolutely continuous with respect to μ) must be locally compact. Thus one must make do with Baire category methods.

Theorem 2.15 (Pettis [Pet50])**.** *Let A, B be subsets of a Baire topological group; then $U(A)U(B) \subseteq AB$. In particular, if A is a Baire-measurable non meager subset of G then AA^{-1} contains a neighborhood of 1_G.*

Consequently, any Baire-measurable homomorphism from a Baire topological group to a second-countable topological group must be continuous.

We recall that $U(A)$ denotes the union of all open subsets of X in which A is comeager, that A is comeager in $U(A)$, and that A is Baire measurable if, and only if, $A \setminus U(A)$ is meager. The fact that Baire-measurable homomorphisms between Polish groups are continuous was first proved by Banach [Ban55].

Proof. Let A, B be two subsets of X, and pick $g \in U(A)U(B)$. Equivalently, $U(A) \cap g(U(B))^{-1} = U(A) \cap U(gB^{-1})$ is nonempty; this is an open set in which A and gB^{-1} are both comeager, hence the fact that open subsets of Baire spaces are Baire implies that $A \cap gB^{-1} \neq \emptyset$, i.e. $g \in AB$.

Now, if A is Baire-measurable and nonmeager, then $U(A)U(A^{-1})$ is a nonempty open neighborhood of 1_G which is contained in AA^{-1}; the automatic continuity of Baire-measurable homomorphisms with range in a second countable group is deduced from this exactly as in the case of Haar-measurable homomorphisms. □

Let us point out a few structural facts about Polish groups.

Theorem 2.16. *Let G be a Polish group, and H be a subgroup of G. Then H, endowed with the relative topology, is a Polish group iff H is closed in G.*

Proof. One implication is obvious. Assume that H is a subgroup of G which is Polish when endowed with the relative topology. Then H is a G_δ subset of $\overline{H}$; thus, for any $k \in \overline{H}$, H and kH are dense G_δ subsets of $\overline{H}$, so the Baire category theorem implies that $H \cap kH$ is nonempty for all $k \in \overline{H}$, so $H = \overline{H}$. □

Theorem 2.17. *Let G, H be Polish groups, and $\varphi \colon G \to H$ be a Baire-measurable isomorphism (of abstract groups). Then φ is an isomorphism of topological groups.*

Proof. Being Baire-measurable, φ is automatically continuous. So its graph is closed, and so is the graph of φ^{-1}; hence φ^{-1} is Borel, hence Baire-measurable, hence continuous. □

Thus, if (G, τ) is a Polish group and $\tilde{\tau}$ is a Polish group topology on G such that each τ-open set is $\tilde{\tau}$-Baire-measurable, then necessarily $\tau = \tilde{\tau}$.

We now turn to a quick panorama of the Polish groups we will encounter in this memoir, as well as an example of a seemingly nice group which cannot be made Polish.

2.3. **Isometry groups.** Whenever (X, d) is a Polish metric space, one can consider its isometry group $\mathrm{Iso}(X)$; it is tempting to turn it into a topological group by endowing it with the metric of uniform convergence. While this is a perfectly reasonable thing to do, the resulting topology will often have too many open sets to be useful - an extreme example of this is obtained when one considers the isometry group of the space of natural integers endowed with the discrete metric or, equivalently, the group of all permutations of $\mathbf{N}$. Then any two distinct permutations are at (uniform) distance 1, so the topology of uniform convergence is discrete in that case.

If uniform convergence is too much to ask, then the next best thing is pointwise convergence. When endowed with the topology of pointwise convergence, $\mathrm{Iso}(X)$ is a Polish group whenever X is a Polish metric space. Given a countable dense subset A of X, $\mathrm{Iso}(X)$ equipped with this topology is homeomorphic (via the map that associates to an isometry its restriction to A) to a subset of X^A, so the topology is metrizable and separable. It is easy to check that group operations are continuous on $\mathrm{Iso}(X)$; an abstract way to see that $\mathrm{Iso}(X)$ is a Polish group is to notice that the image of the restriction map from $\mathrm{Iso}(X)$ to X^A is the set of all elements $g \in X^A$ which satisfy:

- $\forall a, b \in A \quad d(a, b) = d(g(a), g(b))$
- $\forall \varepsilon > 0\ \forall a \in A\ \exists b \in A \quad d(a, g(b)) < \varepsilon$.

The first condition expresses that g preserves the distance, and defines a closed subset of X^A; the second condition means that $g(A)$ is dense in X, and is a G_δ condition. Hence $\mathrm{Iso}(X)$ is homeomorphic to a G_δ subset of X^A, thus is a Polish topological space.

A more down-to-earth way to show the same thing goes as follows: let $A = \{a_n\}_{n<\omega}$, and define a metric ρ on $\mathrm{Iso}(X)$ by setting

$$\rho(g,h) = \sum_{n=0}^{\infty} \min(2^{-n}, d(g(a_n), h(a_n)))$$

This metric induces the topology of pointwise convergence on $\mathrm{Iso}(X)$, and is left-invariant; unfortunately it is not complete in general, but the metric $\tilde{\rho}$ defined by $\tilde{\rho}(g,h) = \rho(g,h) + \rho(g^{-1}, h^{-1})$ is complete. This is a general phenomenon: while any Polish group, and indeed any first-countable Hausdorff topological group, admits a compatible left-invariant metric by the Birkhoff–Kakutani theorem, most Polish groups do not admit a compatible left-invariant *complete* metric.

In a sense, isometry groups are all there is when it comes to Polish groups: Gao and Kechris [GK03a] proved that, given any Polish group G, there exists a Polish metric space X such that G is isomorphic, as a topological group, to $\mathrm{Iso}(X)$ equipped with the topology of pointwise convergence.

2.4. **The unitary group.** Consider an infinite-dimensional, separable Hilbert space $\mathcal{H}$ and denote by $U(\mathcal{H})$ its unitary group, i.e. the set of all $\mathbf{C}$-linear bijections of $\mathcal{H}$ whose inverse coincides with their adjoint. Equivalently, a map is unitary if it is a $\mathbf{C}$-linear isometry of $\mathcal{H}$ onto itself. As above, the first idea that comes to mind might be to endow $U(\mathcal{H})$ with the topology induced by the operator norm: $d(g,h) = \|g-h\|$. This is the topology of uniform convergence on the unit ball of $\mathcal{H}$ and, not unexpectedly, is "almost" discrete: letting $(e_i)_{i<\omega}$ denote a Hilbert basis of $\mathcal{H}$, any permutation σ of the set of natural integers induces a unitary operator $u_\sigma \colon e_i \mapsto e_{\sigma(i)}$, and whenever $\sigma \neq \tau$ one has $\|u_\sigma - u_\tau\| = \sqrt{2}$. Thus the topology induced by the operator norm is certainly not separable (it will play a role later on, though).

The example of isometry groups discussed above shows that, when endowed with the *pointwise* convergence topology with regard to the norm topology on $\mathcal{H}$, the isometry group of $\mathcal{H}$ is a Polish group; being $\mathbf{C}$-linear is closed under pointwise convergence, so this is a Polish topology on $U(\mathcal{H})$, called the *strong topology*. One could equip $U(\mathcal{H})$ with the topology of pointwise convergence with regard to the *weak* topology on $\mathcal{H}$; when it comes to unitary operators, the difference is immaterial since both topologies coincide. This is a hint of a broader phenomenon: there exists a unique Polish group topology on $U(\mathcal{H})$, a fact first proved by Atim and Kallman [AK12] and generalized by Tsankov [Tsa13], who proved that $U(\mathcal{H})$ has the *automatic continuity property*: any homomorphism from $U(\mathcal{H})$ to a Polish group is continuous. This is very much related to our concerns, and we will discuss this phenomenon in some detail later on.

From now on, $\mathcal{H}$ will denote an infinite-dimensional, separable Hilbert space, and $U(\mathcal{H})$ will be its unitary group.

2.5. **Measure-preserving automorphisms.** The notation (X,μ) will stand for a standard atomless probability space throughout the text. This is a fancy way of speaking of the unit interval endowed with the Lebesgue measure; the reason the more abstract notation (X,μ) is useful is that standard atomless probability spaces occur in many different guises, for instance any infinite compact metrizable group endowed with its Haar measure is one.

The group we are concerned with here is made up of all measure-preserving bijections of (X, μ), *identified if they coincide outside of a set of measure* 0; thus one should really speak of *classes* of measure-preserving bijections. This abuse of terminology must be kept in mind, but will not cause us any significant trouble, and we will simply ignore sets of measure 0 whenever it does not cause confusion - these sets are called negligible for a reason, after all. We denote this group by $\mathrm{Aut}(X, \mu)$, or simply $\mathrm{Aut}(\mu)$. Again there seem to be several reasonable choices of topology: one could consider the *uniform* topology, induced by the metric

$$d_u(S, T) = \mu(\{x \colon S(x) \neq T(x)\}).$$

This metric is bi-invariant; unfortunately, it is again far from separable - for instance, see (X, μ) as the unit circle with its usual measure; then two rotations with different angles are at distance 1. One could do even worse: embedding $\mathrm{Aut}(X, \mu)$ into the unitary group $U(L^2(X, \mu))$ and endowing it with the operator norm, one obtains a discrete group.

Of course, we know what went wrong: we considered uniform metrics, which should not be separable; the right choice if one wants to obtain a Polish group is to consider pointwise convergence. The measure algebra $MALG_\mu$ of all measurable subsets of (X, μ) (identified if their symmetric difference has measure 0) is a complete separable metric space when endowed with the distance $d(A, B) = \mu(A \Delta B)$; and measure-preserving bijections are the same as isometries of $MALG_\mu$ which fix $\emptyset$ (Sikorski, see [Kec95, Theorem 15.9]). Thus one obtains a Polish topology by considering the topology of pointwise convergence relative to this metric, which is the topology induced by the maps $g \mapsto \mu(g(A)\Delta A)$ as A ranges over all measurable subsets of X.

As in the case of the unitary group, this is the unique Polish topology on $\mathrm{Aut}(\mu)$ which is compatible with its group structure; one of the results presented below is the fact that $\mathrm{Aut}(\mu)$ satisfies the automatic continuity property which, combined with a result of Glasner [Gla12] and the simplicity of $\mathrm{Aut}(\mu)$ [Fat78], shows that there are only two second-countable topologies on $\mathrm{Aut}(\mu)$: the coarse topology, and the Polish topology we just defined. The fact that there is a unique compatible Polish group topology on $\mathrm{Aut}(\mu)$ is due to Kallman [Kal85].

As a general reference regarding $\mathrm{Aut}(\mu)$, [Kec10] is particularly well-suited to our purposes.

2.6. **Permutation groups.** Both $U(\mathcal{H})$ and $\mathrm{Aut}(\mu)$ are connected, indeed they are both homeomorphic to an infinite-dimensional separable Hilbert space. Since both groups have the automatic continuity property, they cannot act nontrivially on a countable set: the action would have to be continuous with respect to the discrete topology on the countable set, so by connectedness the action must be trivial.

Still, groups acting on countable sets are interesting objects. The first example is the permutation group of the integers, denoted by S_∞; we already met it when discussing isometry groups, and know that it is a Polish group when endowed with the topology of pointwise convergence relatively to the discrete topology on $\mathbf{N}$ - equivalently, this is a group topology such that the family of subgroups of the form $\{\sigma \colon \forall x \in F \ \sigma(x) = x\}$, where F ranges over all finite subsets of $\mathbf{N}$, is a basis of open neighborhoods of 1.

Again, this topology is the unique second-countable group topology on S_∞ (Kechris–Rosendal [KR07], extending a theorem of Gaughan [Gau67]); a Polish group is isomorphic, as a topological group, to a (necessarily closed) subgroup of S_∞ if, and only if, it admits a basis of neighborhoods of 1 made up of open subgroups. When this happens, we say that the group is a *permutation group*.

These groups naturally appear in model theory; we will discuss this in some detail later on. For now, we simply note that when encountering a countable "structure", one can consider its automorphism group, which is made up of all bijections preserving the structure; identifying the universe of the structure with $\mathbf{N}$, its automorphism group is then a closed subgroup of S_∞. Conversely, all Polish permutation groups are automorphism groups of countable structures. As examples, one can cite the automorphism group of the random graph, the automorphism group of a countable free group...

The topology of the permutation group comes from its action on the structure; in some cases, knowing the topology is enough to recover a lot of information about the structure (we will also come back to this later). One is then led to wondering when it is possible to reconstruct the topology when knowing only the algebraic structure of the group, motivating the study of the automatic continuity properties of permutation groups.

2.7. **Full groups.** To close this section, we discuss *full groups*. To motivate our interest in those groups, we begin by recalling the more classical notion of full group of a countable measure-preserving equivalence relation; these equivalence relations are those that arise from a measure-preserving action of a countable group on a standard probability space (X, μ); as usual when dealing with measures, we ignore sets of measure 0.

Given such an action of a countable group Γ, we denote by R_Γ the associated equivalence relation; its full group, denoted by $[R_\Gamma]$, is the group of all measure-preserving bijections $g \in \mathrm{Aut}(X, \mu)$ such that $g(x) R_\Gamma x$ for (almost) all $x \in X$. These groups were introduced by Dye ([Dye59], [Dye63]); the full group completely remembers the relation, in a way made precise by the following definition and theorem [i].

Definition 2.18. Consider two countable groups Γ_1, Γ_2 acting by measure-preserving bijections on a standard probability space (X, μ). We say that the associated equivalence relations are *orbit equivalent* if there exists $g \in \mathrm{Aut}(X, \mu)$ such that

$$\forall x, y \in X \quad \left(x R_{\Gamma_1} y\right) \Leftrightarrow \left(g(x) R_{\Gamma_2} g(y)\right)$$

Orbit equivalence is the natural notion of isomorphism of measure-preserving equivalence relations: up to an isomorphism of the space, the relations coincide. Recall that a measure-preserving action is *ergodic* if it does not admit any nontrivial invariant sets.

Theorem 2.19 (Dye). *Assume that Γ_1, Γ_2 are two countable groups acting by measure-preserving transformations on a standard probability space (X, μ), and that there exists an isomorphism $\Phi \colon \left[R_{\Gamma_1}\right] \to \left[R_{\Gamma_2}\right]$. Then there exists $g \in \mathrm{Aut}(X, \mu)$ such that for all $h \in \left[R_{\Gamma_1}\right]$ one has $\Phi(h) = ghg^{-1}$.*

In particular, g must realize an orbit equivalence between R_{Γ_1} and R_{Γ_2} - thus R_{Γ_1} and R_{Γ_2} are orbit equivalent iff their full groups are isomorphic (as abstract groups).

[i] The definition of orbit equivalence actually appeared after Dye's work, in work of Mackey [Mac66] so Dye's theorem is formulated differently than the original.

Thus, full groups are complete invariants for orbit equivalence. Dye's theorem is also related to considerations of automatic continuity; to explain this we need to discuss topologies on full groups of ergodic, probability measure-preserving actions of countable groups. The first that comes to mind is the topology induced from the Polish topology of $\mathrm{Aut}(X,\mu)$; of course this is a second-countable group topology, but it is never Polish for ergodic relations: indeed, ergodicity of the action $\Gamma \curvearrowright X$ is equivalent to the fact that $[R_\Gamma]$ is dense in $\mathrm{Aut}(X,\mu)$. Not being a closed subgroup, it cannot be Polish for the induced topology; still, it is never too complicated a subset of $\mathrm{Aut}(X,\mu)$: Wei [Wei05] proved that full groups of ergodic actions of countable groups are always countable intersections of countable unions of closed subsets of $\mathrm{Aut}(X,\mu)$, in particular they are Borel subsets of $\mathrm{Aut}(X,\mu)$.

What about the topology induced from the uniform topology? It is easy to see that $[R_\Gamma]$ is closed in $\mathrm{Aut}(X,\mu)$ with respect to the uniform topology; perhaps more surprisingly, $[R_\Gamma]$ is also separable, so it is a Polish group (see e.g. [Kec10, Proposition 3.2]). Then, Dye's theorem implies that an isomorphism between two full groups is necessarily continuous with respect to their Polish topologies; given the examples discussed above, the reader will probably not be surprised to learn that Kittrell and Tsankov [KT10] proved that full groups of relations induced by an ergodic action of a countable group have the automatic continuity property.

We turn to full groups in topological dynamics, which for us means the study of actions of countable groups by homeomorphisms of a Cantor space X. The analogue of ergodicity in that context is *minimality*.

Definition 2.20. Let Γ be a countable group acting by homeomorphisms on a Cantor space X. The action is said to be *minimal* if all orbits are dense.

Then one can define the full group of an action $\Gamma \curvearrowright X$ in the natural way: it is made up of all the homeomorphisms of X which map each Γ-orbit onto itself. Similarly, two actions of countable groups Γ_1, Γ_2 by homeomorphisms of a Cantor space X are orbit equivalent if there exists a homeomorphism of X which maps each orbit for the first group's action onto an orbit for the second group's action. The natural analogue of Dye's theorem holds in that context.

Theorem 2.21 (Giordano–Putnam–Skau [GPS99])**.** *Assume that Γ_1, Γ_2 are two countable groups acting minimally*[i] *by homeomorphisms of a Cantor space X, and that $\Phi\colon [R_{\Gamma_1}] \to [R_{\Gamma_2}]$ is an isomorphism. Then there exists a homeomorphism g of X which is such that for all $h \in [R_{\Gamma_1}]$ one has $\Phi(h) = ghg^{-1}$.*

This g must realize an orbit equivalence between R_{Γ_1} and R_{Γ_2} - thus R_{Γ_1} and R_{Γ_2} are orbit equivalent iff their full groups are isomorphic (as abstract groups).

Now, given an action by homeomorphisms of a countable group Γ on a Cantor space X, we would like to turn $[R_\Gamma]$ into a Polish group; first, what kind of topologies can one put on the homeomorphism group $\mathrm{Homeo}(X)$? As with the homeomorphism group of any compact metric space[ii], one can use the uniform topology: given a compatible metric d on X, this topology is induced by the uniform metric d_u, defined by

$$d_u(g,h) = \sup(\{d(g(x),h(x))\colon x \in X\}) .$$

[i] The minimality assumption is stronger than what is needed, see [Med11].

[ii] An open problem (as far as I know): which Polish groups are isomorphic, as a topological group, to the homeomorphism group of a compact metric space?

This metric is not complete, but the metric $\tilde{d}_u$ defined by $\tilde{d}_u(g,h) = d_u(g^{-1},h^{-1}) + d_u(g,h)$ is complete and induces the same topology on $\mathrm{Homeo}(X)$. It was proved by Rosendal and Solecki [RS07] that $\mathrm{Homeo}(X)$, with this topology, has the automatic continuity property; thus this is the unique Polish group topology on $\mathrm{Homeo}(X)$. Actually, it follows from the simplicity of $\mathrm{Homeo}(X)$ [And58] and a result of Gamarnik [Gam91] that this is the unique second-countable group topology on $\mathrm{Homeo}(X)$ besides the coarse topology. It is actually a permutation group topology; $\mathrm{Homeo}(X)$ naturally acts on the countable set of all clopen subsets of X, and the permutation group topology induced by that action is the same as the one we just described; a basis of neighborhoods of 1 is given by sets of the form

$$\{g \in \mathrm{Homeo}(X) \colon \forall A \in \mathbf{A}\ g(A) = A\}$$

where $\mathbf{A}$ ranges over all finite clopen partitions of X.

Unsurprisingly, the full group of a minimal action of a countable group is not closed in $\mathrm{Homeo}(X)$; in the case of a minimal $\mathbf{Z}$-action, one can describe its closure. Below we say that an homeomorphism φ of a Cantor space X is minimal if the associated $\mathbf{Z}$-action is minimal, and we denote by $[\varphi]$ the associated full group. The following is a consequence of a result of Glasner–Weiss [GW95].

Theorem 2.22 (Glasner–Weiss)**.** *Let φ be a minimal homeomorphism of a Cantor space X. Denote by $\mathbf{M}_\varphi$ the set of all Borel probability measures on X which are φ-invariant. Then the closure of $[\varphi]$ inside* $\mathrm{Homeo}(X)$ *is equal to* $\{g \in \mathrm{Homeo}(X) \colon \forall \mu \in \mathbf{M}_\varphi\ g_*\mu = \mu\}$.

This is somewhat analogous to what happens in the measure-theoretic setting (i.e. the closure of the full group is as large as possible); things already appear to be more complicated in the topological setting, however: two measure-preserving ergodic $\mathbf{Z}$-actions are always orbit equivalent (Dye [Dye59]), while the above result can be used to see that there are continuum many pairwise non-orbit equivalent minimal actions of $\mathbf{Z}$[i]. As far as I know, the closure of the full group of a minimal action of a countable group is not understood in general, even if the group is abelian.

As in the measure-theoretic context, one might expect that there exists a Polish topology on the full group; however, if such a topology existed, it should have a natural definition and none is to be found. This motivated the following result, obtained in collaboration with T. Ibarlucià.

Theorem 2.23 ([IM13])**.** *Let Γ be a countable group acting by homeomorphisms on a Cantor space X; assume that for any nonempty open subset U of X there exists $x \in U$ such that $\Gamma \cdot x$ intersects U in at least two points. Then $[R_\Gamma]$ does not admit a compatible Baire, Hausdorff, second-countable group topology.*

Sketch of proof. The idea behind the proof is fairly standard, see for instance [Ros05]. Assume that $\Gamma \curvearrowright X$ satisfies the assumption above, and that τ is a Baire, Hausdorff group topology on $[R_\Gamma]$. Then, given any clopen U, one can check that $g \in [R_\Gamma]$

[i] I do not know the exact complexity, in the sense of Borel equivalence relations, of the relation of orbit equivalence of minimal homeomorphisms; however, it is known to be fairly complicated since OE for uniquely ergodic homeomorphisms is already not essentially countable as it is bireducible to $=^+$.

is equal to the identity on U if and only if g commutes with all elements which coincide with the identity on $X \setminus U$; thus

$$\{g\colon g_{\restriction U} = id_{\restriction U}\} = \bigcap_{h_{\restriction X\setminus U} = id_{\restriction X\setminus U}} \{g\colon gh = hg\}$$

is an intersection of closed subsets of $[R_\Gamma]$, hence it is closed.

Now, given any clopen U and $g \in [R_\Gamma]$, one has $g(U) \subseteq U$ iff $g^{-1}hg$ coincides with the identity on U for any h which coincides with the identity on U; thus $\{g\colon g(U) \subseteq U\}$ is τ-closed by the same reasoning as above. This means that each set $\{g\colon g(U) = U\}$ is τ-closed, thus the inclusion map from $([R_\Gamma], \tau)$ to $\mathrm{Homeo}(X)$ endowed with its usual Polish topology is Borel, hence continuous since $([R_\Gamma], \tau)$ is assumed to be Baire. We just proved that τ extends the topology induced from the Polish topology of $\mathrm{Homeo}(X)$.

So far, we are in the same situation as in the measure-theoretic context; now, fix $x \in X$ and consider the orbit map $g \mapsto g(x)$ from $([R_\Gamma], \tau)$ to the countable set $\Gamma \cdot x$, which induces a homomorphism from $([R_\Gamma], \tau)$ to the group H_x of permutations of $\Gamma \cdot x$. What we proved above shows that this homomorphism is Borel when H_x is endowed with its permutation group topology; using again the fact that τ is Baire, we obtain that this homomorphism is continuous. Equivalently, each subgroup $\{g\colon g(x) = x\}$ is τ-clopen. With a bit of work one can check that this causes the existence of too many clopen subgroups for τ to be Lindelöff, so τ cannot be second-countable.

□

It is then tempting to study the properties of the closure of $([R_\Gamma], \tau)$; we will get back to this topic later on, in the case when $\Gamma = \mathbf{Z}$. Given Wei's result computing the Borel complexity of full groups of measure-preserving equivalence relations mentioned above, which shows in particular that those are always Borel subsets of $\mathrm{Aut}(\mu)$, it is also natural to wonder how complicated a subset of $\mathrm{Homeo}(X)$ $[R_\Gamma]$ is; I do not know the answer in general, but for minimal $\mathbf{Z}$-actions the answer is that it is as complicated as possible.

Theorem 2.24 ([IM13]). *The full group of a minimal* $\mathbf{Z}$*-action on a Cantor space* X *is a coanalytic non Borel subset of* $\mathrm{Homeo}(X)$.

Let me try to give an idea of our approach. First, recall that a *tree* on a countable set A is a subset T of the set $A^{<\omega}$ of all finite sequences of elements of A which is closed under taking initial segments. In particular, any nonempty tree must contain the empty sequence. The space $\mathcal{T}(A)$ of all trees on A may be identified with a subset of $2^{A^{<\omega}}$ (identifying each tree with its indicator function); endowing $2^{A^{<\omega}}$ with the product topology, we obtain a compact topology on $\mathcal{T}(A)$. A tree is *well-founded* if it has no infinite branches; one can then define inductively the rank (relative to T) $\rho_T(s)$ of an element $s \in A^{<\omega}$ as follows:

$$\rho_T(s) = \sup\{\rho_T(s \frown a) + 1\colon s \frown a \in T\}$$

In particular, if s does not belong to T or is a terminal node of T, then $\rho_T(s) = 0$. The rank of T, $\rho(T)$, is the supremum of the ranks of all elements $s \in A^{<\omega}$; when T is nonempty this is equal to $\rho_T(\emptyset)$. The rank of a countable well-founded tree is a countable ordinal.

The reason why this is relevant for our purposes is the following observation. Fix (for the remainder of this section) a minimal homeomorphism φ of a Cantor space X and define, for any $g \in \mathrm{Homeo}(X)$, a tree $T(g)$ on the countable set $\mathrm{Clop}(X)$ of clopen subsets of X by the following condition:

$$((U_0, \dots, U_n) \in T(g)) \Leftrightarrow \Big(\forall i \leq n-1 \; U_{i+1} \subseteq U_i \text{ and } \forall i \leq n \; \forall x \in U_i \; g(x) \neq \varphi^{\pm i}(x)\Big) .$$

The map $g \mapsto T(g)$ is Borel (for the compact topology on the space of trees on $\mathrm{Clop}(X)$ described above), and $g \in [\varphi]$ iff $T(g)$ is well-founded.

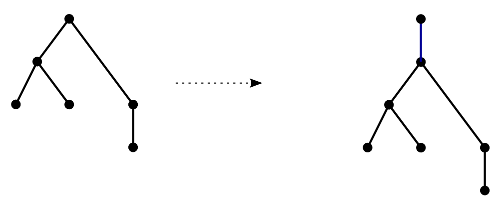
Note that $T(g)$ has finite rank if, and only if, there exists a finite clopen partition $U_0, \dots, U_n$ of X such that g coincides with a fixed power of φ on each U_i or, equivalently, if g belongs to $[\varphi]$ and $\{n \colon \exists x \in X \; g(x) = \varphi^n(x)\}$ is a finite subset of $\mathbf{Z}$. The set of all elements satisfying these conditions is a countable subgroup of $[\varphi]$, which is known as the *topological full group* of φ. Topological full groups of minimal homeomorphisms are important objects in their own right, though we will not say much about them (and not prove any results concerning them); let us simply point out the fact that the rank of $T(g)$ captures whether g belongs to the topological full group of φ as evidence that this rank is a natural and potentially useful invariant.

It is clear from the definition that $[\varphi]$ is co-analytic: $g \in \mathrm{Homeo}(X)$ does not belong to $[\varphi]$ iff

$$\exists x \in X \; \forall n \in \mathbf{Z} \; g(x) \neq \varphi^n(x).$$

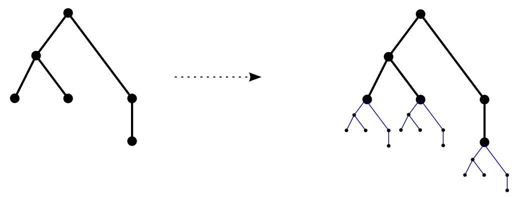
This shows that the complement of $[\varphi]$ is the projection of a G_δ subset of the Polish space $\mathrm{Homeo}(X) \times X$. By Suslin's theorem, $[\varphi]$ being Borel is then equivalent to it being analytic, in which case the set $\mathcal{T}_\varphi = \{T(g) \colon g \in [\varphi]\}$ is an analytic subset of the set of all well-founded trees. This is only possible if the ranks of elements of $\mathcal{T}_\varphi$ are bounded above by a common countable ordinal (this is a classical, non-trivial result of descriptive set theory, see [Kec95, 35.23]).

So we need to prove that the ranks of trees of the form $T(g)$ are not bounded above by a countable ordinal. The usual, simple technique to construct well-founded trees of arbitrarily large rank is to build them "from the root" - for instance, to obtain a tree of rank $\alpha + 1$ from a tree T of rank α, just copy T below a node that is linked to the root of the tree, as in the picture below.



Similarly, to obtain a tree of rank greater than $\sup(\alpha_n)$ from a countable family of trees of rank α_n, just link the root to countably many vertices, each of which is the root of a tree of rank α_n.

This procedure is not adapted to the way our trees $T(g)$ are defined, because changing something high up in the tree (i.e. modifying g on some clopen set U) forces one to also change everything below (g also is modified on any clopen subset contained in U) , thus completely modifying the tree, so one cannot simply copy things easily. This makes the work a bit painful, but also points to our salvation: it is, at least intuitively, possible to modify a tree of the form $T(g)$ "from the bottom" - namely, take a terminal node, and change what g does on the corresponding clopen set to make it as complicated as g is on the whole space. Roughly speaking, this corresponds to replacing a well-founded tree T with a new tree such that any terminal node of T is now the root of a new copy of T - thus increasing the rank. The corresponding picture now looks like this.



Note that just adding a vertex below each terminal node of T would not be enough: it would create a new tree of rank $1 + \rho(T)$, which might be equal to $\rho(T)$ if $\rho(T)$ is infinite, for instance $1 + \omega$ is equal to ω. The set of countable ordinals α such that $\beta + \alpha = \alpha$ for all $\beta < \alpha$ is unbounded, as is easy to see. So one really needs to copy a tree of rank at least $\rho(T)$ below each terminal node to be sure to increase the rank of T.

This intuitive idea can be turned into a (somewhat messy) proof, thus showing that $[\varphi]$ is not analytic. The sketch of proof we discussed seems to adapt easily to any countable group (only the definition of the trees $T(g)$ must be adapted, and this is is not hard), but I do not know if the actual proof can be made to work: our main technical tool to do the "copying" is a result of Glasner–Weiss stating that if A, B are clopen subsets of X such that $\mu(A) < \mu(B)$ for any φ-invariant measure μ, then there exists an element g in the topological full group of φ such that $g(A) \subset B$ (this is also what one needs to prove Theorem 2.22). No analogue of this is known in general, even for countable amenable groups.

This concludes our panorama of Polish groups, with the exception of one important example: the isometry group of the Urysohn space (and its variants), which we turn to now.

3. Urysohn spaces

3.1. **Construction of Urysohn spaces.** After proving that $\ell_\infty(\mathbf{N})$ contains an isometric copy of any separable metric space [Fré10], Fréchet [Fré25] asked the following question: does there exist a *separable* metric space with the same property? This provided the impetus for Urysohn's research and subsequent discovery of the space which now bears his name, published in the posthumous paper [Ury25]. Right after finishing the construction of this space, Urysohn drowned while on vacation in France with Alexandrov; [Ury25] was written by Alexandrov, who along with Brouwer wrote down a large part of Urysohn's work after his untimely death (see the introduction of [Huš08] for a detailed history of the discovery of the Urysohn space, Urysohn's death, and subsequent events; the special volume [LPR$^+$08] is a good general reference about the Urysohn space).

Banach and Mazur [Ban55] shortly thereafter found another space showing that the answer to Fréchet's question is positive: they proved that $C([0,1])$, the space of continuous functions on the unit interval, is isometrically universal. They even proved more, namely, every separable Banach space embeds *linearly* isometrically in $C([0,1])$, and this might have played a part in keeping the Urysohn space out of the spotlight, as a nearly-forgotten curiosity. But this space has another remarkable property: it is characterized up to isometry as being the unique Polish metric space which is both

- *universal*, i.e. **U** contains a copy of every separable metric space (that is the property Fréchet was interested in).
- *homogeneous*, i.e. any isometry between two *finite* subsets of **U** extends to a surjective isometry of **U**.

We will not present Urysohn's original construction; instead we discuss quickly a more recent one, due to Katětov [Kat88]. This construction played a large part in reviving interest in the study of the Urysohn space.

We begin with a convention: by an isometry between two metric spaces X, Y, we mean a *surjective*, distance-preserving map from X to Y. Distance-preserving maps which are not necessarily surjective will be called *isometric embeddings*.

Definition 3.1. Let (X, d) be a metric space. A *Katětov map* on (X, d) is a map $f\colon X \to \mathbf{R}^+$ such that

$$\forall x, y \in X \; |f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y).$$

We let $\mathcal{E}(X)$ denote the set of all Katětov maps on X.

These maps correspond to one-point metric extensions $X \cup \{z\}$ of X, via the correspondence $f(\cdot) \leftrightarrow d(z, \cdot)$. This correspondence was already known to Hausdorff.

One may check that **U** is characterized among Polish metric spaces by the following property, commonly known as *finite injectivity*:

$$\forall A \text{ finite } \subseteq \mathbf{U} \; \forall f \in \mathcal{E}(A) \; \exists z \in \mathbf{U} \; \forall a \in A \; d(z, a) = f(a).$$

In words: any abstract one-point metric extension of a finite subset of **U** is realized inside **U**.

As a way to get used to back-and-forth constructions, let us see why a finitely injective Polish metric space is homogeneous: assume that X is such a space, and that $\varphi\colon A \to B$ is an isometry between finite subsets of X. Let $\{x_i\}_{i<\omega}$ be a countable dense subset of X. Using finite injectivity, one can build inductively finite sets A_n, B_n and isometries $\varphi_n\colon A_n \to B_n$ with the following properties:

- $A_0 = A, B_0 = B, \varphi_0 = \varphi$.
- For all n, $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$ and φ_{n+1} extends φ_n.
- For all n, $x_n \in A_{2n+1}$ ("forth") and $x_n \in B_{2n+2}$ ("back").

Indeed, assume that the process has been carried out up to some rank n, say $n = 2k$ (the case n odd is essentially the same). If $x_k \in A_n$ we have nothing to do; else we may set $A_{n+1} = A_n \cup \{x_k\}$. The one thing we need is to define $\varphi_{n+1}(x_k)$; this must be an element $x \in X$ which satisfies

$$\forall y \in A_n \; d(y, x_k) = d(\varphi_n(y), x) \; .$$

Finite injectivity of X ensures that such an element exists; pick one, call it b_{n+1}, and set $B_{n+1} = B_n \cup \{b_{n+1}\}$.

After ω steps, $\cup\varphi_i\colon \cup A_i \to \cup B_i$ is a densely-defined isometry with dense range which extends φ, and completeness of X ensures that $\cup\varphi_i$ extends to an isometry of X onto itself. As is often the case, the "forth" step ensures that the map we build is defined everywhere, while the "back" step (which, in this case, is just the "forth" step applied to φ^{-1}) ensures that the map is onto.

Using similar ideas, it is not hard to prove that a finitely injective Polish metric space is also universal (here only the "forth" step of the construction is needed), and that any two finitely injective Polish metric spaces must be isometric. Thus, if such a space exists, it is unique up to isometry. It is also not hard to show that homogeneity and universality together imply finite injectivity.

All this reduces the proof of existence of the Urysohn space to the construction of a finitely injective Polish metric space. Katětov's approach is based on the existence of a natural metric on $\mathcal{E}(X)$. Before introducing this metric, we note that X naturally embeds in $\mathcal{E}(X)$ via $x \mapsto d(x,\cdot)$ (this is the degenerate case where we "extend" X by adding a point that was already inside it).

Definition 3.2. The metric on $\mathcal{E}(X)$ is defined by the formula

$$d(f,g) = \sup\{|f(x) - g(x)| \colon x \in X\}.$$

This is indeed a metric (in particular, it takes only finite values); in geometric terms, $d(f,g)$ is the *smallest* possible distance $d(z_f, z_g)$ in a two-point metric extension $X \cup \{z_f, z_g\}$ such that $d(z_f, x) = f(x)$ and $d(z_g, x) = g(x)$ for all $x \in X$.

The map $x \mapsto d(x,\cdot)$ is an isometric embedding from X to $\mathcal{E}(X)$ and in what follows we identify X with the corresponding subspace of $\mathcal{E}(X)$. Then, one has the remarkable relation

$$\forall f \in \mathcal{E}(X)\ \forall x \in X\ d(f,x) = f(x).$$

Unfortunately, $\mathcal{E}(X)$ need not be separable even when X is (see [Mel08] for a detailed discussion of the conditions on X which ensure that $\mathcal{E}(X)$ is separable). Still, all is not lost: to obtain a finitely injective space, we only care about one-point extensions of finite subspaces; and if $Y \subseteq X$ and $f \in \mathcal{E}(Y)$, then f may be extended to an element $\hat{f}$ of $\mathcal{E}(X)$ via the following formula ("shortest path through Y"):

$$\forall x \in X\ \hat{f}(x) = \inf\{f(y) + d(x,y) \colon y \in Y\}$$

This leads to the following definitions.

Definition 3.3. Let (X,d) be a metric space and $f \in \mathcal{E}(X)$. We say that f is *supported by $A \subseteq X$*, or that A is *a support of f*, if one has

$$\forall x \in X\ f(x) = \inf\{f(a) + d(x,a) \colon a \in A\}.$$

Definition 3.4. We denote by $\mathcal{E}(X,\omega)$ the subspace of all $f \in \mathcal{E}(X)$ which have a finite support.

By definition, for any finite $A \subseteq X$ and any $f \in \mathcal{E}(A)$, there exists $\hat{f} \in \mathcal{E}(X,\omega)$ extending f. Hence, if one wants to find an element having prescribed distances to a finite subset of X, then one might do so inside of $\mathcal{E}(X,\omega)$. It is straightforward to check that $\mathcal{E}(X,\omega)$ is separable; the natural embedding from X into $\mathcal{E}(X)$ takes its values in $\mathcal{E}(X,\omega)$, and any isometry φ of X uniquely extends to an isometry of $\mathcal{E}(X)$ defined by

$$\forall f \in \mathcal{E}(X,\omega)\ \forall x \in X\ \tilde{\varphi}(f)(x) = f(\varphi^{-1}(x)).$$

Then $\tilde{\varphi}$ uniquely extends to the completion $\widehat{\mathcal{E}(X,\omega)}$ (we take the completion here to stay inside the domain of Polish metric spaces, but this is inessential), and the homomorphism $\varphi \mapsto \tilde{\varphi}$ is continuous from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(\widehat{\mathcal{E}(X,\omega)})$.

Katětov's construction of the Urysohn space [Kat88] proceeds as follows: start from a given Polish metric space (X, d) and set $X_0 = X$. Then define inductively an increasing sequence of metric spaces X_i by setting $X_{i+1} = \mathcal{E}(X_i, \omega)$. Finally, denote by X_∞ the union of the X_is, and let Y be the completion of X_∞.

Then Y is a Polish metric space, and the construction ensures that Y is approximately finitely injective - that is, for any finite subset A of Y, any $\varepsilon > 0$ and any $f \in \mathcal{E}(Y)$, there exists $y \in Y$ such that $|d(y,a) - f(a)| \leq \varepsilon$ for all $a \in A$. Using completeness and an approximation process, one can prove that Y must actually be finitely injective, and we have built a Urysohn space.

This construction is fairly flexible, which is why the title of this section mentions Urysohn spaces, plural (see [DLPS07] for a full discussion of this flexibility): for instance, one could have done the previous construction using only metric spaces of diameter at most 1, obtaining in the limit the so-called Urysohn sphere, which is the unique Polish space of diameter 1 which is both universal for Polish metric spaces of diameter at most 1 and homogeneous.

We could have also stayed in the countable realm, considering only finite metric spaces whose metric takes only rational values, and used Katětov's tower construction (without taking a completion at the end!) to build the rational Urysohn space $\mathbf{U}_\mathbf{Q}$; this space is the unique countable metric space with rational distances which is homogeneous and universal for countable rational metric spaces. It was originally built by Urysohn, who then proved that its completion is isometric to the Urysohn space; whenever we mention $\mathbf{U}_\mathbf{Q}$ we think of it as sitting densely inside $\mathbf{U}$.

Here we see our first example of a phenomenon that will play an important role later on: a "continuous" structure (in this case, the Urysohn space) is well-approximated by a countable substructure (the rational Urysohn space); further, the *automorphism group* of the structure is well approximated by the automorphism group of the countable substructure, which is a Polish permutation group; for instance in this case, given any isometry φ of $\mathbf{U}$ and any $\varepsilon > 0$, there exists an isometry ψ of $\mathbf{U}_\mathbf{Q}$ such that $d(\varphi(x), \psi(x)) < \varepsilon$ for all x (see for instance [CV06]).

3.2. **Isometry groups of Urysohn spaces.** Uspenskij [Usp90] was the first to put to use a very nice property of Katětov's construction (the notations of which we keep here): any isometry of $X = X_0$ extends to an isometry of $X_1 = \mathcal{E}(X_0, \omega)$, which extends to an isometry of $X_2 = \mathcal{E}(X_1, \omega)$, etc., eventually defining an isometry of $\bigcup X_i$, which in turn extends to its completion, that is as we know isometric to the Urysohn space.

In this way, we obtain an isometric copy of X embedded in the Urysohn space $\mathbf{U}$, with the property that any isometry of X extends naturally to an isometry of $\mathbf{U}$, and the mapping that assigns to a isometry of X its natural extension to $\mathbf{U}$ is a continuous homomorphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(\mathbf{U})$. Since any Polish group is a subgroup of the isometry group of some Polish metric space (actually, any Polish group *is* the isometry group of some Polish metric space, see [GK03a]), this shows that any Polish group embeds, as a topological group, into the isometry group $\mathrm{Iso}(\mathbf{U})$. We just sketched the proof of the following theorem.

Theorem 3.5 (Uspenskij [Usp90]). $\mathrm{Iso}(\mathbf{U})$ *is a universal Polish group, i.e, it contains an isomorphic copy of any Polish group.*

This result rekindled interest in the Urysohn space, which is now a relatively well-known object, at least among logicians. Uspenskij [Usp04] subsequently proved that $\mathbf{U}$ is homeomorphic to the Hilbert space; many results have been proved over the past fifteen years or so, and rather than try to sum all of these up I will simply refer the reader to the special volume [LPR$^+$08] and references therein.

Theorem 3.6 ([Mel10b]). $\mathrm{Iso}(\mathbf{U})$ *is homeomorphic to a separable Hilbert space.*

This is a common feature in large infinite-dimensional groups. Due to a result of Toruńczyk and Dobrowolski [DT81], proving this reduces to showing that $\mathrm{Iso}(\mathbf{U})$ is an *absolute retract*, which in turn follows from the fact that its topology admits a basis which is stable under taking finite intersections, contains the whole space, and is such that all its elements have trivial homotopy type (see [vM89]). The proof is technical and I will not try to explain it here.

It was recently proved by Tent–Ziegler [TZ13a] that $\mathrm{Iso}(\mathbf{U}_1)$ is a simple group; actually, using model-theoretic methods inspired by stability theory, they proved that if $g \in \mathrm{Iso}(\mathbf{U}_1)$ and $n \in \mathbf{N}$ are such that there exists a satisfying $d(a, g(a)) \geq \frac{1}{n}$, then any element of $\mathrm{Iso}(\mathbf{U}_1)$ can be written as a product of at most $n \cdot 2^9$ conjugates of g and g^{-1}.

In the unbounded case, it is clear that $\mathrm{Iso}(\mathbf{U})$ is not simple: the group of *bounded isometries*, i.e. all isometries $g \in \mathrm{Iso}(\mathbf{U})$ such that $d(g(x), x) \leq M$ for some M and all x, is a nontrivial normal subgroup. Tent and Ziegler [TZ13b] showed that the quotient of $\mathrm{Iso}(\mathbf{U})$ by the subgroup of bounded isometries is simple: for any unbounded isometry $g \in \mathrm{Iso}(\mathbf{U})$, every other element of $\mathrm{Iso}(\mathbf{U})$ is a product of at most 8 conjugates of g. As far as I know, it is an open problem whether the group of bounded isometries is simple; it is not very hard to see that $\mathrm{Iso}(\mathbf{U})$ is topologically simple, i.e. has no nontrivial closed normal subgroups. I believe that this fact was first pointed out by K. Tent; at least, I heard it from her.

An interesting, and poorly understood so far, object is the uniform metric on $\mathrm{Iso}(\mathbf{U})$ (or its bounded counterpart $\mathbf{U}_1$), defined by

$$d_u(g, h) = \sup\{d(g(x), h(x)) \colon x \in \mathbf{U}\}$$

Of course this might take the value $+\infty$; replace this d_u by $\frac{d_u}{1+d_u}$ (with the convention $\infty/\infty = 1$) if that causes a philosophical problem. This uniform metric was studied in the last part of [BM13] (joint work with D. Bilge, part of his PhD thesis for which E. Jaligot and I were the advisors), the results of which were partly superseded by Tent–Ziegler's work. It was proved in that paper, using Baire category methods, that any element of $\mathrm{Iso}(\mathbf{U})$ (or $\mathrm{Iso}(\mathbf{U}_1)$) is a commutator and that for all $n \geq 2$ there exists an element g_n of order n in $\mathrm{Iso}(\mathbf{U})$ such that any other element of $\mathrm{Iso}(\mathbf{U})$ is a product of (at most) four conjugates of g_n. In the case of $\mathbf{U}_1$, we proved that there is a 2-Lipschitz homomorphism $F \colon (\mathbf{R}, |\cdot|) \to (\mathrm{Iso}(\mathbf{U}_1, d_u)$ which maps 1 to g_2; as an immediate corollary, one obtains that $(\mathrm{Iso}(\mathbf{U}_1), d_u)$ is path-connected (which of course also directly follows from simplicity of $\mathrm{Iso}(\mathbf{U}_1)$).

The above technique, while fairly successful in the bounded case, does not apply to the study of the group of bounded isometries of the Urysohn space $\mathbf{U}$; the main problem is that this group is not a Polish group (it is a dense, meager subgroup of $\mathrm{Iso}(\mathbf{U})$). The question of whether this group is simple remains open; more generally, there are many examples of Polish groups when one knows that there exists a maximal normal subgroup, which is meager, and investigating its normal subgroup structure is an interesting and challenging problem. In the case of $\text{Iso}(\mathbf{U})$, a natural candidate for a smaller normal subgroup is the path-connected component of 1 in $(\text{Iso}(\mathbf{U}), d_u)$, motivating the following result.

Theorem 3.7. *The group of bounded isometries of* $\mathbf{U}$ *is a geodesic space when endowed with the uniform metric (and in particular it is path-connected).*

Proof. Modulo some easy arguments, it is enough to prove that, for any $g \in \text{Iso}(\mathbf{U})$ with $d_u(g, 1) = 1$, there exists $h \in \text{Iso}(\mathbf{U})$ satisfying $d_u(h, 1) = \frac{1}{2}$ and $d(g(x), h(x)) = \frac{1}{2}$ for all $x \in \mathbf{U}$ (in particular, we have $d_u(g, h) = \frac{1}{2}$, and we have found a midpoint between g and 1 in $(\text{Iso}(\mathbf{U}), d_u)$, which is what we need here; the stronger condition makes the inductive construction that follows easier). To that end, we follow a back-and-forth construction, building an increasing sequence of finite subsets A_n, and isometric maps $h_n \colon A_n \to \mathbf{U}$ satisfying the following conditions:

(1) $\bigcup_n A_n$ and $\bigcup h_n(A_n)$ are dense, and h_{n+1} extends h_n for all n.
(2) For all n and all $a \in A_n$, $d(h_n(a), a) \leq \frac{1}{2}$ and $d(h_n(a), g(a) = \frac{1}{2}$.

For this construction to be possible, it is enough to be able to start with (A, h) satisfying the second point above, and any $x \in \mathbf{U}$, and extend h to $A \cup \{x\}$ in such a way that the second condition is still satisfied (note that this condition is symmetric in h and h^{-1}, so the back step and the forth step are essentially the same). So, we consider $f \in \mathcal{E}(h(A))$ defined by $f(h(a)) = d(x, a)$. We first check that we can realize f inside $\mathbf{U}$ by z which is such that $d(z, g(x)) = \frac{1}{2}$; note that, since the distance from z to each $h(a)$ is already prescribed, the set of possible values for $d(z, g(x))$ is an interval $[\alpha, \beta]$, with

$$\alpha = \max_{a \in A} |d(g(x), h(a)) - d(x, a)| = \max_{a \in A} |d(g(x), h(a)) - d(g(x), g(a))| .$$

Hence $\alpha \leq \max_{a \in A} d(h(a), g(a)) = \frac{1}{2}$; and similarly

$$\beta = \min_{a \in A} d(g(x), h(a)) + d(x, a) = \min_{a \in A} d(g(x), h(a)) + d(g(x), g(a)) \geq \frac{1}{2} .$$

So, it is indeed compatible with the triangle inequality to set $d(z, g(x)) = \frac{1}{2}$. Having done this, we still need to define the distance of z to x; we want to make it as small as possible, and for this the best we can do (since we already set $d(z, g(x)) = \frac{1}{2}$) is to set

$$d(z, x) = \max(\max_{a \in A}(|d(x, a) - d(x, h(a)|, |d(x, g(x) - \frac{1}{2}| .)$$

By assumption on h, we know that for all a we have $|d(x, a) - d(x, h(a))| \leq d(a, h(a)) \leq \frac{1}{2}$; and since $d(x, g(x))$ belongs to $[0, 1]$, we also have $|d(x, g(x) - \frac{1}{2}| \leq \frac{1}{2}$.

So, we can indeed find $z \in \mathbf{U}$ realizing f over $h(A)$, such that $d(z, g(x)) = \frac{1}{2}$ and $d(z, x) \leq \frac{1}{2}$. Setting $h(x) = z$, we are done. □

3.3. **Linearly rigid metric spaces.** We turn to the study of a surprising property of the Urysohn space; we are concerned here with isometric embeddings of metric spaces into Banach spaces. This type of problem goes back at least to Fréchet; as we saw above, he proved in [Fré10] that every separable metric space embeds isometrically in the Banach space $\ell_\infty(\mathbf{N})$. We also mentioned Banach–Mazur's theorem which states that $C([0,1])$ is isometrically universal. They even proved that any separable Banach space *linearly* isometrically embeds in $C([0,1])$; this fact actually follows from the existence of an isometric embedding, but this was proved much later by Godefroy and Kalton [GK03b].

A result analogous to Fréchet's theorem, due to Kuratowski, states that every metric space X embeds in $C_b(X)$, the Banach space of all continuous bounded functions on X endowed with the supremum norm. Such an embedding (often called *Kuratowski embedding*) is easy to describe: fix a basepoint $x_0 \in X$, and consider the map from X to $C_b(X)$ defined by

$$x \mapsto (f_x \colon y \mapsto d(y,x) - d(y,x_0)) .$$

The Kuratowski embedding above depends in a nontrivial way on the choice of basepoint x_0. Another possibility to define an embedding, which was apparently considered first by Kantorovitch [Kan42] in the context of compact metric spaces, and then in general by Arens-Eells [AE56], is to embed X in the so-called *Lipschitz-free Banach space over X*. Let us quickly recall one possible definition of this space. It depends formally on a choice of basepoint $x_0 \in X$; to simplify the notation below, denote by $Lip_0(X,x_0)$ (or just $Lip_0(X)$ when there is no danger of confusion) the space of all Lipschitz maps f on X such that $f(x_0) = 0$, and denote by $K(f)$ the Lipschitz constant of $f \in Lip_0(X,x_0)$ (note that $K(f)$ is a complete norm on $Lip_0(X,x_0)$).

Given $z_1,\dots,z_n \in X$ and $a_1,\dots,a_n \in \mathbf{R}$, define

$$\| \sum_{i=1}^{n} a_i z_i \|_L = \sup\{ \sum_{i=1}^{n} a_i f(z_i) \colon f \in Lip_0(X,x_0),\ K(f) \leq 1\}$$

This is indeed a seminorm on the vector space of all (formal) combinations of elements of X, identifying $x_0 \in X$ with the origin of that vector space. Taking the completion of that space, we obtain the Lipschitz-free Banach space $\mathcal{F}(X,x_0)$. This space is a predual of $Lip_0(X,x_0)$. Note that, in this case, the dependence on the choice of basepoint is inessential: if y_0 is another choice of basepoint, then the spaces $Lip_0(X,x_0)$ and $Lip_0(X,y_0)$ are isometric via the map $f \mapsto f(\cdot) - f(y_0)$, and this induces a canonical isometry of the predual spaces $\mathcal{F}(X,x_0)$ and $\mathcal{F}(X,y_0)$. Accordingly, in the following we shall denote this space simply by $\mathcal{F}(X)$.

For information about Lipschitz-free Banach spaces, we refer the reader to Weaver's book [Wea99] and Godefroy–Kalton's article [GK03b]. Even though it is only tangentially related to our concerns here, let us explicitly state the beautiful result of Godefroy and Kalton [GK03b] alluded to above, the proof of which uses Lipschitz-free Banach spaces: consider a separable Banach space X and a Banach space Y, and assume that there exists an isometric embedding from X into Y. Then there must exist a *linear* isometric embedding from X into Y. This theorem becomes false if one no longer assumes X to be separable.

A curious mind may then ask: can there exist a space X such that all these embeddings coincide? That is, do there exist metric spaces which can be embedded in a Banach space in a unique way (modulo a choice of basepoint)?

Definition 3.8. Let (X,x_0) be a pointed metric space. We say that X is *linearly rigid* if, whenever B,B' are Banach spaces and $\varphi \colon X \to B$, $\varphi' \colon X \to B'$ are isometric embeddings mapping x_0 to 0, one has:

$$\forall a_1, \dots, a_n \in \mathbf{R}\ \forall y_1, \dots, y_n \in X\ \|\sum_{i=1}^{n} a_i \varphi(y_i)\|_B = \|\sum_{i=1}^{n} a_i \varphi'(y_i)\|_{B'}$$

Note that, if X is linearly rigid, then any Banach space generated by X must coincide with $\mathcal{F}(X)$ under the natural identification, so the choice of basepoint is again inessential.

The question of existence of linearly rigid spaces does not seem to have been considered until an example was found by M. Randall Holmes [Hol92]. Working on a question of Sierpinski [Sie45] concerning isometric embeddings of the Urysohn space in $C([0,1])$, he proved the following remarkable result (reformulated to fit our terminology).

Theorem 3.9 (Holmes [Hol92])**.** *The Urysohn space is linearly rigid.*

The original proof of that result is rather intricate and difficult to follow, in large part because Holmes was concerned with Sierpinski's question, and not linear rigidity. The curious reader may consult [Hol08] to read Holmes' account of his proof, how it came about, and the intuition behind it; his result seems to have gone largely unnoticed for fifteen years or so [i]. Then, as interest in the properties of the Urysohn space grew, Holmes' paper was finally noticed and studied (L. Nguyen Van Thé seems to have played a major part in popularizing Holmes' result; it is him who told me about it. I think he is also the originator of the terminology *Holmes space* to denote the unique Banach space spanned by an isometric copy of the Urysohn space) and a natural question appeared: can one give a characterization of linearly rigid metric spaces?

Such a characterization was obtained, simultaneously and independently, by F. Petrov and V. Vershik on one side, and myself on the other side; this led to the publication of a joint paper [MPV08], where our two proofs are presented. Below, I will quickly discuss "my" version of this proof (improved by an anonymous referee). That proof came about by analyzing and simplifying the arguments of [Hol92].

Definition 3.10. Let (X, x_0) be a pointed metric space and $f \in Lip_0(X, x_0)$. Let $F = \{x_0, x_1, \dots, x_n\} \subseteq X$, and f in the unit ball B_F of $Lip_0(F, x_0)$. We say that f is *extremal* if f is an extreme point of B_F.

Remark 3.11. To understand what this means in terms of metric geometry, note that extremality of f is equivalent to the fact that, up to reindexing F, there exists $j \le n$ such that one of the following things happens:

- $f(x_i) = d(x_0, x_i)$ for all $i \le j$, and $f(x_i) = \sup\{d(x_0, x_k) - d(x_i, x_k) \colon k \le j\}$ for all $i > j$.
- $f(x_i) = -d(x_0, x_i)$ for all $i \le j$, and $f(x_i) = \inf\{-d(x_0, x_k) + d(x_i, x_k) \colon k \le j\}$ for all $i > j$.

The first line means that f takes values that are as large as possible (given that f is 1-Lipschitz and $f(x_0) = 0$) on $x_1, \dots, x_j$, and then as small as possible (given the first j values) on $x_{j+1}, \dots, x_n$; the second line means that $-f$ satisfies that condition.

[i] at the time of writing, there are 13 papers citing [Hol92] in the MathSciNet database. The earliest of those was published in 2007.

We are ready to state our characterization of linearly rigid metric spaces.

Theorem 3.12 ([MPV08], Theorem 2)**.** *A pointed metric space X is linearly rigid if and only if it satisfies the following condition:*

For all finite $F \subseteq X$, and all extremal *$f \in Lip_0(F)$, there exist $c \geq 0$ and $z \in X$ such that*

$$\forall x \in F\ d(z,x) = c + f(x)\ .$$

It it immediate from the above theorem that **U** is linearly rigid, indeed we see that linear rigidity has a Urysohn-type flavor. We also see, however, that there are many different examples besides **U**: for instance, the integer-valued Urysohn space $\mathbf{U}_Z$ is also linearly rigid, and one can cook up many different examples using the above characterization and a Katětov-inspired construction.

Let us mention a curious byproduct of the proof. Recall that for $x \in X$ we denote by f_x the 1-Lipschitz map defined by $f_x(y) = d(x,y) - d(x,x_0)$. Then, we let $f_{x,y} = \frac{f_x - f_y}{2}$. These maps are again 1-Lipschitz, and one may define another isometric embedding of X in a Banach space by first setting

$$\|\sum a_i x_i\| = \sup\{\sum a_i f_{x,y}(x_i) \colon x, y \in X\}$$

and then taking the completion of that normed space. We call this embedding the *two-point embedding*, and the corresponding norm the *two-point norm*. Clearly, if X is linearly rigid, then the two-point norm and the Lipschitz-free norm must coincide. Surprisingly, the converse turns out to be true (this is a corollary of the proof of Theorem 3.12 which we present below).

Theorem 3.13 ([MPV08])**.** *A metric space X is linearly rigid if, and only if, the two-point norm and the Lipschitz-free norm coincide.*

Hence, to show that *all* possible norms coincide, one simply must show that *two* of them, explicitly defined, coincide. Since the Lipschitz-free norm $\|\cdot\|_L$ is the largest possible norm defining an isometric embedding of X in a Banach space (this is clear from the Hahn–Banach theorem), its appearance is not surprising. The role of the two-point norm is more mysterious; in particular, it does not seem to be a "minimal" compatible norm in any reasonable sense of the word.

A word to the wise: there exist spaces such that the norm corresponding to the Kuratowski embedding $x \mapsto f_x$ and the Lipschitz-free norm coincide, yet are not linearly rigid. This is why we had to consider the two-point norm above; actually it is obvious that any 3-point metric space is an example of this phenomenon. My co-authors also built a family of examples on 4 points, and we conjectured that there are no other finite examples, which has since been confirmed by Zatitskiĭ [Zat10].

The referee of [MPV08] gave a very nice interpretation of the argument I used to prove Theorem 3.12. The following version of the statement is due to him (her?), as is the functional-analytic proof below, most notably the use of Milman's theorem instead of a cumbersome computation. I am very grateful to the referee for that nice argument; not knowing his/her name it is unfortunately impossible to give proper credit. Unfortunately, this argument did not make it into the published paper, which is part of the reason why I decided to discuss linearly rigid spaces in detail here: I feel that this is the "right" proof of the result, and to my knowledge it was never made publicly available.

Theorem 3.14. *Let B denote the unit ball of $Lip_0(X)$, and D denote the weak* closure of $\{f_x : x \in X\}$. Then the following statements are equivalent:*

(i) *X is linearly rigid.*
(ii) *B is the weak* closed convex hull of D.*
(iii) *The extreme points of B are contained in D.*

It is not hard to see, using the Milman and Krein-Milman theorems, that (ii) and (iii) are equivalent, and that both are equivalent to the criterion appearing in Theorem 3.12 (it is worth noting here that each f_x is an extreme point of B).

Proof. The proof uses in an essential way the fact that $Lip_0(X)$ is the dual of $\mathcal{F}(X)$ (via the natural identification : $\langle f, \sum a_i x_i \rangle = \sum a_i f(x_i)$).

Let us begin by proving that (i) implies (iii). Since X is linearly rigid, the norm on $\mathcal{F}(X)$ must coincide with the two-point norm. Then we must have, for all $\varphi \in \mathcal{F}(X)$, that

$$\sup\{\langle f, \phi \rangle : f \in B\} = \sup\{\langle \frac{f_x - f_y}{2}, \phi \rangle : x, y \in X\}$$

This means (via a standard application of the Hahn–Banach theorem) that the closed convex hull of $\frac{1}{2}(D - D)$ is equal to B, and then Milman's theorem (see e.g [Die84, p.151]) implies that the set of extreme points of B must be contained in $\frac{1}{2}(D - D)$. Since each f_a is an extreme point, we see that $f_a \in -D$, hence $\frac{1}{2}(D - D)$ is a subset of the convex hull of D, and (iii) holds.

Now, let us see why (ii) implies (i). To that end, let X be isometrically embedded in a Banach space Z in such a way that x_0 is mapped to 0 and Z is the closed linear span of X. We identify X with its image in Z (and x_0 with $0 \in Z$).

Denoting by δ_x the element corresponding to x in the natural embedding of X in $\mathcal{F}(X)$, we must show that the map $T : \delta_x \mapsto x$ is an isometry. This is equivalent to showing that its adjoint map is a surjective isometry; in other words, we want to prove that every 1-Lipschitz map f on X such that $f(x_0) = 0$ is the restriction to X of some z^* belonging to the unit ball of Z^*. Since we are assuming that (ii) holds, we must simply show that this is true for every f_x, $x \in X$.

Fix $x \in X$. Since $-f_x$ is extremal, we must have $-f_x \in D$. Fix some finite $F \subseteq X$ containing 0, and $\varepsilon > 0$; we may find $y \in X$ such that

$$\forall z \in F \cup \{x\} \ |f_x(z) + f_y(z)| \leq \varepsilon.$$

Applying this to $z = x$, we obtain $\|x\| + \|y\| \leq \|x - y\| + \varepsilon$ (recall that $f_x(z) = \|z - x\| - \|x\|$).

Hence we have, for all $z \in F$:

$$\|z - x\| + \|z - y\| \leq \varepsilon + \|x\| + \|y\| \leq \|x - y\| + 2\varepsilon.$$

Consequently,

$$\forall z \in F \ \|z - x\| + \|z - y\| - \|x - y\| \leq 2\varepsilon.$$

Using the Hahn–Banach theorem, we may pick $\varphi_F \in Z^*$ such that $\|\varphi_F\| = 1$ and $\varphi_F(y - x) = \|x - y\|$.

We claim that $|\varphi_F(z) - f_x(z)| \leq 2\varepsilon$ for all $z \in F$. To see this, we use the fact that φ_F is 1-Lipschitz, linear, and that points of F look like they are "between" x and y. We have $\varphi_F(z) \leq \varphi_F(x) + \|z - x\|$, which yields

$$\varphi_F(z) - f_x(z) \leq \varphi_F(x) + \|x\|.$$

Similarly, $\varphi_F(z) \geq \varphi_F(y) - \|y - z\|$, and this gives

$$\varphi_F(z) - f_x(z) \geq \varphi_F(y) - \|z - y\| + \|x\| - \|z - x\|.$$

Hence we have

$$\begin{aligned}\varphi_F(z) - f_x(z) &\geq (\varphi_F(y - x) - \|z - y\| - \|z - x\|) + \|x\| + \varphi_F(x) \\ &= (\|y - x\| - \|z - y\| - \|z - x\|) + \|x\| + \varphi_F(x) \\ &\geq \|x\| + \varphi_F(x) - 2\varepsilon\end{aligned}$$

We have obtained the following inequalities, valid for any $z \in F$:

$$\|x\| + \varphi_F(x) - 2\varepsilon \leq \varphi_F(z) - f_x(z) \leq \|x\| + \varphi_F(x)\ .$$

This is in particular true for $z = 0$, so that $0 \leq \|x\| + \varphi_F(x) \leq 2\varepsilon$, and we have proved as promised that, for any finite $F \subseteq X$ and any $\varepsilon > 0$, we may find $\varphi_F \in Z^*$ with $\|\varphi_F\| = 1$ and such that $|\varphi_F(z) - f_x(z)| \leq 2\varepsilon$ for all $f \in F$. Using the compactness of the unit ball of Z^* for the weak topology, we obtain $\varphi \in Z^*$ of norm 1 and such that $\varphi_{\restriction X} = f_x$. □

The unique Banach space spanned by an isometric copy of the Urysohn space seems to be known now as the *Holmes* space, in honor of M.R Holmes. A consequence of the Godefroy–Kalton theorem mentioned above is that this space is linearly isometrically universal for all separable Banach spaces; it would be interesting to know more about its geometry, but the definition makes it hard to approach, and Lipschitz-free Banach spaces are notoriously difficult to understand.

One can use our characterization to show that no bounded metric space can be linearly rigid; as a consequence, the Urysohn sphere is not linearly rigid. Still, from the explicit computations used in my original proof, one sees that it is in some sense "locally" rigid: given any finite set A of sufficiently small diameter (an explicit constant can be computed, $1/10$ works, for instance), the norm of any linear combination of elements of A is uniquely determined.

4. BAIRE CATEGORY IN THE SPACE OF ACTIONS

We now turn to a different type of question, viewing actions of countable groups on some structures via the prism of Baire category. We fist concern ourselves with the problem of extending a measure-preserving action of a subgroup to a measure-preserving action of a larger group, and explain how it fits into a more general framework.

Answering a question of Halmos, Ornstein proved in [Orn72] that there exist elements of $\mathrm{Aut}(\mu)$ without a square root. The proof involved the construction of aperiodic transformations which only commute with their powers; it is clear that such transformations cannot have roots of any order. Different examples, with uncountable centralizer, were subsequently found [FGK88]. Still the question remained: is this a generic phenomenon? Or does a generic element of $\mathrm{Aut}(\mu)$ admit a square root? King [Kin00] provided a positive answer to that question; his proof is fairly long and technical, but was made considerably more accessible shortly thereafter by de la Rue and de Sam Lazaro [dlRdSL03], who built on King's ideas to show that a generic element of $\mathrm{Aut}(\mu)$ embeds in a flow.

The search for n-th roots is part of a more general type of problems. Indeed, consider a countable group Γ, and a subgroup $\Delta \leq \Gamma$. It is natural to wonder whether a Δ-action on (X, μ) (or any other mathematical structure) can be extended to a

Γ-action. If $\Gamma = \mathbf{Z}$ and $\Delta = 2\mathbf{Z}$, this is the same question as asking whether the generator of a given $2\mathbf{Z}$-action admits a square root. The Baire-category version of that question also makes sense once one has introduced the right definitions, which we recall now.

Given a Polish group G, and a countable group Γ, the space $\mathrm{Hom}(\Gamma, G)$ of all homomorphisms from Γ to G is a closed subspace of G^Γ, thus is a Polish space in its own right. When G is the automorphism group of some mathematical structure, $\mathrm{Hom}(\Gamma, G)$ coincides with the *space of actions* of Γ on that structure. One can consider Baire category notions inside this space; it will be important for us that G acts naturally on $\mathrm{Hom}(\Gamma, G)$ by conjugacy:

$$(g \cdot \pi)(\gamma) = g\pi(\gamma)g^{-1} .$$

When $\Gamma = F_n$ is a free group on n generators, $\mathrm{Hom}(\Gamma, G)$ may be identified with G^n; when $\Gamma = \mathbf{Z}^d$, $\mathrm{Hom}(\Gamma, G)$ can be identified with the set $\mathbf{C}_n(G)$ of all commuting n-tuples of elements of G:

$$\mathbf{C}_n(G) = \{(g_1, \dots, g_n) \in G^n \colon \forall i, j\ g_i g_j = g_j g_i\} .$$

Under these identifications, the conjugacy action of G on $\mathrm{Hom}(\Gamma, G)$ coincides with the diagonal conjugacy action of G on G^n and $\mathbf{C}_n(G)$:

$$g \cdot (g_1, \dots, g_n) = (gg_1g^{-1}, \dots, gg_ng^{-1}) .$$

Below, we will use the notation $\mathbf{C}(G)$ to denote $\mathbf{C}_2(G)$, i.e. the set of commuting couples of elements of G. We also denote by $\mathbf{C}(g)$ the centralizer of an element g of G; whenever A is a subset of a Polish group G, we let $\langle A \rangle$ denote the *closed* subgroup generated by A.

Before discussing in more detail questions related to extensions of generic actions, and studying generic properties of monothetic subgroups, we need to expand our Baire category toolbox.

4.1. **Some more Baire category notions.**

Definition 4.1. Let X be a topological space, and G be a group acting on X by homeomorphisms. The action is said to be *topologically transitive* if, for any nonempty open subsets U, V of X, there exists $g \in G$ such that $gU \cap V \neq \emptyset$.

When X is a second-countable Baire space, topological transitivity of the action $G \curvearrowright X$ is equivalent to the existence of a dense G-orbit: indeed, the assumption of topological transitivity is the same as saying that, for any nonempty open $U \subseteq X$, the set $\{x \colon G \cdot x \cap U \neq \emptyset\}$ is dense, and this set is open since the action is by homeomorphisms. Thus, taking the intersection of all these sets as U ranges over a basis for the topology of X, one obtains a dense G_δ set, each element of which has a dense orbit.

Using the fact that a Baire-measurable, non meager subset of a Polish space X must be comeager in a nonempty open set, one obtains the following fact.

Theorem 4.2 (first $0-1$ topological law)**.** *Let X be a Polish space, and $G \curvearrowright X$ be a topologically transitive action. Then, any Baire-measurable, conjugacy-invariant subset A of X is either meager or comeager.*

Whenever G is a topological group acting on a topological space X, we will make the assumption that the maps $g \mapsto g \cdot x$ and $x \mapsto g \cdot x$ are continuous; when G and X are Polish, this is equivalent to the map $(g, x) \mapsto g \cdot x$ being continuous (see e.g. [Kec95, 9.14]). From the first $0-1$ topological law, we see that if G is a Polish group acting continuously and topologically transitively on a Polish space X, then the G-orbits are either meager or comeager (orbits are clearly analytic, thus Baire-measurable; actually orbits are Borel in this setting but we do not need this here).

Now seems like a good time to mention an important result of Effros [Eff65].

Theorem 4.3 (Effros [Eff65])**.** *Let X be a Polish space, and G be a Polish group acting continuously on X. Then the following are equivalent, for any $x \in X$:*

(1) *$G \cdot x$ is comeager in $\overline{G \cdot x}$.*
(2) *$G \cdot x$ is a G_δ subset of X.*
(3) *The map $g \mapsto g \cdot x$ is an open map from G to $G \cdot x$.*

The fact that the third item above implies the second is a consequence of a theorem of Hausdorff stating that a continuous, open, metrizable image of a Polish space is also Polish.

Now, let us come back to our first concern in this section: given countable groups $\Delta \leq \Gamma$ and a Polish group G, does a generic Δ-action extend to a Γ-action? In the cases we will consider, the action $G \curvearrowright \mathrm{Hom}(\Delta, G)$ has a dense orbit; we are asking whether the image of the restriction map $\mathrm{Res}\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Delta, G)$ is comeager and, since this set is analytic and conjugacy-invariant, this is equivalent to proving that it is not meager.

A common approach, popularized by King [Kin86], is via the so-called *Dougherty lemma*.

Definition 4.4. Let X, Y be topological spaces, and $f\colon X \to Y$ be a continuous map. An element $x \in X$ is said to be a *point of local density* for f if, for any neighborhood U of x, $f(x)$ belongs to the interior of $\overline{f(U)}$.

Lemma 4.5 ("Dougherty's lemma")**.** *Assume that X, Y are Baire topological spaces, $f\colon X \to Y$ is continuous and the set of elements of X which are points of local density for f is dense. Then $f(X)$ is not meager.*

As a partial converse, if one assumes additionally that X is second-countable, the image of $\{x \in X\colon x \text{ is not a point of local density of } f\}$ is meager; thus points of local density must exist for $f(X)$ to be nonmeager.

Definition 4.6 ([MT13b])**.** Let X, Y be Polish spaces. We say that $f\colon X \to Y$ is *category-preserving* if it satisfies one of the following equivalent conditions:

(1) For any comeager $A \subseteq Y$, $f^{-1}(A)$ is comeager.
(2) For any nonempty open $U \subseteq X$, $f(U)$ is not meager.
(3) For any nonempty $U \subseteq X$, $f(U)$ is somewhere dense.
(4) $\{x \in X\colon x \text{ is a point of local density of } f\}$ is dense in X.

This definition was introduced in a joint work with T. Tsankov [MT13b], using only the first three items of the list above; the equivalence with the fourth item (and the fact that this notion was fairly classical) was only noticed in [Mel12]. The term "category-preserving" is meant to recall the classical notion of "measure-preserving"maps. This choice is motivated by the following result, which is to the measure disintegration theorem the same as the Kuratowski–Ulam theorem is to the Fubini theorem.

Theorem 4.7 ([MT13b]). *Let X, Y be Polish spaces, and $f\colon X \to Y$ be a category-preserving map. Let also A be a subset of X with the property of Baire. Then the following assertions are equivalent:*

(1) *A is comeager in X.*
(2) *$\{y\colon A \cap f^{-1}(y)$ is comeager in $f^{-1}(y)\}$ is comeager in Y.*

Using symbols:

$$(\forall^* x \in X\ A(x)) \Leftrightarrow (\forall^* y \in Y\ \forall^* x \in f^{-1}(y)\ A(x)) .$$

This result seems to have been formulated for the first time in [MT13b], which is a bit surprising since it is both useful (as we will soon see) and not very hard to prove. Particular cases of if appear in various places in the literature. The proof works by showing that, if $f\colon X \to Y$ is a category-preserving map from a Polish space to another, then there exists a dense G_δ subset A of X such that $f\colon A \to f(A)$ is open; $f(A)$ must be comeager since f is category-preserving, and is Polish since it is a continuous, open image of a Polish space. Noting that the proof of the classical Kuratowski–Ulam theorem as presented for instance in [Kec95] extends to continuous, open maps between Polish spaces, one obtains the desired result.

4.2. **Centralizers of generic elements.** The most basic infinite, countable group is certainly $\mathbf{Z}$; understanding actions of $\mathbf{Z}$ on some structure is of course the same thing as understanding elements of the automorphism group of that structure. In this section we describe an approach to proving that, in some Polish groups, centralizers of generic elements are as small as possible. This phenomenon first appeared in work of Chacon–Schwartzbauer [CS69], who proved that, for a generic $g \in \mathrm{Aut}(\mu)$, the centralizer of g coincides with $\langle g \rangle$ - in other words, a generic monothetic subgroup of $\mathrm{Aut}(\mu)$ is maximal abelian (recall that we only consider *closed* subgroups here). It is easy to prove that the same is true in $U(\mathcal{H})$, using spectral theory; T. Tsankov and I also managed to establish the same result for $\mathrm{Iso}(\mathbf{U})$ [MT13b]. Let us discuss a general simple approach that can be used to prove this type of result for a Polish group G.

Our approach uses properties of category-preserving maps; crucially, we establish that the restriction map $P\colon \mathbf{C}(G) \to G$ (identified with the projection on the first coordinate) preserves category as soon as

$$\{(g,h)\colon \langle g \rangle = \langle g,h \rangle\} \text{ is dense in } \mathbf{C}(G) . \tag{4.1}$$

Indeed, assume (4.1) holds, and let O be dense and open in G. Let U be a nonempty open subset of $\mathbf{C}(G)$; we may find $(g_1, g_2) \in U$ such that $\langle g_1 \rangle = \langle g_1, g_2 \rangle$, so for some n we have $(g_1, g_1^n) \in U$. Using the density of O and the continuity of group operations, there must exist g close to g_1, belonging to O, and such that $(g, g^n) \in U$. This shows that $P^{-1}(O)$ is dense. Hence, assuming (4.1), the restriction map from $\mathbf{C}(G)$ to G preserves category.

Clearly, (4.1) holds as soon as the set of all (g,h) generating a finite cyclic group with g as a generator is dense, or (since the map $(g,h) \mapsto (gh,h)$ is a homeomorphism of $\mathbf{C}(G)$) as soon as

$$\{(g,h)\colon g,h \text{ have coprime finite orders}\} \text{is dense in } \mathbf{C}(G). \tag{4.2}$$

This property is satisfied both for $G = \mathrm{Aut}(\mu)$ and $G = \mathrm{Iso}(\mathbf{U})$; this is a consequence of a multi-dimensional version of Rokhlin's lemma in the first case [Con73], and of a modification of a construction of Pestov and Uspenskij in the second case (see [PU06] for the original result and [MT13b] for the required modification). Note in passing that (4.2) is very unlikely to hold in a permutation group, so this technique can only be successfully applied in "continuous" structures.

Let us explain now why (4.1) implies that the centralizer of a generic element coincides with the closure of its powers; assume that G is a Polish group satisfying the condition. Then we have

$$\forall^*(g,h) \in \mathbf{C}(G)\ h \in \langle g \rangle \ .$$

From this, and the fact that the restriction map is category-preserving, we deduce that

$$\forall^* g \in G\ \forall^* h \in \mathbf{C}(g)\ h \in \langle g \rangle \ .$$

The above sentence says that, for a generic $g \in G$, the closed subgroup generated by g is comeager in the centralizer of g - thus the two must coincide, and we have proved that the centralizer of g coincides with $\langle g \rangle$. Let us sum up.

Theorem 4.8 (reformulation of ideas from [MT13b])**.** *Let G be a Polish group such that $\{(g,h) \in \mathbf{C}(G) \colon \langle g \rangle = \langle g,h \rangle\}$ is dense in $\mathbf{C}(G)$. Then the centralizer of a generic element of G coincides with the closed subgroup it generates - in other words, a generic monothetic subgroup of G is maximal abelian.*

The groups $\mathrm{Aut}(\mu)$, $U(\mathcal{H})$ and $\mathrm{Iso}(\mathbf{U})$ all satisfy these conditions.

4.3. **Extensions of generic actions.** Fix a Polish group G, a countable group Γ and a subgroup $\Delta \leq \Gamma$. As we saw, one cannot expect in general that any element of $\mathrm{Hom}(\Delta, G)$ extends to an element of $\mathrm{Hom}(\Gamma, G)$ - for instance, we saw that there exist elements of $\mathrm{Aut}(\mu)$ without square roots, and I proved in [Mel08] that the same is true for $\mathrm{Iso}(\mathbf{U})$. Here we focus on the question of whether generic elements of $\mathrm{Hom}(\Delta, G)$ can be extended to generic elements of $\mathrm{Hom}(\Gamma, G)$; we describe a way to tackle this kind of problem when Γ is abelian, and G is a Polish group such that there exist dense conjugacy classes in $\mathrm{Hom}(\Gamma, G)$ for any abelian Γ. This problem usually reduces fairly easily to the case when Γ is finitely generated, simply because open sets in $\mathrm{Hom}(\Gamma, G)$ only impose conditions on finitely many elements of Γ, so we add the assumption that Γ is finitely generated. Then one can use the structure theory of finitely generated abelian groups to decompose the problem into easier sub-problems.

First, one needs to understand the case when Δ is finite; for $G = \mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$, this is easy: there exists an element π_0 with comeager conjugacy class in $\mathrm{Hom}(\Delta, G)$ for any finite abelian Δ, and this element can be extended to an action of any finitely generated abelian supergroup of Δ. From this one obtains that the restriction map $\mathrm{Res} \colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Delta, G)$ is category-preserving: given any nonempty open U in $\mathrm{Hom}(\Gamma, G)$, the Effros theorem implies that $\mathrm{Res}(U) \cap G \cdot \pi_0$ is open and nonempty in the comeager set $G \cdot \pi_0$, so it cannot be meager in $\mathrm{Hom}(\Delta, G)$.

Next comes the case where $\Delta = n\mathbf{Z}$ and $\Gamma = \mathbf{Z}$; as we saw when discussing King's theorem, we are asking whether the map $g \mapsto g^n$ is category-preserving. To my knowledge, we lack efficient general techniques to solve this type of question; King's theorem shows that this property holds for $G = \mathrm{Aut}(\mu)$, and it is an open problem for $G = \mathrm{Iso}(\mathbf{U})$.

Then, one needs to understand what happens when $\Delta = \mathbf{Z}^d$, $\Gamma = \mathbf{Z}^k$. Using the same argument as when we studied the restriction map from $\mathbf{C}(G)$ to G, one sees that the restriction map from actions of $\mathbf{Z}^k$ to actions of $\mathbf{Z}^d$ preserves category as soon as

$$\{(g_1,\dots,g_k)\colon \langle g_1,\dots,g_k\rangle = \langle g_1\rangle\} \text{ is dense in } \mathbf{C}_k(G)\ . \tag{4.3}$$

If (4.3) is satisfied, then a similar line of reasoning enables us to deal with the case of $\mathrm{Res}\colon \mathrm{Hom}(\mathbf{Z}^k \times F, G) \to \mathrm{Hom}(\mathbf{Z}^d \times F, G)$ for $d \le k$ and F a finite abelian group (we skip the details).

We turn to the case $\mathbf{Z}^d \le \mathbf{Z}^d \times F$. An obvious necessary condition for an action π of $\mathbf{Z}^d$ to extend to an action of $\mathbf{Z}^d \times F$ is that there exists a copy of F in the centralizer of $\pi(\mathbf{Z}^d)$; thus this has to be true for a generic $\pi \in \mathrm{Hom}(\mathbf{Z}^d, G)$ if we are to hope that the restriction map $\mathrm{Res}\colon \mathrm{Hom}(\mathbf{Z}^k \times F, G) \to \mathrm{Hom}(\mathbf{Z}^d, G)$ is category-preserving. It follows from the conditions we have imposed thus far on our group that a generic $\overline{\pi(\mathbf{Z}^d)}$ is the same as a generic $\overline{\pi(\mathbf{Z})}$, so we need to know that a generic $\overline{\pi(\mathbf{Z})}$ contains infinitely many elements of order n for all $n \ge 2$. This is known to hold for $\mathrm{Aut}(\mu)$ [SE04]. Using the fact that a generic action in $\mathrm{Hom}(\mathbf{Z}^k, G)$ is free ergodic, and that the conjugacy class of such an action is dense (see e.g. [Kec10] for details), one sees that the image of Res being nonmeager is enough to ensure that Res is category-preserving in that case.

The final step is to understand $\mathbf{Z}^d \times \mathbf{F}_1 \le \mathbf{Z}^d \times F_2$, where $F_1 \le F_2$ are finite abelian groups. It is not hard (using Theorem 4.7) to see that the corresponding restriction map preserves category as soon as a generic $\overline{\pi(\mathbf{Z}^d)}$ is divisible, which is equivalent under our current assumptions to a generic $\overline{\pi(\mathbf{Z})}$ being divisible. This follows from assumptions we already made on the group, namely that the restriction map from $\mathbf{Z}^2$ to $\mathbf{Z}$ preserves category, and that a generic element admits roots of any order. As another example of application of Theorem 4.7, let us give details of this proof. Our starting assumption is that, for any integer n,

$$\forall^* g \in G\ \exists f \in G\ g = f^n\ .$$

We know that, for a generic pair $(g,h) \in \mathbf{C}(G)$, we have $\langle g,h\rangle = \langle g\rangle = \langle h\rangle$, so we can write

$$\forall^*(g,h) \in \mathbf{C}(G)\ \mathbf{C}(g) = \mathbf{C}(h) = \langle g\rangle \text{ and } \exists f \in G\ h = f^n$$

The f in the above line must belong to $\mathbf{C}(h)$, and using category-preservation we may write the above line as

$$\forall^* g \in G\ (\forall^* h \in \langle g\rangle\ \exists f \in \langle g\rangle\ h = f^n)\ .$$

Thus, for a generic g, the homomorphism $f \mapsto f^n$ of the abelian Polish group $\langle g\rangle$ has a comeager range, hence it is surjective, proving that $\langle g\rangle$ is divisible.

Let us sum up the properties of G that were used to prove that the restriction map $\mathrm{Res}\colon \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu)) \to \mathrm{Hom}(\Delta, \mathrm{Aut}(\mu))$ preserves category for any pair of finitely-generated abelian groups $\Delta \le \Gamma$:

- $\{(g_1,\dots,g_k)\colon \langle g_1\rangle = \langle g_1,\dots,g_k\rangle\}$ is dense in $\mathbf{C}_k(G)$ for all k.
- The map $g \mapsto g^n$ preserves category for all $n \ge 1$ (which, along with the previous assertion, is enough to obtain that a generic $\overline{\pi(\mathbf{Z})}$ is divisible and coincides with the centralizer of $\pi(1)$)
- The centralizer of a generic element contains a copy of any finite abelian group (equivalently it contains infinitely many elements of order n for any integer $n \ge 2$).

For a general Polish G, the third condition might be too weak to show that the restriction map $\mathrm{Res}\colon \mathrm{Hom}(\mathbf{Z}^d \times F, G) \to \mathrm{Hom}(\mathbf{Z}^d, G)$ preserves category for every finite F. I believe that in "natural" cases this condition (along with the two others) should be sufficient.

These three conditions all hold for $\mathrm{Aut}(\mu)$, and the first is known to hold for $\mathrm{Iso}(\mathbf{U})$ while the other two are open. These questions essentially reduce to understanding the maps $g \mapsto g^n$ in $\mathrm{Iso}(\mathbf{U})$; a proof that these maps are category-preserving would probably lead to a complete positive solution of the problem.

In the case of $G = \mathrm{Aut}(\mu)$, we obtain the following result.

Theorem 4.9 ([Mel12]). *Let Γ be a countable abelian group and Δ be a finitely-generated subgroup of Γ. Then the restriction map $\mathrm{Res}\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Delta, G)$ is category-preserving.*

I asked in [Mel12] whether this result extends to non-finitely generated Δ; Ageev [Age12] shows that such is not the case. Given a countable abelian group G, he completely described the set of its subgroups H for which it is true that a generic H-action can be extended to a free G-action (in particular, his results extend the results of [Mel12]). Also, one cannot expect such a result to hold in general outside of the domain of abelian groups; for instance, Ageev [Age89] proved that a generic element of $\mathrm{Aut}(\mu)$ is not conjugate to its inverse[i]. Hence, a generic measure-preserving $\mathbf{Z}$-action cannot be extended to an action of a nontrivial semidirect product $\mathbf{Z} \rtimes F$.

In [Mel12], I also pointed out an example (found with the help of B. Sévennec) of a polycyclic group Γ with a central subgroup $\Delta \cong \mathbf{Z}$ such that a generic measure-preserving Δ-action cannot be extended to a Γ-action. This example depends on the result of Chacon–Schwartzbauer [CS69] stating that the centralizer of a generic $g \in \mathrm{Aut}(\mu)$ coincides with $\overline{\{g^n : n \in \mathbf{Z}\}}$.

4.4. **Extreme amenability.** A topological group G is *extremely amenable* if any continuous action of G on a compact space has a global fixed point. The first examples of extremely amenable groups, obtained in 1975 by Herer–Christensen [HC75] , were examples of abelian "exotic" groups, which do not admit strongly continuous unitary representations; note that exotic groups are amenable iff they are extremely amenable, and all abelian groups are amenable. The question of the existence of extremely amenable groups was first raised by Mitchell [Mit70]). In the early eighties Gromov and Milman proved that $U(\mathcal{H})$ is extremely amenable, as a consequence of the phenomenon of concentration of measure on euclidean spheres of large dimensions, an avatar of the isoperimetric inequality [GM83]. Since then, many large topological groups have been proved to be extremely amenable, for instance $\mathrm{Aut}(\mu)$ (Giordano–Pestov [GP07]) and $\mathrm{Iso}(\mathbf{U})$ (Pestov[Pes02]). A comprehensive discussion of extremely amenable Polish groups may be found in Pestov's book [Pes06].

Definition 4.10. Let K be a compact metrizable group, and (X, μ) a standard probability space. The group $L^0(K)$ is the group of all measurable maps from (X, μ) to K, identified if they coincide outside of a set of measure 0, endowed with the

[i] I only recently noticed that this also follows from an earlier result of del Junco, who proved that the powers of a generic transformation form a disjoint family [dJ81].

topology of convergence in measure, which in this case is induced by the metric

$$d(f,g) = \int_X d(f(x), g(x)) d\mu(x) \ .$$

(d is any compatible distance on K)

Then $L^0(K)$ is a Polish group, and Azuma's inequality may be used to prove that $L^0(K)$ is a *Lévy group* (Glasner [Gla98]; Furstenberg–Weiss), which implies that it is extremely amenable (for the definition of a Lévy group and other facts related to extreme amenability that we do not discuss in detail, see e.g. [Pes06]).

Definition 4.11. A Polish group G is said to be *monothetic* if there exists $g \in G$ such that $\langle g \rangle$ is dense. We say that G is *generically monothetic* if this holds for a generic $g \in G$.

It follows from a classical result of Halmos–Samelson [HS42] stating that any compact abelian connected metrizable group is generically monothetic, that $L^0(K)$ is generically monothetic for any such K ([Gla98] for $K = \mathbf{T}$). This can be proved with a simple Baire category argument similar to one that can be found in [Kec10, p.26]. Along the same lines, $L^0(K)$ is topologically 2-generated whenever K is compact, metrizable and connected, this time as a consequence of the same Baire category argument and the fact that any such K is topologically 2-generated (Schreier–Ulam [SU33]). The fact that $L^0(\mathbf{T})$ is monothetic was first noticed by Glasner [Gla98].

The starting point of the work that led to [MT13b] was the following observation.

Theorem 4.12 ([MT13b])**.** *Let G be a Polish group, and Γ be a countable group. Then the set*

$$\{\pi \in \mathrm{Hom}(\Gamma, G) \colon \overline{\pi(\Gamma)} \textit{ is extremely amenable}\}$$

is G_δ in $\mathrm{Hom}(\Gamma, G)$.

Sketch of proof. Fix a compatible left-invariant metric d on G. It follows from ([Pes06], 2.1.11) and an easy argument that $\overline{\pi(\Gamma)}$ is extremely amenable if, and only if, the following condition is satisfied:

$$\forall \varepsilon > 0 \ \forall A \text{ finite } \subseteq \Gamma \ \exists B \text{ finite } \subseteq \Gamma \ \forall c \colon B \to \{0,1\}$$
$$\exists i \in \{0,1\} \ \exists \gamma \in \Gamma \ \forall a \in A \ \exists \delta \in c^{-1}(i) \ d(\pi(\gamma a), \pi(\delta)) < \varepsilon.$$

At first glance, there are too many quantifiers involved for this to be a G_δ condition; however this intuition is false, because many of the quantifiers range over finite sets. □

Corollary 4.13. *Assume that G is a Polish group such that*

$$\{(g_n) \colon \langle (g_n) \rangle \textit{ is extremely amenable } \}$$

is dense in G^ω. Then G is extremely amenable.

Actually, to show that G is extremely amenable, it is enough to prove that

$$\{(g_n) \colon \langle (g_n) \rangle \textit{ is contained in an extremely amenable subgroup of } G\}$$

is dense in G^ω.

Sketch of proof. The set $\{(g_n) \colon \langle (g_n) \rangle = G\}$ is dense G_δ in G^ω for any Polish G, so if the first assumption is satisfied the Baire category theorem along with Theorem 4.12 for actions of the free group on countably many generators give the desired conclusion.

Now, assume that the second, weaker assumption holds, and U is a nonempty open subset of G^ω. Without loss of generality, we may assume that $U = V \times G^\omega$ where V is open in some G^m. Our assumption gives us $(g_0, \dots, g_{m-1}) \in V$ which generate a subgroup contained in an extremely amenable $H \leq G$. Let $(h_i)_{i<\omega}$ be dense in H. Then the sequence $(g_0, \dots, g_{m-1}, h_0, h_1, \dots)$ is dense in H and belongs to V. □

This enabled T. Tsankov and I to give in [MT13b] a new proof of the extreme amenability of $\mathrm{Iso}(\mathbf{U})$, $U(\mathcal{H})$ and $\mathrm{Aut}(\mu)$. The same scheme applies in all three cases, and the most complicated fact used in the proofs is the extreme amenability of groups of the form $L^0(K)$ where K is compact metric. For instance, let us give a proof along those lines of the extreme amenability of $U(\mathcal{H})$.

Sketch of proof of the extreme amenability of $U(\mathcal{H})$. It is enough to show that

$$\{(g_1, \dots, g_n) \colon \langle g_1, \dots, g_n \rangle \text{ is contained in an isomorphic copy of some } L^0(U(m))\}$$

is dense in $U(\mathcal{H})^n$ for all n. So we start by picking a nonempty open $O \subseteq U(\mathcal{H})^n$; we fix a Hilbert basis (e_i) of $\mathcal{H}$. One can find $(g_1, \dots, g_n) \in O$ and $m < \omega$ such that $(g_1, \dots, g_n)$ acts trivially on $\mathcal{H}(m)^\perp = \mathrm{Span}(e_i)_{i \geq m}$, and any element of $U(\mathcal{H})$ which agrees with $(g_1, \dots, g_m)$ belongs to O; we identify $U(m)$ with the pointwise stabilizer of $\mathcal{H}(m)^\perp$.

The action $U(m) \curvearrowright \mathcal{H}(m)$ extends to an action $L^0(U(m)) \curvearrowright L^2(\mathcal{H}(m))$, and the latter is a Hilbert space, which we can identify with $\mathcal{H}$ in such a way that constant functions in $L^0(\mathcal{H}(m))$ are identified with $\mathcal{H}(m)$. The image of $(g_1, \dots, g_m)$ under this identification is an element $(h_1, \dots, h_m)$ of $U(\mathcal{H})$ which coincides with $(g_1, \dots, g_m)$ on $\mathcal{H}(m)$, thus belongs to O, and $\langle (h_1, \dots, h_m) \rangle$ is contained in an isomorphic copy of $L^0(U(m))$. □

4.5. **Generic monothetic subgroups.** The fact that extreme amenability is a G_δ condition naturally leads one to wonder whether a generic element of some fixed Polish group G generates an extremely amenable subgroup.

Recall that a countable, abelian group is *unbounded* if there is no upper bound on the order of its elements.

Theorem 4.14 ([MT13b]). *Let Γ be a countable, unbounded abelian group and G be one of* $\mathrm{Aut}(\mu)$, $U(\mathcal{H})$ *or* $\mathrm{Iso}(\mathbf{U})$. *Then the set* $\{\pi \colon \overline{\pi(\Gamma)} \cong L^0(\mathbf{T})\}$ *is dense in* $\mathrm{Hom}(\Gamma, G)$; *therefore, the generic $\overline{\pi(\Gamma)}$ is extremely amenable.*

Actually, the above result holds in somewhat greater generality, and one can write down an algebraic condition such that the extreme amenability of a generic $\overline{\pi(\Gamma)}$ holds (for Γ abelian) exactly when Γ satisfies this condition, and Γ cannot densely embed in an extremely amenable group if this condition fails. I will not go into this level of detail here and focus on the case of $\mathbf{Z}$ below.

The "therefore" part above follows from the facts that extreme amenability is a G_δ condition, and that $L^0(\mathbf{T})$ is extremely amenable. In each case, the proof proceeds by showing first that $\{\pi\colon \overline{\pi(\Gamma)}$ is contained in a closed copy of $L^0(\mathbf{T})\}$ is dense, then by perturbing slightly π so that $\pi(\Gamma)$ becomes dense in $L^0(\mathbf{T})$.

In the case of $U(\mathcal{H})$, using spectral theory, we actually proved a much more precise result: if Γ is unbounded abelian, then the set $\{\pi\colon \overline{\pi(\Gamma)} \cong L^0(\mathbf{T})\}$ is comeager in $\mathrm{Hom}(\Gamma, U(\mathcal{H}))$ (we will see a different proof of this fact below). This leads to the following question (and now, we focus, as promised, on the case $\Gamma = \mathbf{Z}$ for clarity of the exposition): what can one say about the generic properties of monothetic subgroups of G, when G is $\mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$?

We already know that, in all three cases, a generic monothetic subgroup is maximal abelian and extremely amenable. In $U(\ell_2)$, this completely characterizes the group up to isomorphism: the spectral theorem tells us that a maximal abelian subgroup of $U(\ell_2)$ must coincide with the unitary group of a separable abelian von Neumann algebra, and only one of those is extremely amenable: the unitary group of $L^\infty(X,\mu)$ when (X,μ) is atomless, and this group is isomorphic to $L^0(\mathbf{T})$. Thus, we obtain in a fairly soft way (much more painlessly than in our original proof of this result, at any rate) that a generic monothetic subgroup of $U(\mathcal{H})$ is isomorphic to $L^0(\mathbf{T})$.

Question 4.15. Does the same property hold in $\mathrm{Aut}(\mu)$?

Evidence towards a positive answer to that question was recently found by Solecki [Sol14], who proved that the closed subgroup generated by a generic element of $\mathrm{Aut}(\mu)$ is a continuous homomorphic image of a closed subspace of $L^0(\mathbf{R})$, and contains an increasing chain of finite-dimensional tori whose union is dense.

One could also ask whether the same property holds in $\mathrm{Iso}(\mathbf{U})$; we do not even know whether a generic monothetic subgroup is divisible, a property which is presumably much easier to establish. However, it was pointed out by C. Rosendal (private communication) that a generic element of $\mathrm{Iso}(\mathbf{U})$ does not generate a copy of $L^0(\mathbf{T})$. To see why, first recall that a Polish group has *property (OB)* if, whenever it acts by isometries on a metric space (X,d) such that for all x the map $g \mapsto gx$ is continuous, then every orbit is bounded (in the case of Polish groups, this is equivalent to saying that all continuous isometric actions on separable metric spaces have bounded orbits).

Let me sketch Rosendal's argument: to prove that $L^0(\mathbf{T})$ has property (OB), it follows from [Ros09b] that we need to prove that for any neighborhood V of 1 there exists a finite subset F and an integer n such that $G = (FV)^n$. A basis of neighborhoods of 1 in $L^0(\mathbf{T})$ is given by sets of the form

$$V_\varepsilon = \{g\colon \mu(\{x\colon d(g(x),1) > \varepsilon\}) < \varepsilon\} .$$

Now, it is easy to see, by cutting (X,μ) into n pieces each of measure $< \varepsilon$, that for such an n one has $L^0(\mathbf{T}) = V_\varepsilon^n$.

However, a generic isometry of the Urysohn space has unbounded orbits (for instance, because for any fixed x and N the set $\{g \in \mathrm{Iso}(\mathbf{U})\colon \forall n d(x, g^n(x)) \le N\}$ is closed and has empty interior, so the set of isometries for which the orbit of x is bounded is meager in $\mathrm{Iso}(\mathbf{U})$), thus generates a group which does not satisfy property (OB). Hence a generic element of $\mathrm{Iso}(\mathbf{U})$ does not generate a subgroup isomorphic to $L^0(\mathbf{T})$; this property might hold for $\mathrm{Iso}(\mathbf{U}_1)$, as far as I know, but there is no compelling evidence towards that being true (again, one should first understand whether a generic monothetic subgroup is divisible in that case).

Above, it was essential for us that the maximal abelian subgroups of $U(\mathcal{H})$ are easy to classify; the same would be true in unitary groups of separable von Neumann algebras: the point is that when the spectral theorem holds it is very useful, unfortunately it is specific to von Neumann algebras (though I. Farah pointed out to me that a weaker form of spectral theorem holds for general C^*-algebras). This leads to the following, probably hopeless, problem.

Question 4.16. Can one classify the maximal abelian subgroups of $\mathrm{Aut}(X,\mu)$? Of $\mathrm{Iso}(\mathbf{U})$?

While there may be a faint glimmer of hope that something can be said in $\mathrm{Aut}(X,\mu)$, I would not be surprised if the situation in $\mathrm{Iso}(\mathbf{U})$ were extremely wild - for instance, if any (noncompact?) abelian Polish group were isomorphic to a maximal abelian subgroup of $\mathrm{Iso}(\mathbf{U})$.

We have not addressed perhaps the simplest, most natural question about Baire category in $\mathrm{Hom}(\Gamma, G)$: when are conjugacy classes meager? When do comeager classes exist? It turns out that this is harder to investigate than it looks at first; we will come back to this at the end of the next section, after introducing the language of metric model theory and some related ideas.

5. First-order logic and Polish groups

5.1. **Classical first-order logic and Fraïssé classes.**

5.1.1. *Basics of classical first-order logic.* What is a mathematical structure? Certainly, it is a set, along with various operations or relations of particular interest. For instance, a graph could be defined as a set, along with a binary relation (which is, say, irreflexive and symmetric); a group is a set, endowed with operations of multiplication and inverse, and perhaps it makes sense to distinguish the neutral element too.

Definition 5.1. A *first-order structure* is a tuple $\mathbf{M} = (M, (R_i)_{i\in I}, (f_j)_{j\in J})$ where:

- M, I, J are sets.
- Each R_i is a subset of some M^{n_i}.
- Each f_j is a function from some M^{k_j} to M.

The R_i's are called *relations*, and the f_j's are called *functions*. Of course, one might also want to consider functions from some M^k to some M^l, but considering their coordinates these functions reduce to M-valued functions. Also, one might want to consider distinguished constants (for instance, the neutral element of a group); we consider them as functions from M^0 to M. So, above, n_i is to be understood as being a positive integer, while k_j is a nonnegative integer.

Definition 5.2. A *first-order language* is a tuple $\mathcal{L} = ((R_i, m_i)_{i\in I}, (f_j, k_j)_{j\in J})$ where

- I is a set, and each m_i is a positive integer.
- J is a set, and each k_j is a nonnegative integer.

Each R_i is called a *relation symbol*, and each f_j is called a *function symbol*; in the particular case when $k_j = 0$ we say that f_j is a *constant symbol*

Given a structure $\mathbf{M}$, one can then consider its language; conversely, given a language $\mathcal{L}$, one can introduce the class of $\mathcal{L}$-structures, which are all the first-order structures whose language is equal to $\mathcal{L}$.

Given a language $\mathcal{L}$, a $\mathcal{L}$-structure $\mathbf{M}$ and a symbol belonging to $\mathcal{L}$, we call the corresponding relation or function on M the *interpretation* of that symbol. We always assume that our languages contain a special binary symbol, which is interpreted by the equality relation, and will not mention that symbol in our notations.

Note that there is a choice of language to be made when deciding how to turn a mathematical structure into a first-order structure; for instance, as we pointed out above, one might want to include a particular symbol to denote the neutral element of a group, or be content with symbols for multiplication and inverse, or even just for multiplication. Why would one make one choice rather than the other? This is where semantics come into play - so far, our discussion is purely on a syntactical level.

Definition 5.3. Let $\mathcal{L}$ be a first-order language. *Formulas* are built inductively; first one defines *terms* as follows:

- Any variable symbol is a term.
- Any expression $f(t_1, \dots, t_n)$ is a term, where f is a n-ary function symbol of $\mathcal{L}$, and $t_1, \dots, t_n$ are terms.

(in particular constant symbols are terms)

- If R is a n-ary relational symbol and $t_1, \dots, t_n$ are terms then $R(t_1, \dots, t_n)$ is a formula.
- For any formula φ, its negation $\neg\varphi$ is a formula.
- For any formulas φ, ψ their conjunction $\varphi \wedge \psi$ and disjunction $\varphi \vee \psi$ are formulas.
- For any formula φ and any variable symbol x, $\forall x\ \varphi$ and $\exists x\ \varphi$ are formulas.

Here, what matters most is that we do not allow quantification on subsets of the structure, or on "external" objects, such as the integers for instance, nor do we allow infinite conjunctions and disjunctions. So, while the formula $\exists i \in \{1, \dots, N\}\ g^i = 1$ is a valid first-order formula in the language of groups (once it is written in the form $\bigvee_{i=1}^n g^i = 1$), $\exists i \in \mathbf{N}^*\ g^i = 1$ is not. This is a crucial point for the development of first-order logic and the validity of the compactness theorem; we will not really use any first-order logic (except as a guide for intuition), so I will not go into detail here (see [Hod93], [Poi85] or [TZ12] for an introduction to first-order logic and model theory).

There is a natural notion of meaning of a formula inside a model, defined inductively; for instance, consider the formula in the language of groups $(\times, {}^{-1}, e)$ $\varphi(x,y)\colon x \times y = e$. This formula (with two free variables x, y) is true in the group $(\mathbf{Z}, +, -, 0)$ when $x = 2, y = -2$; we write $(\mathbf{Z}, +, -, 0) \models \varphi(2, -2)$. It is false for instance for $x = 14, y = 3$. So one can express the fact that x and y are inverses of one another by the first-order formula φ. If a formula has no free variables, for instance the formula ψ: $\forall x\ x^2 = 1$, then this formula will simply be *true* (or *satisfied*) or *false* in any given $\mathcal{L}$-structure. The formula ψ above is satisfied by a group exactly when all elements of that group have order 2. We write $\mathbf{M} \models \varphi$ if φ is true in $\mathbf{M}$.

The choice of language clearly influences which formulas one can write in a first-order way, thus affecting the *theory* of a $\mathcal{L}$-structure $\mathbf{M}$, which is the set of all first-order formulas (without free variables) which are true in $\mathbf{M}$.

We can also define what a *substructure* of a first-order structure $\mathbf{M}$ is: $\mathbf{N}$ is a substructure of $\mathbf{M}$ if the universe of $\mathbf{N}$ is contained in the universe of $\mathbf{M}$, both structures have the same language, and the relations and functions of $\mathbf{N}$ are the restrictions of the relations and functions of $\mathbf{M}$. Here again, the choice of language influences what substructures are: if we include inverse and neutral element along with multiplication in the language of groups, for instance, then a substructure of a group is exactly a subgroup; this is no longer true if we remove one of these symbols.

This concludes the first part of our crash-course on first-order logic; mostly what needs to be remembered from the above are the notions of language, structure, and substructure. Now we come to the reason why we are interested in first-order structures here: their automorphism groups.

Definition 5.4. Let $\mathbf{M} = (M, (R_i)_{i\in I}, (f_j)_{j\in J})$ be a first-order structure. An *automorphism* of $\mathbf{M}$ is a bijection $g \colon M \to M$ such that:

- For all $i \in I$, for all $\bar{m} = (m_1, \dots, m_k)$ such that $n_i = k$, one has $\mathbf{M} \models R_i(\bar{m}) \Leftrightarrow \mathbf{M} \models R_i(g(\bar{m}))$ (where $g(\bar{m}) = (g(m_1), \dots, g(m_k))$.)
- For all $j \in J$, for all $\bar{m} = (m_1, \dots, m_k)$ such that $k_j = k$, one has $f_j(\bar{m}) = f_j(g(\bar{m}))$.

These groups are of particular interest to us when the structure is *countable*, that is, when it universe is. Below we use the word *countable* to mean an infinite set equinumerous with ω; we say that a set is at most countable if it is finite or countable.

Definition 5.5. Let $\mathbf{M}$ be a countable first-order structure. The *permutation group topology* on its automorphism group $\mathrm{Aut}(\mathbf{M})$ is the topology of pointwise convergence with respect to the discrete topology on $\mathbf{M}$; explicitly, a basis of open neighborhoods of the identity is given by subsets of the form

$$\{g \in \mathrm{Aut}(\mathbf{M}) \colon \forall a \in A \; g(a) = a\}$$

where A ranges over all finite subsets of M.

When $\mathbf{M}$ is $(\mathbf{N}, =)$, the corresponding automorphism group is the permutation group of the integers S_∞, endowed with the topology we discussed in the first section of this text. In general, one may always assume that $M = \mathbf{N}$. Since any automorphism must induce a bijection of the universe, the automorphism group $\mathrm{Aut}(\mathbf{M})$ is then a subgroup of S_∞; the permutation group topology is the topology on $\mathrm{Aut}(\mathbf{M})$ that is induced from the Polish topology on S_∞. It is easy to check that $\mathrm{Aut}(\mathbf{M})$ is a *closed* subgroup on S_∞ or, equivalently, that $\mathrm{Aut}(\mathbf{M})$ endowed with this topology is a Polish group in its own right.

Theorem 5.6 (folklore)**.** *Let G be a closed subgroup of S_∞. Then there exists a first-order structure $\mathbf{M}$ (with a countable language) such that G is isomorphic, as a topological group, to* $\mathrm{Aut}(\mathbf{M})$ *endowed with its permutation group topology.*

Proof. As a subgroup of S_∞, G naturally acts on $\mathbf{N}$ and more generally on any $\mathbf{N}^k$. For any integer $k \geq 1$, let $\mathcal{O}_k = \{O_{k,i}\}_{i \in I_k}$ be an enumeration of all orbits of the action $G \curvearrowright \mathbf{N}^k$. Then consider a language with exactly $|I_k|$ relational symbols $R_{k,i}$ for each k, and turn $\mathbf{N}$ into a $\mathcal{L}$-structure $\mathbf{M}$ by setting

$$\mathbf{M} \models R_{k,i}(\bar{n}) \Leftrightarrow \bar{n} \in O_{k,i} \, .$$

Clearly, G is a subgroup of $\mathrm{Aut}(\mathbf{M})$. Consider a finite set $A = \{n_1, \dots, n_k\} \subseteq \mathbf{N}$ and an automorphism f of $\mathbf{M}$. Then $(n_1, \dots, n_k) \in O_{k,i}$ for some i, and $f(O_{k,i}) =$

$O_{k,i}$, which means exactly that there exists $g \in G$ such that $g(\bar{n}) = f(\bar{n})$. We have just shown that, given any finite subset A of $\mathbf{N}$ and any $f \in \mathrm{Aut}(\mathbf{M})$, there exists $g \in G$ which coincides with f on A. In other words, G is dense in $\mathrm{Aut}(\mathbf{M})$; since G is closed in S_∞ it is also closed in $\mathrm{Aut}(\mathbf{M})$, so $G = \mathrm{Aut}(\mathbf{M})$. □

Automorphism groups of first-order structures actually are *nonarchimedean* Polish groups, which means that the neutral element admits a basis of open neighborhoods made up of open *subgroups*. This is clearly not true for all Polish groups (for instance, a connected Polish group cannot have a nontrivial open subgroup), so not all Polish groups are isomorphic, as topological groups, to automorphism groups of first-order structures. But those are actually completely characterized by being Polish and nonarchimedean: a Polish nonarchimedean group naturally embeds into S_∞ (for any open subgroup V of G, one can consider the natural action of G on G/V, and glue all these actions along each other as V ranges over a basis of neighborhoods of 1; the corresponding action of G on a countable set gives us the desired embedding).

Note that one could still wonder whether any Polish group is isomorphic, as an *abstract* group, to a subgroup of S_∞; this question was asked by Ulam [Ula60] for $SO(3, \mathbf{R})$ and more generally for Lie groups; the answer was proved to be positive for many matrix groups by Kallman [Kal00]; see also Thomas [Tho99]. Now we know plenty of examples of Polish groups which do not admit any nontrivial homomorphism to S_∞, and we will see some of those later on.

One last remark before forging ahead: there are many ways to turn a nonarchimedean Polish group into the automorphism group of a countable first-order structure; some properties of the structure will be independent of this choice, but others will not, which is important to keep in mind.

5.1.2. *Fraïssé classes.* It is often the case that "universal" structures have very large automorphism groups. One way to quantify this "largeness" is via the action of the group on the structure.

Definition 5.7. A first-order structure $\mathbf{M}$ is said to be *homogeneous* if any isomorphism between two finitely generated substructures extends to an isomorphism of the whole structure.

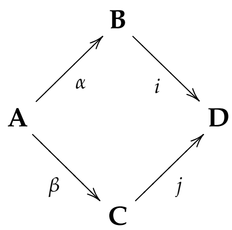
When the structure is *relational* (that is, there are only relation symbols in its language), finitely generated structures are always finite; this is false in general when the language contains functions. For instance, the substructure of $(\mathbf{Z}, +, 0)$ generated by 1 is $\mathbf{N}$; the substructure of $(\mathbf{Z}, +, -, 0)$ generated by 1 is $\mathbf{Z}$.

A crucial observation, due to Fraïssé, is that one can characterize homogeneous first-order structures by properties of their finitely generated substructures.

Definition 5.8. Let $\mathcal{L}$ be a countable first-order language, and $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures. We say that:

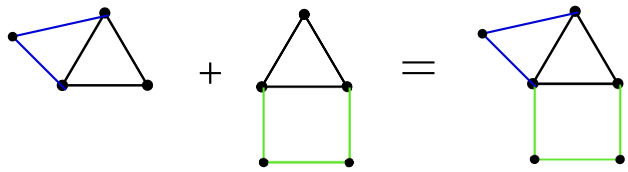
- $\mathcal{K}$ is *countable* if it contains only countably many elements up to isomorphism.
- $\mathcal{K}$ is *hereditary* if, whenever $\mathbf{B}$ belongs to $\mathcal{K}$ and $\mathbf{A}$ is a finitely generated $\mathcal{L}$-structure which embeds in $\mathbf{B}$, $\mathbf{A}$ must belong to $\mathcal{K}$.
- $\mathcal{K}$ has the *joint embedding property* (JEP) if any two elements of $\mathcal{K}$ embed in a third one.

- $\mathcal{K}$ has the *amalgamation property* (AP) if, given $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $\alpha \colon \mathbf{A} \to \mathbf{B}$, $\beta \colon \mathbf{A} \to \mathbf{C}$, there exists $\mathbf{D} \in \mathcal{K}$ and embeddings $i \colon \mathbf{B} \to \mathbf{D}$, $j \colon \mathbf{C} \to \mathbf{D}$ such that $i \circ \alpha(a) = j \circ \beta(a)$ for all $a \in A$. Schematically, the following diagram commutes.



A class satisfying all the properties above is called a *Fraïssé class*.

The amalgamation property is probably the most mysterious at first glance; note that it does not necessarily imply the joint embedding property, because we are not assuming that the empty structure belongs to $\mathcal{K}$, or even that $\mathcal{K}$ contains an initial object. Let us discuss a simple example: the class of finite graphs, which for us are structures in a language with a binary relation symbol R which is irreflexive and symmetric. Countability and hederitarity are obvious in that case; we allow graphs to be empty so the amalgamation property will imply the joint embedding property in that case. It is easy to amalgamate two graphs Γ_1, Γ_2 over a common subgraph Δ: simply form the disjoint union $\Gamma_1 \sqcup \Gamma_2$, then identify both copies of Δ; keep the edges of Γ_1 and Γ_2 and add no new ones.



Amalgamating groups over a common subgroup is somewhat more complicated, but of course the amalgamated free product is the construction we need in that case. However, there are uncountably many finitely-generated groups up to isomorphism, so they do not form a Fraïssé class. *Finite* groups do, however, and the reader is invited to think up a good way to amalgamate finite groups in such a way that the amalgam remains finite and, more generally, to try to come up with examples of Fraïssé classes of her own.

Definition 5.9. Let $\mathbf{M}$ be a countable first-order structure, with language $\mathcal{L}$. The *age* of $\mathbf{M}$ is the class of all $\mathcal{L}$-structures which are isomorphic to a finitely generated substructure of $\mathbf{M}$.

Clearly, the age of any countable structure is countable, hereditary and satisfies the joint embedding property.

Theorem 5.10 (Fraïssë [Fra54])**.** *The age of a homogeneous countable structure is a Fraïssé class.*

Proof. We only need to check the amalgamation property; let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be three elements of the age of $\mathbf{K}$ and α, β be embeddings from $\mathbf{A}$ to $\mathbf{B}, \mathbf{C}$ respectively. We may assume that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are substructures of $\mathbf{K}$, and (by homogeneity) that α, β are restrictions to $\mathbf{A}$ of automorphisms of $\mathbf{K}$, which we still denote by α, β.

Then, let $\mathbf{D}$ denote the substructure of $\mathbf{K}$ generated by $\alpha^{-1}(\mathbf{B})$ and $\beta^{-1}(\mathbf{C})$. $\mathbf{D}$ is finitely generated, and is an amalgam of $\mathbf{B}, \mathbf{C}$ over $\mathbf{A}$. □

Using a back-and-forth construction, Fraïssé proved a converse of the above result, in the following strong sense.

Theorem 5.11 (Fraïssë [Fra54]). *Let $\mathcal{K}$ be a Fraïssé class in a countable language $\mathcal{L}$. Then there exists a $\mathcal{L}$-structure $\mathbf{K}$ which is homogeneous and whose age is equal to $\mathcal{K}$. This structure is unique up to isomorphism and is called the* Fraïssé limit *of $\mathcal{K}$.*

Uniqueness up to isomorphism is easy to obtain from countability and homogeneity. The structure $\mathbf{K}$ is characterized by the following property, which should make more or less clear how to construct $\mathbf{K}$ via repeated embeddings and amalgamations (recall Katětov's construction of the Urysohn space): for any finitely generated substructure $\mathbf{A}$ of $\mathbf{K}$, any $\mathbf{B} \in \mathcal{K}$, and any embedding $i \colon \mathbf{A} \to \mathbf{B}$, there exists $\tilde{\mathbf{B}} \subseteq \mathbf{K}$ containing $\mathbf{A}$ and an isomorphism from $\tilde{\mathbf{B}}$ to $\mathbf{B}$ which coincides with i on $\mathbf{A}$.

This characterization is sometimes called *Alice's restaurant axiom*: everything you can imagine is already there.

Let us consider again the class of finite graphs: the Fraïssé limit of the class of finite graphs is the *Radó graph* $\mathbf{R}$, which is characterized among all countable graphs by the following property: for any disjoint subsets A, B of $\mathbf{R}$, there exists an element x of $\mathbf{R}$ such that there is an edge from x to every element of A and to no element of B. This is the translation of Alice's restaurant axiom for graphs; interestingly, there is also a probabilistic construction of this object. Consider a graph on $\mathbf{N}$ built in the following way: for each $i < j$, flip a coin; if the coin lands on heads, put an edge between i and j, and do not put an edge otherwise. Clearly, with probability 1, the axiom we just wrote down will be satisfied, because, once we only consider $j > \max(A, B)$, the probability to put an edge between j and all elements of A and no element of B is a fixed strictly positive number, hence such a j will appear with probability 1. In general, one cannot hope for such simple probabilistic constructions of Fraïssé limits, but this is still an interesting area, with some promising recent developments which unfortunately fall outside the scope of this text.

We saw other examples of homogeneous structures: Urysohn spaces. For instance, the rational Urysohn space $\mathbf{U}_{\mathbf{Q}}$ may be seen as a homogeneous structure in the language with countably many binary relational symbols $(d_q)_{q \in \mathbf{Q}^+}$, by setting

$$\mathbf{U}_{\mathbf{Q}} \models d_q(x, y) \Leftrightarrow d(x, y) = q \, .$$

We let the reader think of how one can amalgamate metric spaces.

In the end of this section, we will discuss some other examples. Going back to Polish groups for a moment, we note that the construction of Theorem 5.6 actually shows that, for any nonarchimedean Polish group G, there exists a *homogeneous* countable structure $\mathbf{M}$, in a countable relational language, such that G is isomorphic (as a topological group) to $\mathrm{Aut}(\mathbf{M})$.

5.1.3. *Free amalgams.* There is a situation where amalgams are particularly simple: *free amalgamation*. In this section, we assume that all languages are relational and countable; the material discussed here comes from [BM13].

Definition 5.12. Let $\mathcal{L}$ be a countable relational language, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be three $\mathcal{L}$-structures and $\alpha\colon \mathbf{A} \to \mathbf{B}$, $\beta\colon \mathbf{A} \to \mathbf{C}$ be two embeddings. The *free amalgam* of $\mathbf{B}, \mathbf{C}$ over α, β is the structure $\mathbf{M}$, where:

- The universe M of $\mathbf{M}$ is the quotient of the disjoint union $B \sqcup C$ by the equivalence relation which identifies $\alpha(a)$ and $\beta(a)$ for all elements of A (and identifies only those elements)
- Relations in $\mathbf{M}$ come only from tuples entirely contained in B and tuples entirely contained in C, and are such that the natural inclusion maps from B, C to M induce embeddings of $\mathcal{L}$-structures.

Informally: glue together the two copies of $\mathbf{A}$, copy the relations from $\mathbf{B}$ and $\mathbf{C}$, and add no other relations. Below we will simply say that this structure is the free amalgam of $\mathbf{B}, \mathbf{C}$ over $\mathbf{A}$ (the embeddings should always be clear from the context). Using the same idea, we can freely amalgamate any family of $\mathcal{L}$-structures over a common substructure $\mathbf{A}$.

Some classes are stable under free amalgamation (we say that they satisfy the *free amalgamation property*), for instance the class of all graphs is; the amalgamation procedure we described to show that the class of finite graphs is a Fraïssé class was exactly free amalgamation. Most classes are not stable under free amalgamation: for instance, the class of finite rational metric spaces certainly is not, since there must be a distance between any pair of elements, so we must add relations between elements of $B \setminus A$ and $C \setminus A$.

Free amalgamation behaves very well with respect to automorphisms, in the sense that it enables one to glue automorphisms together (which is also possible for rational metric spaces, and indeed in many cases where there exists a "natural" amalgamation procedure). It turns out to be possible to reproduce Katětov's construction of the Urysohn space in any free amalgamation class.

Definition 5.13. Let $\mathcal{K}$ be a Fraïssé class in a countable relational language $\mathcal{L}$, with the free amalgamation property. We let $\mathcal{K}_\omega$ denote the class of all at most countable $\mathcal{L}$-structures whose age is contained in $\mathcal{K}$.

For instance, if $\mathcal{K}$ is the class of all finite graphs, then $\mathcal{K}_\omega$ is the class of all (at most) countable graphs. Assuming that $\mathcal{K}$ has the free amalgamation property, it is easy to check that $\mathcal{K}_\omega$ is also stable under free amalgamation.

Definition 5.14. Let $\mathbf{A}, \mathbf{B}$ be $\mathcal{L}$-structures such that $B = A \cup \{b\}$. The *quantifier-free type* of b over $\mathbf{A}$ is the set of all formulas φ with at most one free variable x, with parameters in A, such that $\mathbf{B} \models \varphi(b)$.

A quantifier-free type (q.f type for short) over some $\mathcal{L}$-structure $\mathbf{A}$ is a set of formulas with at most one free variable x such that there exists a structure $\mathbf{B}$ containing $\mathbf{A}$ and an element b of B such that our set of formulas is exactly the q.f type of b over $\mathbf{A}$.

We recall that all our Fraïssé classes are assumed to be infinite. In particular, for any $\mathbf{A} \in \mathcal{K}$, there exists at least one q.f type over it which does not come from an element of A. To each q.f type p over $\mathbf{A}$ one can associate a unique $\mathcal{L}$-structure whose universe is of the form $A \cup \{b\}$, where b realizes the q.f. type we started from; we call this the structure *associated to* p (if p contains a formula $x = a$ for some $a \in A$, then $B = A$, otherwise A is strictly contained in B).

For $\mathbf{A} \in \mathcal{K}_\omega$, we say that a q.f. type p over $\mathbf{A}$ is *finitely induced* if there exists a finite substructure $\mathbf{M}$ of $\mathbf{A}$ and a q.f. type q over $\mathbf{M}$ such that the structure associated to p is the free amalgam over $\mathbf{M}$ of $\mathbf{A}$ and the structure associated to q. Note that there are only countably many finitely induced q.f. types over a given $\mathbf{A} \in \mathcal{K}_\omega$.

Definition 5.15. Let $\mathcal{K}$ be a Fraïssé class of $\mathcal{L}$-structures with the free amalgamation property, and $\mathbf{A} \in \mathcal{K}_\omega$. We let $\mathcal{E}(\mathbf{A})$ denote the $\mathcal{L}$-structure obtained by forming the free amalgam over $\mathbf{A}$ of the structures associated to all the finitely induced q.f. types over $\mathbf{A}$.

There is an obvious natural embedding of $\mathbf{A}$ into $\mathcal{E}(\mathbf{A})$, and we always see $\mathbf{A}$ as a substructure of $\mathcal{E}(\mathbf{A})$ via this embedding. Now, given a Fraïssé class $\mathcal{K}$ with the free amalgamation property, we can simply mimic Katětov's construction of the Urysohn space: start from any $\mathbf{A} \in \mathcal{K}$, and construct a tower of elements in $\mathcal{K}_\omega$ by setting $\mathbf{M}_0 = \mathbf{A}$, $\mathbf{M}_{i+1} = \mathcal{E}(\mathbf{M}_i)$. The limit $\mathbf{M}_\infty = \cup \mathbf{M}_i$ must then be the Fraïssé limit of $\mathcal{K}$. It is clear that automorphisms of $\mathbf{M}$ extend to automorphisms of $\mathbf{M}_\infty$, and one can check that this induces a continuous embedding of permutation groups. Let us sum up.

Theorem 5.16 ([BM13])**.** *Let $\mathcal{K}$ be a Fraïssé class with the free amalgamation property, with Fraïssé limit $\mathbf{K}$, and $\mathbf{M}$ be a countable structure whose age is contained in $\mathcal{K}$. Then there exists an embedding $i \colon \mathbf{M} \to \mathbf{K}$ such that all automorphisms of $\mathbf{M}$ extend to automorphisms of $\mathbf{K}$, and this extension map can be taken to be a continuous group embedding from* $\mathrm{Aut}(\mathbf{M})$ *to* $\mathrm{Aut}(\mathbf{K})$.

This construction can be tweaked a little bit. For instance, at each step, we could let $\mathbf{M}_{i+1}$ be the free amalgam of two copies of $\mathcal{E}(\mathbf{M}_i)$ over $\mathbf{M}_i$; then any automorphism of $\mathbf{M}_i$ extends uniquely to an automorphism of $\mathbf{M}_{i+1}$ that swaps the two copies of $\mathcal{E}(\mathbf{M}_i)$. This idea can be used to prove the following result.

Theorem 5.17 ([BM13])**.** *Let $\mathcal{K}$ be a Fraïssé class with the free amalgamation property, with Fraïssé limit $\mathbf{K}$, $\mathbf{M}$ be a countable structure whose age is contained in $\mathcal{K}$, and $n \geq 2$ an integer. There exists an automorphism $\varphi_{\mathbf{M}}$ of $\mathbf{K}$ such that $\varphi_{\mathbf{M}}^n = 1$, and the set of fixed points of $\varphi_{\mathbf{M}}$ is isomorphic to $\mathbf{M}$.*

The construction ensures that $\mathbf{M}$ and $\mathbf{N}$ are isomorphic iff $\varphi_{\mathbf{M}}$ and $\varphi_{\mathbf{N}}$ are conjugate; the map $\mathcal{M} \mapsto \varphi_{\mathbf{M}}$ can be turned into a Borel reduction of the isomorphism relation of elements of $\mathcal{K}_\omega$ to the relation of conjugacy in $\mathrm{Aut}(\mathbf{K})$.

We do not discuss definitions of Borel reducibility, and refer the reader to [BK96], [Gao09], [Hjo00] or [Kec02] for background. The remainder of this section can be safely skipped by readers unfamiliar with this theory. Let us note that the above result implies that the relation of conjugacy among, say, involutions of the random graph, is universal among Borel actions of S_∞; the same result is true if one replaces the random graph by any one of the Henson graphs. The reason this holds is that, in each of these cases, the isomorphism relation among elements of $\mathcal{K}_\omega$ is universal for actions of S_∞, and the above result gives us a reduction of this relation to the conjugacy relation of involutions in $\mathrm{Aut}(\mathbf{K})$ (involutions could be replaced by elements of any fixed finite order, of course). The fact that the relation of conjugacy in the automorphism group of the random graph is universal among Borel actions of S_∞ was originally proved, differently, by Coskey–Ellis–Schneider [CES11].

The main reason why I discussed the constructions above in some detail is that they lead to a question I find intriguing: what are the possible complexities for the relation of isomorphism of elements of $\mathcal{K}_\omega$?

Question 5.18. Let $\mathcal{K}$ be a Fraïssé class with the free amalgamation property. Is it true that the relation of isomorphism of elements of $\mathcal{K}_\omega$ is either smooth or universal for Borel actions of S_∞?

5.1.4. *An example: good measures.* Fraïssé classes may appear in somewhat unexpected places; we discuss an intriguing example related to topological dynamics. First, consider the class of all finite boolean algebras, say in the language with constant symbols for the emptyset and the whole space, as well as function symbols for union, intersection and complement. This is a Fraïssé class - the amalgamation procedure may be checked by using product algebras, for instance: indeed, let $\mathbf{A}$ be a common subalgebra of two finite boolean algebras $\mathbf{B}, \mathbf{C}$. Then $\mathbf{D} = \mathbf{B} \times \mathbf{C}$ is a finite boolean algebra, and the diagonal embedding $a \mapsto (a, a)$ amalgamates $\mathbf{B}$ and $\mathbf{C}$ over $\mathbf{A}$. The limit of this class is easily seen to be the countable atomless boolean algebra $\mathbf{B}_\infty$, whose Stone space is a Cantor space X (the Boolean algebra of clopen subsets of X is isomorphic to $\mathbf{B}_\infty$). So, the automorphism group $\mathrm{Aut}(\mathbf{B}_\infty)$ and the homeomorphism group $\mathrm{Homeo}(X)$ are isomorphic as topological groups.

Now, let us increase the complexity of our class a little bit.

Definition 5.19 (Akin [Aki05])**.** Let X be a Cantor space. A *good measure* on X is a Borel measure μ which is atomless, has full support, and is such that for any clopen A, B such that $\mu(A) \leq \mu(B)$, there exists a clopen $C \subseteq B$ such that $\mu(A) = \mu(C)$.

It follows from a result of Glasner–Weiss [GW95, Proposition 2.6] we already mentioned that, whenever φ is a uniquely ergodic homeomorphism of a Cantor space X, the unique φ-invariant measure is a good measure. A beautiful theorem of Akin [Aki05] states that the converse is also true: given a good measure μ on a Cantor space X, there exists a homeomorphism φ on X such that μ is the unique φ-invariant measure.

Definition 5.20. Given μ a good measure on a Cantor space X, we define its *clopen value set* $V(\mu)$ as the set of all values $\mu(V)$ as V ranges over clopen subsets of X.

Whenever V is the clopen value set of a good measure, it is easy to see that V is countable, contains 0 and 1, is the intersection of a subgroup of $\mathbf{R}$ and $[0, 1]$, and is dense in $[0, 1]$. Any such set will be called a *good value set*.

Akin pointed out in [Aki05] that for any good value set V there exists a good measure μ such that $V = V(\mu)$. Let us see this from the point of view of Fraïssé theory: fix a good value set V, and consider the language $\mathcal{L}_V$ made up of the language of boolean algebras expanded by unary relational symbols μ_r for all $r \in V$. We may then consider the class of $\mathcal{L}_V$-structures $\mathbf{A}$ which are finite Boolean algebras and are such that, when one sets $(\mu(a) = r)$ iff $\mathbf{A} \models \mu_r(a)$, one defines a probability measure on $\mathbf{A}$. One can check that this defines a Fraïssé class, the limit of which is a countable atomless Boolean algebra endowed with a probability measure whose set of values is equal to V. Looking at the Stone space, one can see the limit as a Cantor space endowed with a good measure μ such that $V(\mu) = V$. We thus see that to any good value set corresponds a good measure.

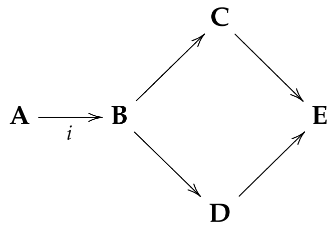
Using back-and-forth as usual, it is straightforward to check that, if μ is a good measure on a Cantor space X and $\mathbf{A}, \mathbf{B}$ are finite subalgebras of clopen subsets of X, any isomorphism from $\mathbf{A}$ to $\mathbf{B}$ extends to a homeomorphism of X which preserves μ. Hence the algebra of clopen sets on X endowed with the measure μ is the Fraïssé limit of the class of finite boolean algebras endowed with a probability measure taking its values in $V(\mu)$. In particular, two good measures with the same clopen value set must be isomorphic, a fact which is proved very differently in [Aki05].

As we saw earlier, there is no Polish topology on the full group of a minimal homeomorphism of a Cantor space X; for $\mathbf{Z}$-actions, the closure of the full group is still a complete invariant for orbit equivalence (this is pointed out in [IM13], and follows easily from results of Giordano–Putnam–Skau [GPS95]), thus a natural object to study.

We focus on the case of a uniquely ergodic homeomorphism φ of a Cantor space X, call μ the unique φ-invariant measure, and H_μ the group of all homeomorphisms of X which preserve μ. Then the same argument we used to prove that there is no Polish group topology on $[\varphi]$ shows that the Polish group topology on H_μ is unique; it follows from the arguments of [BM08] that any nontrivial normal subgroup of H_μ contains its derived subgroup, so to decide whether H_μ is simple as an abstract group we need to know whether every element of H_μ is a product of commutators. A case where this is particularly easy to prove is when H_μ has a comeager conjugacy class: assume that such is the case, call Ω the comeager class, and let k be any element of H_μ. Then $k\Omega \cap \Omega$ must be nonempty, so there exists $g \in \Omega$ and $f \in H_\mu$ such that $kfgf^{-1} = g$, or $k = gfg^{-1}f^{-1}$, i.e. k is a commutator. Using an argument due to Rosendal and Solecki [RS07], one can also see that when H_μ has a comeager conjugacy class then it has the automatic continuity property.

We are led to the question of whether H_μ has a comeager conjugacy class; this is a well-studied question for Polish groups in general, well-understood in the case of subgroups of S_∞ since work of Kechris–Rosendal [KR07] extending a study initiated by Hodges–Hodkinson–Lascar–Shelah [HHLS93] . Kechris and Rosendal approached this problem using a Fraïssé-theoretic point of view: let $\mathcal{K}$ be a Fraïssé class with limit $\mathbf{K}$, and denote by $\mathcal{K}_{aut}$ the class made up of all pairs $(\mathbf{A}, \varphi)$ such that $\mathbf{A}$ is an element of $\mathcal{K}$ and φ is a partial automorphism of $\mathbf{A}$. Then the existence of a dense conjugacy class in $G = \mathrm{Aut}(\mathbf{K})$ is equivalent to saying that $\mathcal{K}_{aut}$ satisfies the joint embedding property. Intuitively, pairs $(\mathbf{A}, \varphi)$ encode basic open sets in G, and the joint embedding property says that any two basic open sets have conjugates which intersect, equivalently, that there exists a comeager set of elements with a dense orbit.

The existence of a comeager conjugacy class may similarly be expressed in terms of the class $\mathcal{K}_{aut}$, but is a bit trickier; say that a class $\mathcal{F}$ of finite structures has the *weak amalgamation property* if, given any $\mathbf{A} \in \mathcal{F}$, there exists an embedding $i \colon \mathbf{A} \to \mathbf{B} \in \mathcal{F}$ such that any two superstructures of $\mathbf{B}$ belonging to $\mathcal{F}$ can be amalgamated over $\mathbf{A}$ - the corresponding diagram is as follows.



Kechris–Rosendal [KR07] proved that there exists a comeager conjugacy class in $\mathrm{Aut}(\mathbf{K})$ if, and only if, $\mathcal{K}_{aut}$ satisfies both the joint embedding and weak amalgamation properties. The weak amalgamation property is sometimes easy to check in presence of the following phenomenon: when, given any $(\mathbf{A}, \varphi) \in \mathcal{K}_{aut}$, there exists $(\mathbf{B}, \psi) \in \mathcal{K}_{aut}$ in which $(\mathbf{A}, \varphi)$ embeds and such that ψ is a *global* automorphism of $\mathbf{B}$. We then say that $\mathcal{K}$ has the *weak extension property*. When looking at things from the angle of the action of $\mathrm{Aut}(\mathbf{K})$ on $\mathbf{K}$, the weak extension property says that elements with finite orbits are dense in $\mathbf{K}$. Typically, the weak extension property is difficult to prove or just plain false, while the joint embedding property holds in many examples. The following result was proved in joint work with T. Ibarlucia.

Theorem 5.21. *Let μ be a good measure on a Cantor space X. Then the set of elements of finite order is dense in the group of homeomorphisms which preserve μ.*

Using Akin's theorem linking good measures and minimal homeomorphisms, this result can be considerably reinforced, as was pointed out by K. Medynets.

Theorem 5.22 (Essentially Grigorchuk–Medynets [GM12])**.** *Let φ be a minimal homeomorphism. Then $[\varphi]$ contains a dense locally finite subgroup.*

The proof of the above theorem uses in an essential way the existence of Kakutani–Rokhlin partitions (and the fact that, up to replacing φ by another minimal homeomorphism which is orbit equivalent to it, one can always assume that $[[\varphi]]$ is dense in $[\varphi]$).

Going back to the existence of dense/comeager conjugacy classes in the automorphism group of a good measure, the previous theorems tell us that the extension property always holds; unfortunately, the joint embedding property is not always satisfied, as the following simple example shows.

Example 5.23. Assume that μ is a good measure, that $1/n \in V(\mu)$, and that α is a cyclic permutation of atoms of measure $1/n$. Let B be any clopen set different from the empty set and the whole space, and let β be an automorphism fixing B and $X \setminus B$. Assume that the partial automorphisms α, β jointly embed in some μ-preserving automorphism δ; identify B with its image via this embedding. Since α embeds in δ, B must be split up in n subsets of equal measure (the trace on B of the atoms which are permuted by α); thus $\mu(B)/n$ must belong to $V(\mu)$.

Thus, we see that if H_μ satisfies the joint embedding property, then $1/n \in V(\mu) \Rightarrow V(\mu)/n = V(\mu)$. This condition is clearly not always satisfied; for instance, it fails when V is the smallest good value set containing $1/2$ and $1/\pi$ and μ is the good measure such that $V(\mu) = V$.

Analyzing the counterexample above, one can give a characterization, in terms of the structure of $V(\mu)$, of exactly when there exists a dense conjugacy class in H_μ.

This is satisfied in particular when $V(\mu)$ is the intersection of a $\mathbf{Q}$-vector subspace of $\mathbf{R}$ and $[0,1]$, (in which case there is a comeager conjugacy class, a fact already proved by [Aki05]) or of a subring of $\mathbf{R}$ and $[0,1]$.

In most cases I am aware of, the conjunction of the joint embedding property and the weak extension property is sufficient to obtain the weak amalgamation property (usually, one can produce a class of finite structures endowed with a global automorphism which is cofinal in the class of finite structures with a partial automorphism, and satisfies the amalgamation property); it appears not to be the case in Fraïssé classes of measures.

One can also use the density of elements of finite order and results of [BM08] to show that H_μ is always topologically simple, or, more generally, that the closure of the full group of any minimal homeomorphism is topologically simple (hence, the same is true for the full group itself), see [IM13]. The following problem remains open.

Question 5.24. Is the full group of a minimal homeomorphism a simple group? What about its closure in $\mathrm{Homeo}(X)$?

There are other intriguing questions; we already mentioned Akin's theorem stating that for any good measure μ on a Cantor space X there exists a minimal homeomorphism φ of X such that μ is the unique φ-invariant measure. Can this result be recovered using the Fraïssé-theoretic approach we used here? Can it be extended to more general situations?

Question 5.25. Given a Cantor space X, can one give a characterization of all the compact, convex sets of measures K such that there exists a minimal homeomorphism of X for which K is the set of all φ-invariant measures?

By a result of Downarowicz [Dow91], any abstract Choquet simplex can be obtained in such a way - so the question is about how the Choquet simplex sits inside the set of measures on the Cantor space (a natural candidate is to ask for a "goodness" condition as in the case of singletons, as well as asking that the extreme points of the simplex be mutually singular; Dahl [Dha08] obtained such a characterization for finite-dimensional simplices in her thesis).

5.2. **Metric structures and Fraïssé classes.**

5.2.1. *Moving from the discrete to the continuous setting.* As we saw, Fraïssé theory provides a fairly versatile tool to approach structures with somewhat different flavors, the unifying feature being homogeneity. However, the class of Polish groups one can capture using classical Fraïssé theory is limited to nonarchimedean Polish groups, so for instance connected Polish groups look unapproachable in this way. Still, at least in an intuitive sense, many classical structures of analysis look just as homogeneous as those from first-order logic: for instance, think of a Hilbert space, or of the Urysohn space... A way to use Fraïssé-theoretic ideas to study the automorphism groups of such structures goes through the formalism of *continuous first-order logic*, or *metric model theory*. This formalism had a precursor in Henson's work on logics adapted to the study of Banach spaces, and was introduced by Ben Yaacov and Usvyatsov in its current form [BU10]; its basic properties were developed in [BYBHU08].

We will not actually be using any tools from logic, (most notably, no compactness theorem), so our definitions are fairly relaxed.

Definition 5.26. A *metric structure* is a tuple $\mathbf{M} = ((M,d),(R_i)_{i\in I},(f_j)_{j\in J})$ such that

- (M,d) is a *complete* metric space.
- Each $R_i \colon M^{k_i} \to \mathbf{R}$ is a Lipschitz map.
- Each $f_j \colon M^{n_j} \to M$ is a Lipschitz map.

When (M,d) is separable, we say that $\mathbf{M}$ is a *Polish metric structure*.

As in the discrete setting, 0-ary functions are considered as named constants. A continuous language is then what one would expect, with the added wrinkle that the language includes a Lipschitz constant for each R_i and each f_j. For instance, the language of real Banach spaces could be written as $(0,+,(\cdot_\lambda)_{\lambda\in\mathbf{R}})$ where 0 is a constant, $+$ is a 2-Lipschitz map, and each $\cdot_\lambda$ is $|\lambda|$-Lipschitz. The distance function plays the same role as equality does in the classical, or discrete, setting; in particular, we always assume that the distance is part of our language, as a distinguished binary 1-Lipschitz predicate.

Many definitions (substructure, embedding ...) extend seamlessly from the discrete setting to the continuous one.

Definition 5.27. Let $\mathbf{M} = ((M,d),(R_i)_{i\in I},(f_j)_{j\in J})$ be a metric structure. An automorphism of $\mathbf{M}$ is a bijection g of M onto itself such that

- For all $\bar{m} \in M^k$ and each i such that $k_i = k$, $R_i(\bar{m}) = R_i(g(\bar{m}))$. In particular, g must be an isometry of (M,d).
- For all $\bar{m} \in M^k$ and each j such that $n_j = k$, $f_j(\bar{m}) = f_j(g(\bar{m}))$

The automorphism group $\mathrm{Aut}(\mathbf{M})$ of a Polish metric structure $\mathbf{M}$ is then a closed subgroup of the isometry group of (M,d) (endowed with the pointwise convergence topology), so is a Polish group itself.

When $\mathbf{M}$ is a metric structure and $(a_1,\dots,a_n)$ is a finite tuple of elements of M, we denote by $\langle a_1,\dots,a_n\rangle$ the substructure of $\mathbf{M}$ generated by $a_1,\dots,a_n$.

Definition 5.28. We say that a Polish metric structure $\mathbf{M}$ is *homogeneous* when it is true that, for any $a_1,\dots,a_n \in M$, for any $\varepsilon > 0$, and for any embedding $f \colon \langle a_1,\dots,a_n\rangle \to \mathbf{M}$, there exists an automorphism g of $\mathbf{M}$ such that $d(g(a_i),f(a_i)) < \varepsilon$ for all i.

In other words: an isomorphism between finitely generated substructures of $\mathbf{M}$ can be approximated arbitrarily well by an automorphism of $\mathbf{M}$, the approximation taking place on the images of the generators of the first substructure. Naming generators is a price to pay when dealing with structures whose language includes functions; of course this is not necessary when the language is relational, since finitely generated substructures of $\mathbf{M}$ must then be finite. Controlling what happens on finitely many elements is really just a way of saying that we are working with the pointwise convergence topology on $\mathrm{Aut}(\mathbf{M})$.

Then, the same argument as for discrete structures leads to the following observation.

Theorem 5.29 ([Mel10a])**.** *Let G be a Polish group. There exists a homogeneous Polish metric structure $\mathbf{M}$ such that G is isomorphic, as a topological group, to* $\mathrm{Aut}(\mathbf{M})$.

Here, one can wonder to what extent the ε in the definition of homogeneity is important: it seems natural to ask for exact homogeneity. Very recently, I. Ben Yaacov answered a question of mine and proved that there exist Polish groups (even, Roelcke precompact) which cannot act transitively, continuously and isometrically on a complete metric space; such a group cannot be realized as the automorphism group of an exactly homogeneous metric structure. Also, there are natural examples of homogeneous metric structures which are not exactly homogeneous (for instance, the Gurarij space, which we will discuss later on), and it certainly seems that accepting the intrusion of ε here is the right thing to do. We will see shortly that this is crucial when working with Fraïssé classes in the metric setting, via the example of the Gurarij space.

For now, let us recall that, when considering topologies on isometry groups, we pointed out two choices: the pointwise convergence topology, and the uniform convergence topology. Given any Polish metric structure $\mathbf{M}$, one can endow its automorphism group G with the metric of uniform convergence d_u, defined by

$$d_u(g,h) = \sup\{d(g(x),h(x)) \colon x \in M\}$$

(truncated for instance at 1 if allowing infinite distances causes moral issues). This d_u is always complete and bi-invariant (i.e. it is impervious to multiplications on the left and the right), which are certainly desirable qualities. But it is in general not separable, and often close to discrete (or outright discrete), and it might seem at first glance that it cannot give much information. It turns out that this metric can sometimes be used in conjunction with the Polish topology; let us make an abstract definition to describe the corresponding object.

Definition 5.30. A *Polish topometric group* is a triple (G,τ,∂), where

1. (G,τ) is a Polish group
2. ∂ is a bi-invariant distance on G, refining τ.
3. ∂ is τ-lower semicontinuous, i.e. each set $\{(g,h) \colon \partial(g,h) \leq r\}$ is τ-closed.

These assumptions (which imply that ∂ is complete) are satisfied when G is the automorphism group of a Polish metric structure, endowed with the topology of pointwise convergence and the metric of uniform convergence. Actually, what matters is not really the *metric* ∂ but the uniformity it generates, but we will describe everything in metric terms (the reader should keep in mind that replacing δ by an equivalent metric, as long as the third assumption remains satisfied, is of no consequence). Starting from any Polish group G, there exists a left-invariant metric d inducing the topology of G (this d is usually not complete, as we saw; any two such distances generate the same uniformity, called the *left uniformity*). Then one can define a metric ∂ by setting

$$\partial(g,h) = \sup\{d(gk,hk) \colon k \in G\}$$

Clearly ∂ is τ-lower semicontinuous, bi-invariant, and refines τ. One can also show that ∂ is always complete when (G,τ) is Polish (one says that Polish groups are *Raikov-complete*). Thus (G,τ,∂) is a Polish topometric group, and ∂ induces the coarsest uniformity among all metrics turning (G,τ) into such a structure. Most of the time we will be working with this ∂. We call the uniformity generated by ∂ the *minimal bi-invariant uniformity*, and will abuse notation somewhat by calling *minimal bi-invariant metric* any metric which generates this uniformity (and satisfies the third topometric axiom).

Two remarks are in order here.

- Given a Polish group (G,τ), we saw in Theorem 5.29 that there exists a Polish metric structure $\mathbf{M}$ whose automorphism group, endowed with the topology of pointwise convergence, is isomorphic to G. If one builds this structure in the same way as we did earlier in the discrete case, then the uniform metric on $\mathrm{Aut}(\mathbf{M})$ induces a minimal bi-invariant metric on G.

- If (G, τ) is a Polish group isomorphic to $\mathrm{Aut}(\mathbf{M})$ for some Polish metric structure $\mathbf{M}$, then the uniform metric on $\mathrm{Aut}(\mathbf{M})$ is not necessarily minimal. For instance, given $G = S_\infty^\omega$, one can embed G into S_∞, and then make G act on $\mathbf{N}$; the associated uniformity is discrete. But it is easy to see that the minimal bi-invariant uniformity on G is the trace on G of the product of the discrete uniformities on each factor, which is not discrete.

Mostly out of curiosity, let us note the following problem.

Question 5.31. Let (G, τ, ∂) be a Polish topometric group. Under which condition does there exist a metric structure $\mathbf{M}$ such that (G, τ, ∂) is isomorphic, as a topometric group, to $\mathrm{Aut}(\mathbf{M})$ endowed with the topology of pointwise convergence and the metric of uniform convergence?

As far as I am aware, it is not even excluded that all Polish topometric groups have this property, even though that seems highly unlikely to me.

Let us now describe what Fraïssé classes become in the metric setting; metric Fraïssé classes were first considered in [Sch07], but our presentation follows a more streamlined and efficient approach presented in [Ben12]; what we present here is more restrictive than what can be found in [Ben12] but is sufficient for our purposes.

As in the classical, discrete setting, we consider a class $\mathcal{K}$ of finitely generated metric structures in some fixed metric language $\mathcal{L}$, and we want to state conditions on $\mathcal{K}$ that are equivalent to being the age of a homogeneous structure (the age of a continuous structure being defined exactly as in the discrete setting). Some properties must be satisfied by the age of any structure.

Definition 5.32. Let $\mathcal{K}$ be a class of finitely generated metric structures in some metric language $\mathcal{L}$. We say that

(1) $\mathcal{K}$ satisfies the *hereditary property* (HP) if any finitely generated substructure of an element of $\mathcal{K}$ belongs to $\mathcal{K}$.
(2) $\mathcal{K}$ satisfies the *joint embedding property* (JEP) if any two elements of $\mathcal{K}$ embed in a third one.

So far, so good; but we need a condition that bounds the size of $\mathcal{K}$, so that $\mathcal{K}$ can be the age of a separable structure. In the discrete world that condition was countability, clearly in the metric world it must be separability for an appropriately chosen metric. To introduce this metric, and since we allow functions in our languages, it is useful to make the following convention: whenever we write $\mathbf{A} = \langle \bar{a} \rangle$, we mean that $\mathbf{A}$ is generated by the tuple $\bar{a} = (a_1, \ldots, a_n)$; repetitions are allowed in the enumeration $(a_1, \ldots, a_n)$ (and the order in which elements are enumerated matters).

Definition 5.33. Let $\mathcal{K}$ be a class of finitely generated metric structures in some metric language $\mathcal{L}$, satisfying (JEP). We denote by $\mathcal{K}_n$ the class of all structures $\langle a_1, \ldots, a_n \rangle$ belonging to $\mathcal{K}$, and define d_n on $\mathcal{K}_n \times \mathcal{K}_n$ by setting

$$d_n(\langle \bar{a} \rangle, \langle \bar{b} \rangle) = \inf_{(\alpha,\beta)} \sup_{i=1,\ldots n} d(\alpha(a_i), \beta(b_i))$$

where (α, β) ranges over all pairs of embeddings of $\langle \bar{a} \rangle$, $\langle \bar{b} \rangle$ into a common structure $\mathbf{C} \in \mathcal{K}$.

The assumption that (JEP) holds ensures that d_n takes finite values; d_n measures how close two elements of $\mathcal{K}_n$ can be mapped to one another, and saying that $d_n(\langle a \rangle, \langle b \rangle) = 0$ does mean as expected that the two structures are isomorphic; the fact that the Lipschitz constants of the functional symbols are imposed by the language is useful to check this when functions are present. Intuitively, d_n should be a pseudometric, but the triangle inequality need not be satisfied under the assumptions we are working with so far: given a structure witnessing that $\langle \bar{a} \rangle$, $\langle \bar{b} \rangle$ are close, and another structure witnessing that $\langle \bar{b} \rangle$, $\langle \bar{c} \rangle$ are close, one can not necessarily produce a structure witnessing that $\langle \bar{a} \rangle$, $\langle \bar{c} \rangle$ are close - unless one can glue together in some way the copies of $\bar{b}$ appearing in both structures.

Definition 5.34. Let $\mathcal{K}$ be a class of finitely generated metric structures in some metric language $\mathcal{L}$. We say that $\mathcal{K}$ satisfies the *near-amalgamation property* (NAP) if the following condition is satisfied:

For any $\varepsilon > 0$, any $\mathbf{A} = \langle \bar{a} \rangle \in \mathcal{K}$, and any embeddings $\alpha \colon \mathbf{A} \to \mathbf{B} \in \mathcal{K}$, $\beta \colon \mathbf{A} \to \mathbf{C} \in \mathcal{K}$, there exists $\mathbf{D} \in \mathcal{K}$ and embeddings $i \colon \mathbf{B} \to \mathbf{D}$ and $j \colon \mathbf{C} \to \mathbf{D}$ such that $d(i \circ \alpha(a_i), j \circ \beta(a_i)) < \varepsilon$ for all i.

When the class $\mathcal{K}$ satisfies both (JEP) and (NAP), it is easy to check that each d_n is a pseudometric.

Definition 5.35. Let $\mathcal{K}$ be a class of finitely generated metric structures in some metric language $\mathcal{L}$, satisfying (JEP) and (NAP). We say that $\mathcal{K}$ has the *Polish property* (PP) if each d_n is separable and complete.

We have finally listed all the properties characterizing the age of a homogeneous Polish metric structure.

Definition 5.36. Let $\mathcal{K}$ be a class of finitely generated metric structures in some metric language $\mathcal{L}$. We say that $\mathcal{K}$ is a Fraïssé class if $\mathcal{K}$ satisfies (HP), (JEP), (NAP) and (PP)

The following is not hard to prove.

Theorem 5.37. *The age of any homogeneous Polish structure is a Fraïssé class.*

The converse is harder, especially if one allows functions; Ben Yaacov's proof [Ben12] introduces an interesting formalism (leading to a formal weakening of the notion of Fraïssé limit in the metric context), which we do not discuss here.

Theorem 5.38 ([Ben12]). *Let $\mathcal{K}$ be a Fraïssé class of finitely generated metric structures in some metric language $\mathcal{L}$. Then there exists a unique homogeneous Polish metric structure whose age is equal to $\mathcal{K}$. We call this structure the* Fraïssé limit *of $\mathcal{K}$.*

The simplest non-discrete example of a Fraïssé class is given by the class of all finite metric spaces, whose limit is the Urysohn space. Going in the other direction, the infinite-dimensional, separable Hilbert space $\mathcal{H}$ is certainly homogeneous, so its age is a Fraïssé class. The same goes for a standard atomless probability algebra, which is the Fraïssé limit of all finite probability algebras. In all these cases, one can replace near amalgamation by exact amalgamation, and the limit is homogeneous in a stronger sense than what we asked for, namely one can set $\varepsilon = 0$ in the definition of homogeneity; this is not always possible. One of only two examples of this that I know at the moment is the Gurarij space, which we discuss now (the other example is L^p lattices which we will not discuss).

Let us consider the class of all finite-dimensional normed vector spaces, in a language whose symbols (besides the norm/distance) are 0, $+$, and $(\cdot_\lambda)_{\lambda \in \mathbf{Q}}$ (with the appropriate Lipschitz constants). As pointed out in [Ben12], this is a Fraïssé class; let us see why (NAP) holds in this case. Consider three finite-dimensional normed vector spaces X, Y, Z and isometric embeddings $i \colon X \to Y$ and $j \colon X \to Z$. Then endow the direct sum $Y \oplus Z$ with the l^1-norm:

$$\|(y,z)\| = \|y\| + \|z\| \ .$$

Next, let N denote the closed subspace $\{(i(x), -j(x)) \colon x \in X\}$ of $Y \oplus Z$, and let E be the space $(Y \oplus Z)/N$, with the quotient norm

$$\|(y,z)\| = \inf\{\|(y,z) + (i(x), -j(x))\| \colon x \in X\}$$

Then Y isometrically embeds in E via $\alpha \colon y \mapsto [(y,0)]$, Z isometrically embeds in E via $\beta \colon z \mapsto [(0,z)]$, and for any $x \in X$ one has

$$\alpha \circ i(x) = [(x,0)] = [(0,x)] = \beta \circ j(x) \ .$$

Hence the class of finite-dimensional normed vector spaces satisfies (NAP), actually one even has exact amalgamation. Joint embedding follows immediately (take $X = \{0\}$), and separability of each $(\mathcal{K}_n, d_n)$ is an immediate consequence of the existence of a universal separable Banach space. The fact that each $(\mathcal{K}_n, d_n)$ is complete is easy once one knows how to compute the distance between structures: given $E = \langle a_1, \dots, a_n \rangle$ and $F = \langle b_1, \dots, b_n \rangle$, C. Ward Henson (see [BYH14]) proved that

$$d_n(E,F) = \sup\{ \big| \|\sum r_i a_i\| - \|\sum r_i b_i\| \big| \colon \sum |r_i| = 1 \} \ .$$

So the class of finite-dimensional Banach spaces is a Fraïssé class. Its limit is the unique universal homogeneous separable Banach space, an object which was built by Gurarij [Gur66] and whose uniqueness up to isometry was proved by Lusky [Lus76]. A simple proof of existence/uniqueness of the Gurarij space was published recently by Kubis–Solecki [KS13]. Note that the usual Banach-theoretic characterization of the Gurarij space $\mathbf{G}$ is not quite the same as the Fraïssé-theoretic version one obtains via the Fraïssé-theoretic approach, see [Ben12].

An interesting point here is that, while the class of finite-dimensional spaces amalgamates exactly, no universal Banach space can be exactly homogeneous: this is because the norm must have points of differentiability (this is true in any separable space by a classical result of Mazur [Maz33]), while universality implies that it cannot be differentiable everywhere. A linear isometry cannot map a point at which the norm is differentiable to a point at which it is not; so the group of linear isometries of $\mathbf{G}$ cannot act transitively on one-dimensional subspaces, showing that $\mathbf{G}$ is not homogeneous (this line of reasoning was explained to me a long time ago by G. Godefroy). This shows that allowing for small errors in the definition of homogeneity is useful to capture some natural examples.

As a Fraïssé limit, the Gurarij space is certainly analogous, in the setting of Banach spaces, to the Urysohn space; this analogy was taken further by Ben Yaacov [BY14], who adapted Katětov's construction of $\mathbf{U}$, showing in the process that any separable normed space embeds in $\mathbf{G}$ in such a way that all its isometries extend, and the extension map can be taken to be a group homomorphism. Consequently

Aut($\mathbf{G}$) is a universal Polish group. This analogy with the Urysohn space, and the fact that the Urysohn space generates a unique Banach space (the Holmes space, discussed at the end of the second section), makes it tempting to believe that the Holmes space and the Gurarij space are one and the same. Surprisingly, this turns out to be false, see [FW08].

5.2.2. *Hjorth's oscillation theorem revisited.* Hjorth's oscillation theorem is the first example that made me realize that continuous logic could be used to translate results known to hold for closed subgroups of S_∞ to the context of general Polish groups; this process was initiated by a suggestion of S. Solecki while I was a postdoc in the University of Illinois at Urbana-Champaign. The results of this section were published in [Mel10a].

In [KPT05], Kechris, Pestov and Todorcevic established a link between topological dynamics and combinatorics, relating the so-called finite oscillation stability of subgroups G of S_∞ with combinatorial properties of a Fraïssé class of which G is the automorphism group (we will get back to this in the next section). This led them to formulate a notion of oscillation stability for isometric actions of topological groups. The discussion below is mostly taken from [Pes06].

Definition 5.39. Let G be a metrizable topological group with a compatible left-invariant distance δ. The *left-completion* of G, denoted by $\widehat{G}$, is simply the metric completion of (G,δ).

Note that G naturally acts on $\widehat{G}$ by isometries; $\widehat{G}$ does not depend on the choice of left-invariant metric δ, in the sense that any two left-invariant metrics on G (compatible with its topology) will produce isomorphic $\widehat{G}$. This happens because, as we already mentioned, Cauchy sequences are the same for all left-invariant distances (it would probably be more natural to work with uniformities here, since what we are really using is the left uniformity of G).

Also, $\widehat{G}$ is in general not a group but is always a semigroup in which multiplication is jointly continuous. By a *right ideal* of $\widehat{G}$ we mean a subset of $\widehat{G}$ which is invariant under multiplication on the right.

If (X,d) a Polish metric space and G is a subgroup of the isometry group of (X,d), one can naturally view $\widehat{G}$ as a semigroup of isometric embeddings of (X,d) into itself.

Definition 5.40. Let G be a Polish group, and $f\colon G \to \mathbf{R}$ be a left-uniformly continuous function, which one may then uniquely extend to $\widehat{G}$. Say that f is *oscillation stable* if for every $\varepsilon > 0$ there exists a right ideal $\mathcal{I}$ of $\widehat{G}$ such that the oscillation of f on $\mathcal{I}$ is less than ε.

Definition 5.41. Let a Polish group G act continuously and by isometries on a Polish metric space X. Say that the action of G is *oscillation stable* if every Lipschitz function $f\colon X \to \mathbf{R}$ is oscillation stable. If the action of G is not oscillation stable, say that it has *distortion*.

For instance, saying that the action of the unitary group $U(\ell_2)$ on the unit sphere of ℓ_2 has distortion turns out to be equivalent to Odell and Schlumprecht's celebrated solution to the distortion problem for ℓ_2 [OS94] (note, however, that ℓ_2 is the only separable Banach space in which the notion of distortion as presented here and the classical functional-analytic notion of distortion coincide).

It is then natural to ask, given some action of a Polish group G, whether it has distortion or not. In particular, Kechris, Pestov and Todorcevic asked whether there exists a nontrivial group G such that the action of G on $\widehat{G}$ does not have distortion. Answering this question, Hjorth proved the following result.

Theorem 5.42 (Hjorth [Hjo08]). *Let (X,d) be a complete separable metric space, and $G \leq \mathrm{Iso}(X,d)$ be a group of cardinality bigger than one. Then there exists $x_0, x_1 \in X$ and uniformly continuous*

$$f\colon \overline{\{(\pi.x_0, \pi.x_1)\colon \pi \in G\}} \to [0,1]$$

such that for any $\rho \in \widehat{G}$ there exist

$$(y_0,y_1),(z_0,z_1) \in \{(\rho(\pi(x_0)), \rho(\pi(x_1)))\colon \pi \in \widehat{G}\}$$

with $f(y_0,y_1) = 0$ and $f(z_0,z_1) = 1$.

As was pointed out by Hjorth, this theorem has as an easy corollary the fact that for any non-trivial Polish group G the left-translation action of G on $\widehat{G}$ has distortion.

In the same paper, Hjorth proves a version of this theorem for automorphism groups of first-order countable relational structures.

Theorem 5.43 (Hjorth [Hjo08]). *Let $\mathcal{M}$ be a homogeneous relational countable first-order structure such that $|Aut(\mathcal{M})| > 1$. Then there exist a function $f\colon M^2 \to \{0,1\}$ and $(a_0,a_1) \in M^2$ such that for any morphism $\rho\colon \mathcal{M} \to \mathcal{M}$ one can find (b_0,b_1) and (c_0,c_1) in the image of ρ^2, with the same quantifier-free type as (a_0,a_1) and such that $f(b_0,b_1) = 1$ while $f(c_0,c_1) = 0$.*

After stating Theorem 5.43, Hjorth points out that "a weaker form can be derived from the final theorem [i], [and] its proof is easier" . Looking at Theorem 5.43 with continuous logic in mind, it is tempting to formulate the following statement.

Theorem 5.44. *Let $\mathcal{M}$ be a homogeneous relational Polish metric structure such that $|Aut(\mathcal{M})| > 1$. Then there exist a uniformly continuous $f\colon M^2 \to [0,1]$ and $(a_0,a_1) \in M^2$ such that for any morphism $\rho\colon \mathcal{M} \to \mathcal{M}$ one can find (b_0,b_1) and (c_0,c_1) in the image of ρ^2, with the same quantifier-free type as (a_0,a_1) and such that $f(b_0,b_1) = 1$, $f(c_0,c_1) = 0$.*

(Actually, one can take f to be Lipschitz in the above statement and in Hjorth's theorem, but I stick to uniform continuity since this was Hjorth's original formulation).

It is clear that this result implies Theorem 5.43: given an homogeneous countable first-order relational structure $\mathcal{M}$, one may use the same idea as in the proof of Theorem 5.6 to turn it into a homogeneous relational Polish metric structure (denoted by $\mathcal{M}_{met}$) by endowing the universe of $\mathcal{M}$ with the discrete metric and, for any relation symbol R of the language of $\mathcal{M}$, introducing a $\{0,1\}$-valued predicate symbol R_{met} defined by $R_{met}(m_1,\dots,m_k) = 0 \leftrightarrow \mathcal{M} \models R(m_1,\dots,m_k)$. Then $\mathcal{M}_{met}$ satisfies the assumptions of Theorem 5.44, and morphisms of $\mathcal{M}$ and $\mathcal{M}_{met}$ are the same. If f is the function yielded by Theorem 5.44, then $\tilde{f}$ defined by $\tilde{f}(m,m') = 0$ if $f(m,m') < 1$, and 1 otherwise, shows that the conclusion of Theorem 5.43 holds.

[i] i.e, Theorem 5.42 in our notation.

It is also easy to see that Theorem 5.42 implies Theorem 5.44. We do not detail the proof here, but it is a straightforward consequence of the fact that, when $\mathcal{M}$ is approximately homogeneous, the left-completion of $\mathrm{Aut}(\mathcal{M})$ coincides with the set of morphisms from $\mathcal{M}$ into itself, and morphisms preserve quantifier-free type. Perhaps more interestingly, Theorem 5.44 implies Theorem 5.42: indeed, assume that we are in the situation of Theorem 5.42. Then, one may find a countable family of predicates (R_i) such that $(X, d, (R_i))$ becomes an approximately homogeneous Polish metric structure $\mathcal{M}$, with G as its automorphism group, and Theorem 5.44 enables one to show that the conclusion of Theorem 5.42 holds.

To sum up this brief discussion: Theorem 5.44, whose statement is just the continuous logic translation of Theorem 5.43, unsurprisingly implies Theorem 5.43 and turns out to be equivalent to Hjorth's oscillation theorem. Interestingly, one may combine Hjorth's ideas from his proof of Theorem 5.43, and some of his arguments to establish Theorem 5.42, to provide a proof of Theorem 5.44 which is simpler (at least, shorter) than the original proof. I will not go into detail here; work on Hjorth's theorem is what convinced me that the language of metric structures could be useful to study properties of Polish groups.

5.3. **Extremely amenable Polish groups.** Recall that a Polish group is extremely amenable if any continuous action of G on a compact space has a fixed point. Earlier, we gave a proof that extreme amenability of a countable group was a G_δ condition (in the right framework); this was based on an intrinsic characterization of extreme amenability of a Polish group G in terms of the left translation of G on itself.

Definition 5.45. Let G be a group acting by isometries on a metric space (X, d), and let f be a function from X to $\mathbf{R}$. We say that f is *finitely oscillation stable* if for every finite $F \subseteq X$ and every $\varepsilon > 0$ there exists $g \in G$ such that the oscillation of f on gF is less than ε.

We say that the action $G \curvearrowright X$ is finitely oscillation stable if every bounded Lipschitz function $f \colon (X, d) \to \mathbf{R}$ is finitely oscillation stable.

Theorem 5.46 (Pestov [Pes06]). *Let G be a Polish group, and $\{d_i\}_{i \in I}$ be a directed collection of left-invariant pseudometrics inducing the topology of G. Then G is extremely amenable if, and only if, each action $G \curvearrowright (G, d_i)$ is finitely oscillation stable.*

Of course, one could simply consider *one* left-invariant metric in the characterization above; but, if G is the automorphism group of some metric structure $\mathbf{M}$, then there is a natural collection of pseudometrics inducing the topology of G.

Definition 5.47. Let $\mathbf{M}$ be a Polish metric structure and G be its automorphism group. For any finite $A \subseteq M$ we define a pseudometric d_A on G by setting

$$d_A(g, h) = \sup\{d(g(a), h(a)) \colon a \in A\}$$

The family $\{d_A\}$, as A ranges over finite subsets of M, induces the topology of G.

One could let A vary only over some dense subset of M and still induce the topology of G. What matters to us is that extreme amenability of $\mathrm{Aut}(\mathbf{M})$ depends on how $\mathrm{Aut}(\mathbf{M})$ acts on its finitely generated substructures; when $\mathbf{M}$ is homogeneous, this means that one can expect a characterization of extreme amenability in terms of the properties of the age of $\mathbf{M}$. In the discrete setting, such a characterization was obtained in the seminal [KPT05], following earlier work of Pestov. To see in action, in a simple setting, some of the ideas behind that work, let us discuss a striking combinatorial proof of extreme amenability of a Polish group.

Theorem 5.48 (Pestov [Pes98]). *The automorphism group of $(\mathbf{Q}, \leq)$ is extremely amenable.*

Proof. Let $G = \mathrm{Aut}(\mathbf{Q}, \leq)$, and A be a finite subset of $\mathbf{Q}$, of cardinality n. We need to show that the left-translation action of G on (G, d_A) is finitely oscillation stable; (G, d_A) naturally identifies with the set X of all n-element subsets of $\mathbf{Q}$, endowed with the discrete metric, and on which G acts diagonally. Since we are looking at Lipschitz functions on a discrete set, we may as well focus on functions taking values in $\{0,1\}$; so, what we are aiming to prove is that, for any map $f \colon X \to \{0,1\}$, and any finite subset F of X, there exists $g \in G$ such that f is constant on gF.

Let B denote the (finite) union of all the elements of F, and denote its cardinality by m. The map f is a coloring of all subsets of $\mathbf{Q}$ of cardinality n, with two colors, and the finite version of the Ramsey theorem tells us that there exists N such that, whenever we color n-elements subsets of an N element set with two colors, there exists a m-element subset which is monochromatic. Let B be any subset of $\mathbf{Q}$ of cardinality N; there exists a subset $\tilde{B}$ of B of cardinality m such that f is constant on subsets of $\tilde{B}$ with cardinality n. One can pick $g \in G$ such that $gF = \tilde{B}$, and f is constant on gF, as desired. □

The appearance of the Ramsey theorem in the proof above, and of maps defined on the space of copies of a given finitely generated substructure, is not a coincidence: indeed, if $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$, and $G = \mathrm{Aut}(\mathbf{K})$, then the oscillation stability of each action $G \curvearrowright (G, d_A)$ is equivalent to a Ramsey-theoretic property of $\mathcal{K}$.

Definition 5.49. Let $\mathcal{K}$ be a Fraïssé class of discrete finitely generated structures. Given $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, we let $\binom{\mathbf{B}}{\mathbf{A}}$ denote the set of substructures of $\mathbf{B}$ which are isometric to $\mathbf{A}$.

Say that $\mathcal{K}$ has the *Ramsey property* if, for any $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$, and any $k \in \mathbf{N}$, there exists $\mathbf{C} \in \mathcal{K}$ such that, for any map $c \colon \binom{\mathbf{C}}{\mathbf{A}} \to \{1, \dots, k\}$, there exists $\mathbf{B}_0 \in \binom{\mathbf{C}}{\mathbf{B}}$ such that c is constant on $\binom{\mathbf{B}_0}{\mathbf{A}}$.

The map c above is usually called a *coloring*; the Ramsey property could be stated equivalently using colorings with only 2 colors instead of any finite number of colors.

Whenever $G \leq S_\infty$ is a closed subgroup, G acts on the compact set of orders on $\mathbf{N}$; so, if G is extremely amenable, then G must fix an ordering on $\mathbf{N}$ since the space of orderings is a compact space on which G acts continuously. In particular, S_∞ is not extremely amenable, a fact which was first observed in [Pes98]. This observation also implies that, whenever $G = \mathrm{Aut}(\mathbf{K})$, where $\mathbf{K}$ is the Fraïssé limit of some Fraïssé class $\mathcal{K}$, all elements of $\mathcal{K}$ must be *rigid*, i.e. have trivial automorphism group, and one may as well assume that the language of $\mathcal{K}$ contains a binary symbol $\prec$ which is interpreted by a total ordering in $\mathbf{K}$. Following [KPT05], we then say that $\mathcal{K}$ is a *Fraïssé order class*. One of the main results of [KPT05] is the following.

Theorem 5.50 ([KPT05]). *Let G be a closed subgroup of S_∞. Then G is extremely amenable if, and only if, $G = \mathrm{Aut}(\mathbf{K})$, where $\mathbf{K}$ is the Fraïssé limit of a Fraïssé order class with the Ramsey property.*

As pointed out in [KPT05], every Fraïssé order class such that G is the automorphism group of its limit must have the Ramsey property, so the above result does not depend on the way G is represented as the automorphism group of a Fraïssé limit. One could replace the statement that $\mathcal{K}$ is an order class by asking that $\mathcal{K}$ is made up of rigid structures.

Now, our task is to translate Theorem 5.50 to the context of general Polish groups. At first glance, something seems to go awry: many natural metric Fraïssé limits whose automorphism group is known to be extremely amenable (the standard atomless probability algebra, the separable infinite-dimensional Hilbert space, the Urysohn space ...) are made up of very much *non-rigid* structures, and no ordering is to be found. As it turns out, the ordering, which plays a very important role in the discrete setting, is a bit of a red herring here: what one needs to understand is that, if $\mathbf{A}$ is a rigid structure, then the set of copies of $\mathbf{A}$ inside $\mathbf{B}$ is the same thing as the set of *embeddings* from $\mathbf{A}$ to $\mathbf{B}$. So, the Ramsey property could be restated in terms of embeddings.

Definition 5.51. Let $\mathcal{K}$ be a metric Fraïssé class. For any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, let ${}^{\mathbf{A}}\mathbf{B}$ denote the set of all embeddings from $\mathbf{A}$ to $\mathbf{B}$, and turn it into a metric space by setting

$$\forall \alpha, \beta \in {}^{\mathbf{A}}\mathbf{B},\ d(\alpha, \beta) = \sup\{d(\alpha(a), \beta(a)) \colon a \in A\} .$$

A *coloring* of ${}^{\mathbf{A}}\mathbf{B}$ is a 1-Lipschitz map from ${}^{\mathbf{A}}\mathbf{B}$ to $[0, 1]$.

The fact that colorings are asked to take values in $[0, 1]$, and to be 1-Lipschitz, is somewhat inessential - all that really matters is that they take value in a compact metric space and their behavior is controlled by the metric on embeddings.

Definition 5.52. Let $\mathcal{K}$ be a metric Fraïssé class, and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be elements of $\mathcal{K}$. For any $\beta \in {}^{\mathbf{B}}\mathbf{C}$, set

$${}^{\mathbf{A}}\mathbf{C}(\beta) = \{\beta \circ \alpha \colon \alpha \in {}^{\mathbf{A}}\mathbf{B}\}$$

the set of embeddings of $\mathbf{A}$ in $\mathbf{C}$ which factor through β.

Once we agree that we should be coloring embeddings when working in the continuous setting, the analogue of finding a copy $\mathbf{B}_0$ of $\mathbf{B}$ in $\mathbf{C}$ such that a coloring is constant on $\binom{\mathbf{B}_0}{\mathbf{A}}$ is finding $\beta \in {}^{\mathbf{B}}\mathbf{C}$ such that a coloring has small oscillation on ${}^{\mathbf{A}}\mathbf{C}(\beta)$. With this in mind, the Ramsey property naturally translates to the following.

Definition 5.53. Let $\mathcal{K}$ be a metric Fraïssé class. We say that $\mathcal{K}$ has the *approximate Ramsey property for embeddings* (ARP) if the following condition is satisfied:

For any $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$, and any $\varepsilon > 0$, there exists $\mathbf{C} \in \mathcal{K}$ such that, for any coloring c of ${}^{\mathbf{A}}\mathbf{C}$, there exists $\beta \in {}^{\mathbf{B}}\mathbf{C}$ such that the oscillation of c on ${}^{\mathbf{A}}\mathbf{C}(\beta)$ is less than ε.

When the class is made up of discrete structures, we are just reformulating the Ramsey property in terms of embeddings rather than substructures. When the class is made up of rigid structures, we recover the usual Ramsey property; however, the Ramsey property for embeddings is stronger, since it actually forces the structures to be rigid.

As in the discrete setting, this condition turns out to be a reformulation of the finite oscillation stability of the action of $(\mathrm{Aut}(\mathbf{K}), d_A)$ for any finite $A \subseteq \mathbf{K}$, and we obtain the following result (unpublished joint work with T. Tsankov).

Theorem 5.54 ([MT13a]). *Let $\mathcal{K}$ be a metric Fraïssé class, and G be the automorphism group of its limit. Then G is extremely amenable if, and only if, $\mathcal{K}$ has the approximate Ramsey property.*

Taking advantage of the continuous setting, one can formulate a formal weakening of the approximate Ramsey property which is equivalent to it (for instance, this enables one to work with a dense subclass of $\mathcal{K}$ rather than the whole of $\mathcal{K}$). Unfortunately, even this weakening seems very hard to prove, and we were unable to use Theorem 5.54 to obtain interesting new examples of extremely amenable Polish groups. One obvious candidate would seem to be the automorphism group of the Gurarij space; while I failed to prove that it is extremely amenable, this was recently achieved by Bartošová, Lopez-Abád and Mbombo, who proved that the class of finite-dimensional Banach spaces has the approximate Ramsey property.

Theorem 5.55 ([i]). *The automorphism group of the Gurarij space is extremely amenable.*

It is not clear to me whether Theorem 5.54 can really be useful; it was of some use as a guide towards obtaining the following result, joint with Nguyen Van Thé and Tsankov.

Theorem 5.56 ([MNT14]). *Let G be a Polish group. Then the following are equivalent.*

(1) *The universal minimal flow of G is metrizable and has a comeager orbit.*
(2) *There exists a closed subgroup G^* such that the right uniformity on G/G^* is precompact, and the universal minimal flow of G is the action $G \curvearrowright \widehat{G/G^*}$.*

Shortly after we proved this theorem, Andy Zucker [Zuc14] announced results that imply in particular that in the important case of subgroups of S_∞ one can remove the assumption of existence of a comeager orbit in the first item above; that is, this assumption is always satisfied for nonarchimedean Polish groups when their universal minimal flow is metrizable. It is an open problem whether one can do away with this assumption in general. One could also wonder whether some of the ideas presented above could be used to make Zucker's approach work for general Polish groups; there appear to be significant difficulties to overcome before achieving this.

5.4. **Ample generics.** When looking at the question of simplicity of full groups of minimal homeomorphisms, and their closures, we already noticed that the existence of an element with a comeager conjugacy class was a desirable, and strong, property for a Polish group G to have. This property is usually not satisfied (for instance P. Wesolek [Wes13] recently proved that no nontrivial locally compact Polish group can have a comeager conjugacy class); it can be particularly enlightening when one thinks of G as the automorphism group of some structure. Indeed, the action of the generic element on the structure should be intimately linked with the

[i] There seems to be no preprint yet; to be updated.

structure's properties, despite the fact that having a comeager orbit is expressible purely in terms of the group. Actually, as was first noted by Hodges–Hodkinson–Lascar–Shelah [HHLS93] , existence of generic tuples in G^n for all n provides a tool to reconstruct the structure from its automorphism group as an *abstract* group (in a model-theoretic sense that I will not go into; this also depends on earlier results of Ahlbrandt–Ziegler [AZ86]).

Definition 5.57 ([KR07])**.** Let G be a Polish group. We say that G has *ample generics* if for all $n \in \mathbf{N}$ there exists $(g_1, \dots, g_n) \in G^n$ such that the diagonal conjugacy class $\{(kg_1k^{-1}, \dots, kg_nk^{-1}) \colon k \in G\}$ is comeager in G^n.

The notion above was introduced, using a somewhat more flexible (and opaque to me) definition, in the context of permutation groups in [HHLS93] ; the above formulation, which makes sense for general Polish groups, comes from [KR07].

Recall that a Polish group G has the *automatic continuity property* if, whenever H is a separable topological group, any homomorphism from G to H must be continuous. Any Polish group with the automatic continuity property must have a unique Polish topology compatible with its group structure (since an abstract group isomorphism between Polish groups which is continuous must also have a continuous inverse), and automatic continuity is a strictly stronger property. To see that the two properties are different, one can for instance note that Kallman [Kal76] proved that the group of p-adic integers has a unique Polish topology compatible with its group structure; but, as observed in [Ros09a, Example 1.6], any uncountable abelian compact Polish group admits a non-continuous homomorphism into S_∞. Indeed, any infinite abelian group has a subgroup of countable, infinite index; if the ambient group G is compact then this subgroup cannot be open, and the left-translation action of G on the coset space produces a discontinuous action of G on a countable set, which is the same thing as a non-continuous mapping from G into S_∞.

It was proved in [HHLS93] that, whenever G is a closed subgroup of S_∞ with ample generics, G must satisfy the *small index property*, i.e. any subgroup of G with countable index must be open. This last property is equivalent to saying that any homomorphism from G to S_∞ is continuous (in one direction, use the fact that the topology of S_∞ has a basis consisting of open subgroups, which are of countable index; in the other direction, look at the action of G on its quotient by some countable subgroup). The following stronger result is due to Kechris and Rosendal [KR07].

Theorem 5.58 (Kechris–Rosendal [KR07])**.** *Let G be a Polish group with ample generics. Then G satisfies the automatic continuity property.*

Using the weak amalgamation property we mentioned in an earlier section, Kechris and Rosendal also provided a Fraïssé-theoretic characterization of closed subgroups of S_∞ with ample generics. These appear to be fairly common among automorphism groups of highly homogeneous discrete structures - for instance, S_∞ has ample generics (which is easy to show by hand), as do the automorphism group of the random graph, the isometry group of the rational Urysohn space...

It is only very recently that examples of Polish groups with ample generics and which are not isomorphic to a subgroup of S_∞ have been discovered, by Malicki

[Mal] and Kaïchouh–Le Maître [KLM]. In the automorphism groups of nondiscrete metric structures that we encountered so far, conjugacy classes are meager. For $\mathrm{Iso}(\mathbf{U})$ this is a result of Kechris (published in a paper of Glasner–Weiss [GW08]). For $U(\mathcal{H})$, one can find a proof of meagerness of conjugacy classes in [GW08], while Kechris [Kec10] refers to Nadkarni's book ([Nad98], Chapter 8); I do not know who first proved the result. For $\mathrm{Aut}(\mu)$, [GW08] points out that meagerness of conjugacy classes follows from a result of del Junco [dJ81], and Kechris [Kec10] attributes meagerness of conjugacy classes there to Rokhlin.

So, automatic continuity via ample generics seems to be a non-starter in those cases. However, these groups do have dense conjugacy classes, at least (Kechris–Rosendal [KR07] for $\mathrm{Iso}(\mathbf{U})$, Rokhlin for $\mathrm{Aut}(\mu)$); and we already noticed that the uniform metric could be of interest - in analysis, one is used to neglecting small, uniformly controlled errors, or at least to working with them.

Definition 5.59. Let (G, τ, ∂) be a Polish topometric group, and A be a subset of G. We set

$$(A)_{<\varepsilon} = \{g \in G \colon \exists a \in A \ \partial(g, a) < \varepsilon\}$$

.

Then, the next best thing after a conjugacy class is the *uniform closure* of a conjugacy class. To make notation a bit simpler below, we denote by $\mathrm{Conj}(\bar{g})$ the diagonal conjugacy class of $\bar{g} \in G^n$.

Definition 5.60. Let (G, τ, ∂) be a Polish topometric group. We say that G has *ample generics* if, for any $\varepsilon > 0$ and any n, there exists $\bar{g} \in G^n$ such that $(\mathrm{Conj}(\bar{g}))_{<\varepsilon}$ is comeager.

If (G, τ) is a Polish group, ∂ is the coarsest bi-invariant distance refining τ, and (G, τ, ∂) has ample generics then we say that (G, τ) has *metric ample generics*.

Note that saying that (G, τ, ∂) has ample generics iff there exists $\bar{g}$ such that the uniform closure of $\mathrm{Conj}(\bar{g})$ is comeager; we call such elements *metric generics*.

It seems somewhat unlikely at first that G might have metric ample generics if it does not have ample generics to start with: indeed, if we assume that there exists a dense conjugacy class, then the fact that ∂ is τ-Baire measurable and bi-invariant imposes that there exists some $r > 0$ such that $\{(g, h) \colon \partial(g, h) = r\}$ is comeager. Thus ∂ looks to be almost constant (and discrete) from the point of view of τ. Also, if there are dense conjugacy classes and no comeager one, then they are all meager; so, we are hoping to take a meager set, expand it by taking an arbitrarily small tubular neighborhood for an almost discrete metric, and obtain something comeager. As it turns out, this can actually happen, as shown by the following examples.

Theorem 5.61 ([BYBM13]). *The Polish groups* $\mathrm{Aut}(\mu)$, $U(\mathcal{H})$ *and* $\mathrm{Iso}(\mathbf{U})$ *all have ample metric generics.*

The key point to prove this is that, in each case, there is a countable substructure sitting inside the continuous one, whose automorphism group has ample generics (when endowed with its permutation group topology) and is a very good approximation of the automorphism group of the continuous structure. One way to formalize this is as follows.

Definition 5.62. Let $\mathbf{M}$ be a Polish metric structure, and $\mathbf{N}$ be a (classical) countable structure. We say that $\mathbf{N}$ is a *good approximating substructure* if the following conditions are satisfied:

- The universe of $\mathbf{N}$ is a countable dense subset of the universe of $\mathbf{M}$.
- Any automorphism of $\mathbf{N}$ extends to an automorphism of $\mathbf{M}$ and (under the obvious identification) $\mathrm{Aut}(\mathbf{N})$ is dense in $\mathrm{Aut}(\mathbf{M})$.
- For every open subset U of $\mathrm{Aut}(\mathbf{N})$ (in its permutation group topology) and any $\varepsilon > 0$, $(U)_{<\varepsilon}$ is open in $\mathrm{Aut}(\mathbf{M})$.

For instance, the countable atomless rational probability algebra (the Fraïssé limit of all finite probability algebras with measure taking only rational values) is a good approximating substructure of the standard atomless probability algebra; the rational Urysohn space is a good approximating substructure of the Urysohn space. By playing Banach-Mazur games, one can then show the following result, which implies in particular that, if $\mathbf{N}$ is a good approximating substructure of $\mathbf{M}$, and $\mathrm{Aut}(\mathbf{N})$ has ample generics as a permutation group, then $\mathrm{Aut}(\mathbf{M})$ has ample metric generics.

Theorem 5.63. *Let $\mathbf{N}$ be a good approximating substructure of a Polish metric structure $\mathbf{M}$. Then, whenever $A \subseteq \mathrm{Aut}(\mathbf{N})$ is comeager (for the permutation group topology of $\mathrm{Aut}(\mathbf{N})$), the uniform closure of A is comeager in $\mathrm{Aut}(\mathbf{M})$ (for the Polish topology of $\mathrm{Aut}(\mathbf{M})$).*

So far, all our examples of Polish groups with ample metric generics come from structures with a good approximating substructure, making one wonder whether this is a general phenomenon. This might simply be a consequence of our lack of examples.

Ample metric generics for a Polish topometric group can be used, in some sense, to translate questions about the topology to (formally easier, and trivial when the metric is discrete) questions about the metric. For instance, using the ideas of [KR07] and some additional work to take care of the ε's, we proved the following result in [BYBM13].

Theorem 5.64. *Let (G, τ, ∂) be a Polish topometric group with ample generics, H be a separable topological group, and $\varphi \colon (G, \partial) \to H$ be a continuous homomorphism. Then $\varphi \colon (G, \tau) \to H$ is continuous.*

This applies in particular to $\mathrm{Iso}(\mathbf{U})$, $U(\mathcal{H})$ and $\mathrm{Aut}(\mu)$. We claimed earlier that the uniform metric in these groups was almost discrete; since it should not be hard to prove continuity of homomorphisms starting from an almost discrete group, we look well on our way to proving automatic continuity for these groups. The situation is actually somewhat more complicated.

Theorem 5.65.

(1) *The group* $\mathrm{Aut}(\mu)$ *has the automatic continuity property (*[BYBM13]*).*
(2) *The group* $U(\mathcal{H})$ *has the automatic continuity property (*Tsankov [Tsa13]*).*
(3) *The group* $\mathrm{Iso}(\mathbf{U})$ *has the automatic continuity property (*Sabok [Sab13]*).*

In the first two cases, the original proof uses Theorem 5.64 (even though, as was pointed out to me by M. Malicki, one could bypass the notion of ample metric generics and work directly with the good approximating substructure and the uniform metric; still, the interplay of metric and topology is fundamental in this argument) to reduce the question to continuity of homomorphism to separable groups when the source Polish group is endowed with its uniform metric. In the case of $\mathrm{Aut}(\mu)$, it turns out to be not too hard to obtain the desired result, by following an argument of Kittrell–Tsankov [KT10] which they used to prove automatic continuity of full groups of ergodic, probability-measure-preserving actions of countable groups. The case of $U(\mathcal{H})$ requires more ingenuity and technical skill, and was dealt with by T. Tsankov [Tsa13].

Automatic continuity for the isometry group of the Urysohn space was proved very recently by Sabok [Sab13], using a different method; his method can be used to obtain automatic continuity for $\mathrm{Aut}(\mu)$ and $U(\mathcal{H})$ as well, though this leads to more complicated, less transparent proofs (to my tastes at least). Still, his technique appears to be more versatile, in that it captures the example of the Urysohn space; both techniques seem powerless to tackle some natural classes of candidates for the automatic continuity property, for instance, full groups of aperiodic, non-ergodic probability-measure-preserving equivalence relations with countable classes.

Once one is convinced that metric generic elements are interesting objects, it becomes worthwhile to try and give an "intrinsic" characterization of them. Rather than try to define formally what I mean here, let me recall the following theorem of Effros, which answers that question for generic elements.

Theorem 5.66 ([Eff65])**.** *Let G be a Polish group acting continuously on a Polish space X. Let $x \in X$ have a dense orbit. Then, the following are equivalent:*

(1) *The orbit $G \cdot x$ is comeager in X.*
(2) *The orbit map $g \mapsto g \cdot x$ is an open map from G to $G \cdot x$.*
(3) *The orbit $G \cdot x$ is a G_δ subset of X.*

The characterization we are interested in, for metric generic elements, is similar to the equivalence of the first two items above. The last item is interesting in its own right, in that it shows that the set of generic elements is G_δ; one may then wonder whether the same is true of metric generic elements in a Polish topometric group.

The second condition above says that, for any open $U \subseteq G$, the set $U \cdot x$ is open in $G \cdot x$. The natural analogue of $U \cdot x$ in the topometric setting is given by sets of the form $(U \cdot x)_{<\varepsilon}$; $G \cdot x$ could be left unchanged, replaced by its uniform closure, or, more ambitiously, replaced by $(G \cdot x)_\varepsilon$. So we have three somewhat natural candidates for a generalization of the Effros theorem to the topometric setting.

The use of ε's threatens to be cumbersome, so it is useful to subsume all of them into a single object: the distance function. Instead of $g \in G$, what we are really working with is the distance function $\partial(g, \cdot)$, and g being a metric generic element is a property of the orbit of that function under the natural shift action of G. It turns out to be possible to think of $\partial(g, \cdot)$ as being a point, in a setting where ∂ plays the role of the diagonal. We turn to a discussion of this approach before going back to the promised topometric version of the Effros theorem.

5.5. **Grey sets.** Material in this section comes from [BYM13], joint with I. Ben Yaacov.

Definition 5.67. Let X be a set. A *grey subset* of X is a function $A \colon X \to [0, +\infty]$.

Given A, B two grey subsets, we write $A \sqsubseteq B$ to mean that $B(x) \leq A(x)$ for all $x \in X$.

The terminology is meant to evoke scales of grey: rather than dealing with sets, where things are black or white (belonging to the set or not), we want to deal with distance to sets, where one can be more or less close to belonging. Of course, subsets can be seen as grey subsets, via their *zero-indicator* functions: given $A \subseteq X$, define

$$\mathbf{0}_A(x) = \begin{cases} 0 & \text{if } x \in A \\ \infty & \text{else} \end{cases}$$

We use "square" versions of usual set-theoretic symbols when working in the grey setting; thus, $\sqcup$ denotes the infimum operation (analogous to the union), while $\sqcap$ denotes the sup (and, unfortunately, this runs contrary to the usual symbols $\vee$ for max and $\wedge$ for min).

The plan is to introduce a variant of descriptive set theory where subsets are replaced with grey subsets, in order to avoid getting bogged down in epsilon-tracking during proofs taking place in the topometric setting. When applied to zero-indicators, the new notions should boil down to the usual notions. An obvious problem with this approach is that there is no complementation operation when dealing with grey subsets; this can be overcome but makes a few definitions somewhat awkward.

5.5.1. *Grey topology.* Throughout, we assume that X is a completely metrizable topological space.

A subset is open iff its zero-indicator is upper semi-continuous, closed iff its zero-indicator is lower semi-continuous, and we have our first definition.

Definition 5.68. Let A be a grey subset of X. We say that A is *open* (respectively, *closed*) if it is upper (respectively, lower) semi-continuous. We write $A \sqsubseteq_o X$ when A is an open grey subset of X.

It is straightforward to check that a union of open grey subsets is open, an intersection of closed grey subsets is closed; consequently one can define the interior A° and closure $\bar{A}$ of a grey subset A, and check the formulas

$$\forall x \in X \quad A^\circ(x) = \limsup_{y \to x} A(y) \quad \text{and} \quad \bar{A}(x) = \liminf_{y \to x} A(y)$$

Definition 5.69. A grey subset $A \sqsubseteq X$ is *meager* if there exists $r > 0$ such that $\forall^* x \in X\ A(x) \geq r$. It is *comeager* if $\forall^* x \in X\ A(x) < r$ for all $r > 0$, equivalently, if $A(x) = 0$ for a comeager set of x.

In the above definition we feel the effect of the lack of a complementation operation, as we cannot say that a grey subset is meager iff its complement is comeager, and the two definitions have a somewhat different flavor.

We write $A \sqsubseteq^* B$ to mean that $\forall^* x \in X\ A(x) \geq B(x)$, similarly for $\sqsupseteq^*$, $=^*$. To do descriptive set theory, we want to define Baire-measurable grey subsets; they should be those which coincide almost everywhere (in the sense of Baire category) with open sets.

Definition 5.70. Let A be a grey subset of X. Define

$$U(A) = \bigsqcup \{O \sqsubseteq_o X \colon O \sqsubseteq^* A\}$$

Then, as in usual descriptive set theory, it is always the case that $U(A) \sqsubseteq^* A$, and we define A to be Baire-measurable if the reverse inclusion holds, namely, $A \sqsubseteq^* U(A)$. It is not hard to see that this is equivalent to the existence of an open grey subset B such that $A =^* B$ and, perhaps more interestingly, also equivalent to the fact that A is a Baire-measurable function from X to $[0, +\infty]$. I see this as a hint that our definitions are the right ones; of course, an equally viable point of view is that our definitions have so far only enabled us to recover a well-known concept that certainly did not need grey subsets to be introduced.

Similarly, one could define a G_δ grey subset either as the (grey) intersection of countably many open sets, or a function $A\colon X \to [0, +\infty]$ such that $A_{\leq r}$ is G_δ for all r. Again, the two definitions coincide.

There is a notion of a grey subset being dense in another: simply say that $A \sqsubseteq B$ is dense if $\bar{A} \sqsupseteq B$; similarly, one can define the relative closure of A in B as being equal to $\bar{A} \sqcap B$. These notions have the expected properties; it is more tedious to define what a relative open subset is, again due to the fact that there is no notion of "grey complement" of a grey subset, so one cannot simply dualize the notion of relative closure. However, a definition can be made to work: define the *relative interior* of $A \sqsubseteq B$ as being equal to $(A - B)^\circ + \overline{B}$. Then one can say that a grey subset of B is relatively open if it coincides with its relative interior; the important example to keep in mind here is that, if U is an open grey subset of the ambient space, then $B + U$ is a grey open subset of B.

Armed with these definitions, we now can state (and prove rather straightforwardly) a grey version of the Baire category theorem: a countable intersection of dense open grey subsets of a G_δ grey subset B of a complete metric space is dense in B.

All this leads to a new version of the Kuratowski–Ulam theorem. Below, when (Y, τ, ∂) is a Polish topometric space, X is a Polish space and $f\colon X \to Y$ is a continuous map, we define for all $y \in Y$ and all $A \sqsubseteq X$ a grey subset A_y of X (the "fibre of A above y") by setting $A_y(x) = A(x) + \partial(f(x), y)$. Similarly, we define a topometric variant of the image of A under f, by setting

$$(f(A))_\partial(y) = \inf_x A(x) + \partial(f(x), y) = \inf_x A_y(x) \, .$$

In the particular case where f is the identity map from Y to itself, we simply denote $(id(A))_\partial$ by $(A)_\partial$. When A is a "true" subset of X this is equal to the ∂-distance to A.

Theorem 5.71 ([BYM13]). *Let (Y, τ, ∂) be a Polish topometric space, X a Polish space, and $\pi\colon X \to Y$ a continuous map. Assume that:*

- *Whenever $U \sqsubseteq X$ is open, $(\pi U)_\partial$ is open in Y.*
- *Whenever $V \sqsubseteq Y$ is open in Y, $(V)_\partial$ is open in Y.*

Then the following conditions are equivalent, for a Baire-measurable $A \sqsubseteq X$:

(1) *The grey set A is comeager in X.*
(2) *The set $\{y \in Y\colon A_y$ is comeager in $X_y\}$ is comeager in Y.*

The above Kuratowski–Ulam theorem is, so far, the main payoff of grey topology for us; when trying to prove an analogue of the Effros theorem in the topometric setting, it is useful to understand how the uniform metric and Baire category interact (recall that, initially, it seemed unlikely that ample metric generics could even exist outside of "usual" ample generics), in particular one needs to show that, whenever A is comeager in an open set O of a Polish group G and r is some positive real, then $(A)_{<r}$ is still comeager in $(O)_{<r}$. This is the content of the following corollary of our grey Kuratowski–Ulam theorem.

Corollary 5.72. *Let (X, τ, ∂) be a Polish topometric space, and assume that $(V)_\partial$ is open for any open $V \sqsubseteq X$. Assume also that $A \sqsubseteq U \sqsubseteq X$, where U is open and A is comeager in U. Then $(A)_\partial$ is comeager in $(U)_\partial$.*

In particular, if $A \sqsubseteq X$ is 1-Lipschitz (relative to ∂), then $U(A)$ is also 1-Lipschitz.

Note that the compatibility assumption between topology and metric featured above is automatically satisfied in a Polish topometric group.

Now we focus on grey subsets of (completely metrizable, not necessarily separable) groups.

Definition 5.73. Let G be a group. For $A, B \sqsubseteq X$, we define

$$A * B(g) = \inf_{hk=g} A(h) + B(k)\,, \quad A^{-1}(x) = A(x^{-1})\,.$$

The $*$ operation is a form of convolution, and extends to grey subsets the group operation of G as applied to subsets of G (identified with their zero-indicator function). One can then extend to the grey context several classical, and useful, properties of grey subsets of completely metrizable groups.

Lemma 5.74 (Pettis' theorem for grey subsets)**.** *Let A, B be grey subsets of a completely metrizable group G. Then $U(A) * U(B) \sqsubseteq A * B$.*

Now, let us go further, and try to see what a "grey subgroup" should be. A subset H of a group G is a subgroup if the following conditions are satisfied: H is nonempty, and $HH^{-1} \subseteq H$.

Thus, a grey subgroup $H \sqsubseteq G$ should be a grey subset such that $\inf H = 0$, and $H * H^{-1} \sqsubseteq H$. Explicitly, this last condition says that $H(x) + H(y) \geq H(xy^{-1})$ for all $x, y \in G$. These two conditions imply that $H(1) \leq \inf_{xy=1} H(x) + H(y^{-1}) = 2\inf(H) = 0$, from which the fact that $H * H^{-1} \sqsubseteq H$ yields $H = H^{-1}$; finally we see that the conditions are equivalent to writing $H(1) = 0$, $H(g^{-1}) = H(g)$ and $H(gh) \leq H(g) + H(h)$ for all g, h. In other words, our grey subgroups are simply *seminorms* on G, which are themselves in natural bijection with left-invariant pseudometrics on G. So the grey analogue of a subgroup is a left-invariant pseudometric; hence one should expect that results concerning subgroups of permutation groups should translate, using the topometric formalism, to results about left-invariant pseudometrics.

One example of this phenomenon is the translation of the small index property. Recall that a subgroup G of S_∞ has the *small index property* if any subgroup of G of index strictly below the continuum is open. We know by now that this property implies that any homomorphism from G to S_∞ is continuous (and really, for that one only cares about subgroups of countable index). Let us now try to translate the small index property in the grey context. Thinking of a left-invariant pseudometric as the counterpart of a subgroup, the obvious analogue of "index strictly below the continuum", that is, "cardinality of G/H strictly below the continuum", is "the density character of the metric space associated to the pseudometric is strictly below the continuum". Since "open" translates to "continuous", we have a first candidate: "any left-invariant pseudometric on G of density character strictly below $2^{\aleph_0}$ is continuous"; equivalently, any homomorphism from G to a metrizable group of density character strictly below $2^{\aleph_0}$ is continuous. This implies that any homomorphism from G to a separable group is continuous (what we called earlier the automatic continuity property); in the discussion above it would also make sense to replace all instances of the words "strictly below $2^{\aleph_0}$" by the word "countable", and the proposed analogue of the (very) small index property is then exactly the automatic continuity property. There is something wrong in this picture, however: we are not taking the topometric structure into account at all. Some compatibility between the pseudometric under consideration, and the uniform metric on our topometric group, should be assumed.

Definition 5.75. Let (G, τ, ∂) be a Polish topometric group. We say that (G, τ, ∂) has the *small density property* if, whenever d is a left-invariant pseudometric such that the density character of (G, d) is $< 2^{\aleph_0}$, and d is Baire measurable with respect to ∂, d must be continuous.

Equivalently: any homomorphism from G to a metrizable group of density character $< 2^{\aleph_0}$ which is ∂-Baire measurable is τ-continuous.

In the definition above, one would not change anything if one replaced ∂-Baire measurability with ∂-continuity.

Theorem 5.76 ([BYM13])**.** *Let (G, τ, ∂) be a Polish topometric group with ample generics. Then (G, τ, ∂) has the small density property.*

This is essentially a variant of the automatic continuity theorem proved in [BYBM13], though the approach via grey sets makes the proof neater and probably easier to comprehend.

5.5.2. *A topometric version of Effros' theorem.*

Theorem 5.77 ([BYM13])**.** *Assume that (X, τ, ∂) is a Polish topometric space, and that G is a Polish group acting continuously on X by τ-homeomorphisms which are also ∂-isometries. Assume further that, for any U open in X and any $r > 0$, the set $(U)_{<r}$ is open. Assume also that $x \in X$ is such that $G \cdot x$ is dense. Then the following conditions are equivalent:*

(1) $\overline{G \cdot x}^{\partial}$ *is G_δ.*
(2) $\overline{G \cdot x}^{\partial}$ *is comeager.*
(3) *For any open subset U of G and any $r > 0$, $(U \cdot x)_{<r}$ is open in $\overline{G \cdot x}^{\partial}$.*
(4) *There exists $y \in \overline{G \cdot x}^{\partial}$ such that, for any open subset U of G and any $r > 0$, $(U \cdot y)_{<r}$ is open in $G \cdot y$.*

Note that all the assumptions above are satisfied when $X = (G, \tau, \partial)$ is a Polish topometric group (or a power thereof) and G acts by (diagonal) conjugacy.

The only interesting implications here are (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1). As it turns out, to close the implication diagram once one has proved the implication (2) $\Rightarrow$ (3) it is simpler to prove that (4) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1); these implications are both instances of a topometric variant of a well-known theorem of Hausdorff stating that a metrizable space which is a continuous, open image of a Polish space is Polish itself. I will only discuss the proof of (2) $\Rightarrow$ (3).

Proof of (2) $\Rightarrow$ (3). Denote by π the orbit map $g \mapsto g \cdot x$. Fix a countable basis $(O_n)_{n<\omega}$ for the topology of G; for any n $(\pi O_n)_\partial$ is Baire-measurable and 1-Lipschitz (relative to ∂). Then 5.72 shows that $U_n = U((\pi O_n)_\partial)$ is also 1-Lipschitz.

Let $\Omega = \{y \colon \forall n (\pi O_n)_\partial(y) = U_n(y)\}$. This is a τ-comeager, ∂-closed subset. Also, for any $O \sqsubseteq_o G$, $(\pi O)_\partial \sqcap \mathbf{0}_\Omega \sqsubseteq_o \mathbf{0}_\Omega$.

Now, let $B = \{y \colon \forall^* g \in G\, g \cdot y \in \Omega\}$. This set is G-invariant, τ-comeager, and ∂-closed. The first point is obvious, the second follows from the (usual) Kuratowski-Ulam theorem, and to see why the third holds assume that $b_i \in B$ and $b \in X$ are such that $\partial(b_i, b) \to 0$. Then there exists a comeager set of $g \in G$ such that $g \cdot b_i \in \Omega$ for all i, so since Ω is ∂-closed we get $g \cdot b \in \Omega$ for all such g, i.e. $b \in B$.

It follows that $\overline{G \cdot x}^\partial$ is contained in B; to conclude, it is enough to prove that for all $U \sqsubseteq_o G$ $(\pi U)_\partial \sqcap \mathbf{0}_B \sqsubseteq_o \mathbf{0}_B$. To that end, let $b_i \in B$ converge to $b \in B$; there exists $g \in G$ such that $g \cdot b \in \Omega$ and $g \cdot b_i \in \Omega$ for all i.

Since $(\pi g U)_\partial \sqcap \mathbf{0}_\Omega \sqsubseteq_o \mathbf{0}_\Omega$, we have $\limsup (\pi g U)_\partial (g \cdot b_i) \leq (\pi g U)_\partial (gb)$, equivalently $\limsup (\pi U)_\partial (b_i) \leq (\pi U)_\partial (b)$. □

5.6. **Meagerness of conjugacy classes in the space of actions.** We conclude this text by discussing a topometric approach to proving that conjugacy classes are meager in $\mathrm{Hom}(\Gamma, G)$ for some countable groups Γ and Polish groups G - so far this approach only really works when G is the isometry group of the Urysohn space.

Assume that $\mathbf{M}$ is a Polish metric structure, and that G is its automorphism group, which we turn into a Polish topometric group in the usual way. For any finite $A \subset M$, denote by G_A the pointwise stabilizer of A, and assume that for any $\varepsilon > 0$ the set $(G_A)_{<\varepsilon}$ contains 1 in its interior. This is true for instance in the standard atomless probability algebra, the Urysohn space or the Urysohn sphere; the assumption is a bit stronger than what we really need but simplifies exposition somewhat.

Fix a countable group Γ. We may endow $\mathrm{Hom}(\Gamma, G)$ with a very strong uniform metric d_∞, defined by $d_\infty(\pi, \sigma) = \sup_{g \in \Gamma} d_u(\pi(g), \sigma(g))$. Here d_u denotes the uniform metric on $G = \mathrm{Aut}(\mathbf{M})$; note that even if $\Gamma = \mathbf{Z}$ d_∞ is much finer than d_u, since we are taking a supremum over all elements of $\mathbf{Z}$. In the case of $\mathrm{Aut}(\mu)$, this metric is considered in [Kec10], where it is proved that conjugacy classes are clopen in the topology induced by d_∞.

Now, assume that $\pi_0 \in \mathrm{Hom}(\Gamma, G)$ has a comeager conjugacy class. Then, for any neighborhood U of 1, π_0 must belong to the interior of $\overline{U \cdot \pi_0}$ (where closure is relative to the Polish topology on $\mathrm{Hom}(\Gamma, G)$, and $\cdot$ denotes the conjugacy action of G on $\mathrm{Hom}(\Gamma, G)$). Thus, under our assumption on $\mathbf{M}$, π_0 must belong to the interior of the closure of $(G_A)_{<\varepsilon} \cdot \pi_0$ for any finite $A \subset M$ and $\varepsilon > 0$.

Let us focus on $V = (G_A)_{<\varepsilon} \cdot \pi_0$ for a moment: assume that π belongs to this set; then there exists $h \in G_A$ and $g \in (1)_{<\varepsilon}$ such that $\pi = gh \cdot \pi_0$. Thus for any $a, b \in A$ and $\gamma, \delta \in \Gamma$ we have that $|d(\pi(\gamma)a, \pi(\delta)b) - d(\pi_0(\gamma)a, \pi_0(\delta)b)|$ is equal to

$$
\begin{aligned}
&|d(gh\pi_0(\gamma)h^{-1}g^{-1}a, gh\pi_0(\delta)h^{-1}g^{-1}b) - d(\pi_0(\gamma)a, \pi_0(\delta)b)| \\
&\leq 2\varepsilon + |d(h\pi_0(\gamma)h^{-1}a, h\pi_0(\delta)h^{-1}b) - d(\pi_0(\gamma)a, \pi_0(\delta)b)| \\
&= 2\varepsilon + |d(h\pi_0(\gamma)a, h\pi_0(\delta)b) - d(\pi_0(\gamma)a, \pi_0(\delta)b)| \\
&= 2\varepsilon\,.
\end{aligned}
$$

So for any $\pi \in V$, we have for all $\gamma, \delta \in \Gamma$ and all $a, b \in A$ that

$$|d(\pi(\gamma)a, \pi(\delta)b) - d(\pi_0(\gamma)a, \pi_0(\delta)b)| \leq 2\varepsilon \, .$$

Note that the set of all π satisfying these conditions is closed in $\mathrm{Hom}(\Gamma, G)$, while we know that the closure of V contains π_0 in its interior: hence there exists an open neighborhood W of π_0 in $\mathrm{Hom}(\Gamma, G)$ such that

$$\forall \pi \in W \, \forall \gamma, \delta \in \Gamma \, \forall a, b \in A \, |d(\pi(\gamma)a, \pi(\delta)b) - d(\pi_0(\gamma)a, \pi_0(\delta)b)| \leq 2\varepsilon \, .$$

In other words, the map $\pi(\gamma)a \mapsto \pi_0(\gamma)a$ must be an isomorphism from $\pi(\Gamma)A$ to $\pi_0(\Gamma)A$ up to a prescribed error 2ε: the finite number of constraints imposed by the open set W must control the whole orbit $\pi(\Gamma)A$ up to a prescribed error. This seems to be a very strong condition that is unlikely to hold when Γ is infinite.

Theorem 5.78. *For any infinite, countable group* Γ, *conjugacy classes are meager in* $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$ *and* $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}_1))$.

Proof. One can use the above criterion with A a singleton to derive a contradiction. Let us give the proof for $\mathrm{Iso}(\mathbf{U})$: assume for a contradiction that $\pi_0 \in \mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$ has a comeager conjugacy class. There must exist an open subset W of $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$ containing π_0 and such that, for all $\pi \in W$ and all $\gamma \in \Gamma$, one has

$$|d(\pi(\gamma)a, a) - d(\pi_0(\gamma)a, a)| \leq 1 \, .$$

This implies that all elements of W have bounded orbits, or all elements of W have unbounded orbits (depending on how π_0 behaves).

Thus to derive a contradiction it is enough to prove that Γ-actions with bounded orbits and Γ-actions with unbounded orbits are both dense in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$. This is easy to do. Indeed, let

$$O = \{\pi \colon \forall a \in A \forall \gamma \in F \; d(\pi(\gamma)a, \sigma(\gamma)a) < \varepsilon\}$$

be an open subset (with A, F finite, $\varepsilon > 0$). Then consider the supremum M of all distances between elements of $\{\sigma(\gamma)a \colon \gamma \in F, a \in A\}$; let (X, d) be the metric space $\sigma(\Gamma)A$. One can endow it with a new metric $\rho = \min(d, M)$; Γ still acts isometrically on (X, ρ), which may be embedded in $\mathrm{Iso}(\mathbf{U})$; denote this new action by π. Using the homogeneity of $\mathbf{U}$, and the fact that ρ, d agree on elements $\{\sigma(\gamma)a \colon \gamma \in F, a \in A\}$ we obtain that π belongs to O, and has bounded orbits. Thus the set of elements with bounded orbits is dense in $\mathrm{Hom}(\Gamma, G)$ for any countable group Γ.

Now, let ρ be an unbounded left-invariant metric on Γ (this exists because Γ is countable infinite). The left-translation $\Gamma \curvearrowright (\Gamma, \rho)$ extends to an action $\pi \colon \Gamma \curvearrowright \mathbf{U}$ with unbounded orbits; thus we obtain two pseudometrics on $\Gamma \times A$ which are invariant when Γ acts on $\Gamma \times A$ by left-translation on the first coordinate: $d_1((\gamma, a), (\delta, b)) = d(\sigma(\gamma)a, \sigma(\delta)b)$ and $d_2((\gamma, a), (\delta, b)) = d(\pi(\gamma)a, \pi(\delta)b)$. For any $r > 0$ $d_1 + rd_2$ is a pseudometric on $\Gamma \times A$ which is invariant under the left-translation action, and elements have unbounded orbits for $d_1 + rd_2$ under this action. As r goes to 0, the values of $d_1 + rd_2$ on $F \times A$ get arbitrarily close to the values of d_1 on $F \times A$, so for r small enough, using the homogeneity of $\mathbf{U}$, we obtain an action of Γ that belongs to O and has unbounded orbits. This concludes the proof for $\mathrm{Iso}(\mathbf{U})$. One can use similar ideas to deal with $\mathrm{Iso}(\mathbf{U}_1)$ (looking at the behaviour of $d(\pi(\gamma)a, a)$ as γ goes to ∞), though I will not give details here. □

It would be much more interesting to be able to prove the same result for $G = \mathrm{Aut}(\mu)$; this is related to questions about complexities of some classification problems. It is known since work of Foreman and Weiss [FW04] that conjugacy classes are meager in $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ for any infinite amenable Γ. They used entropy for amenable actions as an invariant that contradicts the possibility of a comeager conjugacy class. This makes it tempting to believe that, using the notion of entropy for measure-preserving actions of sofic groups (see e.g. [Bow10] and [Ker13]), one could extend their result to all sofic groups. But entropy for sofic groups is significantly more complicated than for amenable groups and at the moment I do not know whether this approach can be fruitful in this generality.

While the approach discussed above leads to some partial results, it does not seem to be powerful enough to solve the problem of existence of comeager conjugacy classes for all countable groups. In particular, it seems powerless to prove that conjugacy classes are meager in the case of infinite groups with property (T) (or, maybe, it suggests that groups with property (T) are a good place to look for examples of groups for which there exists a comeager conjugacy class in the space of measure-preserving actions).

I did not discuss the case of the unitary group in this section - this is because Kerr–Li–Pichot [KLP10] proved that conjugacy classes are meager in $\mathrm{Hom}(\Gamma, U(\mathcal{H}))$ for any countable infinite group Γ. They actually prove more, using an approach based on operator algebras; maybe the approach discussed above can be used to give a simpler proof than theirs in the case of countable groups.

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