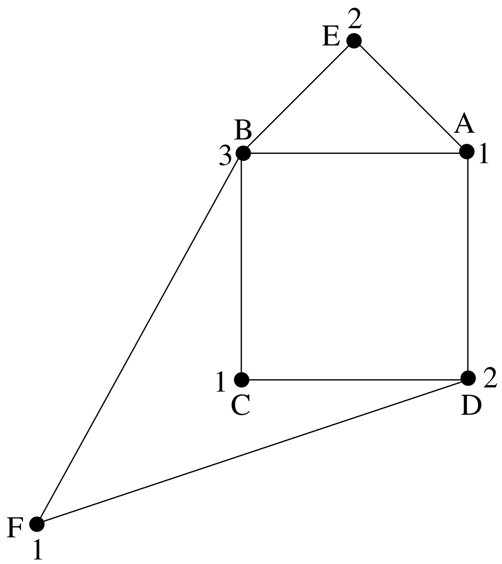
University of Illinois at Urbana-Champaign — Spring 2007
Math 181 — Group F1

Midterm 2.
Correction.

1. The table below shows chemical compounds which cannot be mixed without causing dangerous reactions. Draw the graph that would be used to facilitate the choice of disposal containers for these compounds; what is the minimal number of containers needed?

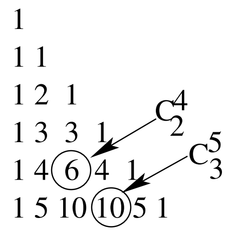
	A	B	C	D	E	F
A		X		X	X	
B	X		X		X	X
C		X		X		
D	X		X			X
E	X	X				
F		X		X		



Answer. The graph is above; a vertex-coloring of this graph will at least use three colors since the graph contains a triangle, and the coloring above shows that 3 colors is enough. So the chromatic number of the graph is 3, which means that the minimal number of containers needed is 3.

2. (a) In designing a security system for its accounts, a bank asks each customer to choose a five-digit number, all the digits to be distinct and nonzero. How many choices can a customer make?
Answer. There are $9 \times 8 \times 7 \times 6 \times 5 = 15120$ possible choices.
(b) A restaurant offers 4 soups, 10 entrees and 8 desserts. How many different choices for a meal can a customer make if one selection is made from each category? If 3 of the desserts are pie and the customer will never order pie, how many different meals can the customer choose?
Answer. If one selection is made from each category, there are $4 \times 10 \times 8 = 320$ different choices; if the customer never orders pie, he has only 5 desserts to choose from and thus can make $4 \times 10 \times 5 = 200$ different choices.
(c)You want to create a mileage grid showing the distance between every pair of the 50 U.S state capitals. How many numbers will you have to compute?
Answer. One needs to compute $(50 \times 49)/2 = 1225$ numbers (the division by 2 comes from the choice of which capital one considers first).

3.(a) In how many ways can a voter rank three candidates, without allowing ties ? And if ties are allowed ?
Answer. Without allowing ties, there are $3! = 6$ different possible choices. If ties are allowed, one must add the case where there is a three-way tie, and the cases when there are two-way ties ; there are $2 \times C_2^3 = 6$ of those (depending on whether the first two or the last two are tied), so overall there are 13 possible ways of ranking three candidates if ties are allowed.
(b) Use Pascal's triangle to compute C_2^4 and C_3^5.
Answer. From the picture below, one obtains $C_2^4 = 6$ and $C_3^5 = 10$.



(c) How many different combinations of YES and NO votes can there be with 5 voters ?
Answer. There are $2^5 = 32$ different combinations of YES and NO votes with 5 voters.

4. (a) Explain the Pareto condition.
Answer. A voting system satisfies this condition provided that if every voter prefers a candidate to another, then it is impossible for the latter candidate to win the election.
(b) Explain independence of irrelevant alternatives.
Answer. A voting system satisfies IIA if the only way a candidate (call him A) can go from losing one election to being a winner of a new election is for at least one voter to reverse his or her ranking of A and the previous winner.
(c) Explain why majority rule is not a good way to choose between four alternatives.
Answer. The problem is that it may happen that no candidate obtains a majority of the vote, in which case there is no winner using majority rule.
(d) Arrow's impossibility theorem states that any voting system can give undesirable outcomes. Explain.
Answer. For any voting system, it is possible to find a set of voters' preference ballots that will cause the voting system to violate a condition deemed desirable for a fair voting system. Examples of such conditions are the Condorcet Winner Criterion, the Pareto condition and the Independence of Irrelevant Alternatives condition.
(e) Explain the Chair's paradox.
Answer, The chair paradox may be described as the fact that, with three voters and three candidates, the voter with tie-breaking power (the "chair") can, if all voters act rationally in their own self-interest, end up with his or her least-preferred candidate as the election winner.

5. Twenty-nine voters must choose from among three alternatives : A,B and C. The voters preference schedules are shown below. The method used is the Borda count.

	Number of voters			
	12	8	6	3
First choice	B	C	A	C
Second choice	C	A	B	B
Third choice	A	B	C	A

(a) What is the winning alternative ?
Answer. The Borda score of A is $8 \times 1 + 6 \times 2 = 20$; the score of B is $12 \times 2 + 6 \times 1 + 3 \times 1 = 33$, and the score of C is $12 \times 1 + 8 \times 2 + 3 \times 2 = 34$. Hence the winning alternative is C.
(b) Can the voters who least prefer the winning alternative change their preference list to produce an outcome they like better (explain why/why not) ?
Answer. Yes : if these voters replace their ballot by the ballot B,A,C (in that order), then B wins ; actually it is enough that just one of these voters changes his/her ballot in that way to have B become the winner.
(c) Which alternative would have won if the Hare system had been used ?
Answer. Candidate A is deleted first, leading to a runoff between B and C in which B obtains 18 first-place votes and C obtains 11 firs-place votes ; so candidate B would have won if the Hare system had been used.

6. Fifty voters who elect one of the five candidates A,B,C,D or E have the preference schedule shown below.

	Number of voters			
	20	14	10	6
First choice	A	B	B	C
Second choice	C	A	A	D
Third choice	E	D	C	B
Fourth choice	B	C	D	A
Fifth choice	D	E	E	E

(a) Which candidate (if any) will be elected using plurality voting ?
Answer. Candidate B has the most first-place votes so he/she would be elected if plurality voting was used.
(b) Same question for the Condorcet method.
Answer. A beats E by 50 to 0, C by 44 to 6 and D by 44 to 6 ; B beats A by 30 to 20, and C beats B by 26 to 24. So each candidate loses to at least one other candidate : there is no Condorcet winner for this election.
(c) Same question for sequential pairwise voting with the agenda A,B,C,D,E.
Answer A loses to B $(20 - 30)$, then B loses to C (24-26), C defeats D $(36 - 14)$ and C defeats E $(50 - 0)$. So with this agenda candidate C wins.
(d) Can you find an agenda that would make candidate A win ?
Answer. Yes : if one uses the agenda B,C,A,D,E, the results of one-one one duels from question (a) show that A wins (the only candidate to whom A loses is B, and B loses to C before meeting A).

7. Eight board members vote by approval voting on four candidates A,B,C and D. An "X" in the table below indicates an approval vote.

Approval ballots

A	X	X	X	X		X	X	X
B		X	X		X	X		X
C		X		X	X		X	X
D	X		X	X	X	X	X	

(a) Which candidate will be chosen by the board if just one of them is to be elected ?
Answer. The candidate with the most approval ballots is A, so he/she is the one that would be chosen if just one of them is to be elected.
(b) Which candidates would be chosen if 75 percent approval was needed ?
Answer. Candidate A has 87.5 percent approval ; B and C both have 62.5 percent, and D has 75 percent. So A and D would be chosen.

8. Use the following election to show that the plurality runoff voting system is manipulable.

A	A	C	C	B
B	B	A	A	C
C	C	B	B	A

Answer. In the election above, the runoff is between A and C, and C wins. However, the leftmost voter would prefer B to win over C, and he can achieve that outcome by changing is ballot to B,A,C (in that order). In the new election, the runoff is between B and C, and B wins. Thus, an unilateral change of ballot by a voter may lead to an outcome that is prefered by that voter, which shows that the plurality runoff voting system is manipulable.

9. True or false ?
Any graph admits a vertex-coloring using 4 colors or less.
Answer. This is **FALSE** ; the 4 Color Theorem states that any *planar* graph admits a vertex-coloring using 4 colors or less. The plurality voting system is not manipulable.
Answer. This is **TRUE** : the plurality voting system is group-manipulable but not manipulable.
The Borda count voting system satisfies the independence of irrelevant alternatives criterion.
Answer. This is **FALSE** (the ballots in exercise 5 provide an example of that !). Sequential pairwise voting satisfies the Condorcet criterion.
Answer. This is **TRUE** : if there is a Condorcet winner then he/she wins all his/her one-on-one duels, so he/she wins no matter which agenda is used.
Every set of voters' preference ballots produces a Condorcet winner.
Answer. This is **FALSE** ; we saw it in class, and this is the reason why we tried to find voting systems that were better than the Condorcet voting system.