University of Illinois at Urbana-Champaign — Spring 2007
Math 181 — Group F1

Midterm 3.

Friday, April 27th.

No documents allowed. Mobile phones, mp3 players, etc., are also forbidden. The one and only piece of equipment you may use is a basic calculator- and you won't need it.
You must provide explanation for all your answers.

NAME ————————————————————————

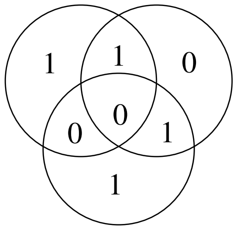
1.Consider the following binary linear code :

0000	→ 0000000	0110	→ 0110010
0001	→ 0001011	0101	→ 0101110
0010	→ 0010111	0011	→ 0011100
0100	→ 0100101	1110	→ 1110100
1000	→ 1000110	1101	→ 1101000
1100	→ 1100011	1011	→ 1011010
1010	→ 1010001	0111	→ 0111001
1001	→ 1001101	1111	→ 1111111

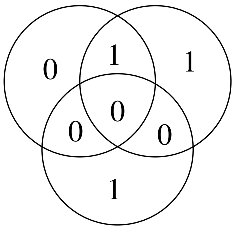
(a) What is the weight of this code ?
The weight of a code is the minimal number of 1's occuring in non-wero code words of the code ; hence here it is equal to 3.
(b) How many errors could this code detect ? How many could it correct ?
Following the formulas that we saw in class, we know that the code could detect any $3-1=2$ errors, and correct any $(3-1)/2=1$ error. This means that if a single-digit error is made during transmission then nearest-neighbor decoding will recover the correct word.
(c) Using nearest-neighbor decoding, decode (or explain why you cannot decode) the message 1011011.
Answer. The message 1011011 is not a code word, but it has a nearest neighbor in the code, which is 1011010 (at distance 1) ; so one decodes it as if one had received 1011010, meaning that it is decoded as 1011.

2. One creates a code for five-digit binary strings by using the parity-check sums $a_1+a_2+a_3$, $a_3+a_4+a_5$ and a_2+a_4.
(a) How many code words would you have to compute if you were to give the code in full ?
Answer. There are $2^5=32$ five-digit binary strings, so one would have to compute 32 code words.
(b) Write down the code words for 01010 and 11000.
Answer. For 01010, one gets $a_1+a_2+a_3=1$, $a_3+a_4+a_5=1$ and $a_2+a_4=2$, so the corresponding code word is 01010110. Similarly, the code word for 11000 is 11000001.

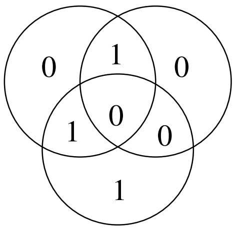
3. (a) Use the Venn diagram method to encode the string 1001.
Answer.



Using the figure above, one obtains that the code word for 1001 is 1001101.
(b) Use the Venn diagram method to decode 1000011 and 1100001 (each time, say whether the code word is correct or not).
Answer.



Both the left–hand circle and the bottom circle are incorrect; so one switches the number that belongs to both of them and not to the other circle, guessing that the original message was 1100011, which is decoded as 1100.



The message is again incorrect; since two circles out of three look correct, one guesses that the erroneous digit is the one that belongs only to the third circle. Thus the guess is that the message received was 1100011, which is decoded as 1100.

4. Recall that the check-digit a_{10} is added to a nine-digit ZIP+4 code in such a way that $a_1+a_2+a_3+\ldots+a_10$ ends with a 0.
(a) Find the check digit for the ZIP+4 code 61820-1309.
Answer. The check-digit a_{10} must be such that $6+1+8+2+0+1+3+0+9+a_{10} = 29+a_{10}$ ends with a 0, so one has $a_{10} = 1$.
(b) Is the number 61801-1405-2 valid ? If not, can you correct the error ? If you know that the fourth digit is incorrect, can you correct the error ?
Answer. The sum is equal to $6+1+8+0+1+1+4+0+5+2 = 28$, which doesn't end with a zero ; we don't have neough information to recover the correct number. If, however, we know that a_4 is the same number, then the sum tells us that $28+a_4$ ends with a 0, so that one has $a_4 = 2$ and the correct number is 61821-1405-2.

5. Consider the following voting system : [54 :45,43,7,5,1]
(a) List minimal winning coalitions.
Answer. Recall that the minimal winning coalitions are those in which each voter is critical ; the list of such coalitions here is : $\{A,B\}$, $\{A,C,D\}$, and $\{B,C,D\}$.
(b) Is there a dictator ? Voters with veto power ? Dummy voters ?
Answer. A dictator is a voter which belongs to every winnng coalition, so there is no dictator here ; similarly, a voter with veto power is one who belongs to every blocking coalition, and from the list of minimal voting coalitions above one can see that no one has this power here. Finally, a dummy voter is one who does not appear in a minimal voting coalition (that's equivalent to saying that his/her vote doesn't matter), so here voter E is a dummy voter.

6. Are the voting systems [11 :10,9,2] and [2 :1,1,1] equivalent ? (explain)
Answer. To decide whether two voting systems are equivalent, one can look at the minimal winning coalitions : in the first system these are $\{A, B\}$, $\{A, C\}$ and $\{B, C\}$; in the second system the minimal winning coalitions are again $\{A, B\}$, $\{A, C\}$ and $\{B, C\}$. So the two voting systems have the same minimal winning coalitions, which implies that they are equivalent.
One could also phrase things differently, noticing that in both systems a motion passes exactly when tvo voters vote "yes" ; thus the condition for a motion to pass is the same in both systems, and this is the same as saying that the systems are equivalent.

7. A weighted voting system has five members.
(a) How many (distinct) coalitions are there ?
Answer. There are $2^5 = 32$ distinct coalitions.
(b) How many (distinct) coalitions are there in which exactly two voters vote YES ?
Answer. There are $C_2^5 = 5.4/2 = 10$ such coalitions.

8. Consider the voting system [8 :5,2,2,2,2,2].
(a) What are the winning coalitions in which A is critical ? (**describe** them, don't give a list !)
Answer. The winning coalitions in which A is critical are those which contain A, have a weight more than or equal to 8, and strictly less than 5 extra votes. So these coalitions are the ones that are composed of A and 2 or 3 other voters.
(b) Compute the Banzhaf power index of A.
Answer. To compute in how many winning coalitions A is critical, we have to count how many coalitions of the form "A + 2 weight-2 voters" there are, and how many coalitions of the form "A+3 weight-2 voters" there are. To form a coalition of the first kind, one must choose two people among 5 ; so there are $C_2^5 = 10$ of them. Similarly, there are $C_3^5 = 10$ coalitions of the second kind. Thus overall A is critical in 20 winning coalitions, so the Banzhaf power index of A is $2.20 = 40$ (don't forget that one needs to double the number of *winning* coalitions in which a voter is critical in order to account for the *blocking* coalitions in which that same voter is critical),

9. Given the weighted voting system [10 :5,5,3,2], give the Shapley-Shubik power index for each voter.
Answer. There are 24 permutations to consider, which is done in the table below.

For each permutation a circle indicates the pivotal voter.

A (B) C D	B (A) C D	C A (B) D	D A (B) C
A (B) D C	B (A) D C	C A (D) B	D A (C) B
A C (B) D	B C (A) D	C B (A) D	D B (A) C
A C (D) B	B C (D) A	C B (D) A	D B (C) A
A D (B) C	B D (A) C	C D (A) B	D C (A) B
A D (C) B	B D (C) A	C D (B) A	D C (B) A

Counting, one obtains that A and B are both pivotal in 8 permutations, and that C and D are both pivotal in 4 permutations. Dividing by the total number of permutations, one obtains that the Shapley-Shubik power index of this system is [8/24,8/24,4/24,4/24]=[1/3,1/3,1/6,1/6].

10. Recall that for practical purposes, ZIP+4 codes (along with their check-digit) are printed using bar codes, with the following correspondence between bar code and digits :

Decimal digit	1	2	3	4	5	6	7	8	9	0
Bar code	ııı||	ıı|ı|	ıı||ı	ı|ıı|	ı|ı|ı	ı||ıı	|ııı|	|ıı|ı	|ı|ıı	||ııı

Is the code below correct ? If not, can you recover the correct ZIP+4 code ?

ı|ı|ı	ı|ı|ı	ı|ıı|	ıı||ı	ı|ı|ı	|ıı|ı	|ıı|ı	ı||ıı	|ı|||	|ıı|ı
5	5	4	3	5	8	8	6	?	8

Answer. The code is not correct because the 9th digit is unreadable (the sequence of short and long bars does not correspond to anything in the table above). Assuming that no other error has been made, we can still recover the correct code : the 9th digit a_9 must be such that $5+5+4+3+5+8+8+6+a_9+8 = 52+a_9$ ends with a 0, so we obtain that $a_9 = 8$. From this we get that the correct ZIP+4 code was 55435-88688.