University of Illinois at Urbana-Champaign — Fall 2006

Math 380 — Group G1

Final Exam.

Friday, December 15.

3 hours

You are not allowed to use your textbook or any other kind of documentation. Calculators, mobile phones and other electronic devices are also prohibited.

NAME ____

SIGNATURE ____

1. (20 points)
Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous partial derivatives. Define a function $g\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$g(x) = f\big(x+1, \ln(1+x^2)\big) \ .$$

Explain why g is differentiable and give a formula for $g'(x)$ (the formula in question should involve the partial derivatives of f).

2. (20 points)

Recall that polar coordinates (r, θ) are linked to cartesian coordinates (x, y) by the formulas $x = r\cos(\theta)$, $y = r\sin(\theta)$. Let now γ be a curve parameterized (in polar coordinates) by the formula $r(\theta) = e^{\theta}$, for $0 \leq \theta \leq 1$.

(a) Find a parametrization for γ in cartesian coordinates.

(b) Compute the length of γ.

3. (20 points)

For $x \geq 0$, set $F(x) = \displaystyle\int_0^1 \ln(1 + xe^t)dt$. Give an expression for $F'(x)$ that doesn't involve an integral.

4. (20 points)
Let γ be the path parameterized by $x(t) = 1 + \sin(2\pi t)e^{2t+3}$, $y(t) = \ln(1+8t)$, $z(t) = t^2 + 4t + 4$, $0 \le t \le 1$. Assume γ is oriented in the direction of increasing t; compute

$$\int_\gamma (3x^2 + 2x\sqrt{z}e^y)dx + (2y + x^2\sqrt{z}e^y)dy + (\frac{x^2 e^y}{2\sqrt{z}})dz \ .$$

5. (25 points)
Let S be the cone of equation $x^2 + y^2 = z^2$, $0 \leq z \leq H$ (viewed as a closed surface). Compute

$$\iint_S (1 - z^2) y^2 \, d\sigma$$

6. (30 points)

Consider the system of equations $\begin{cases} x_1 + 2x_2 + 3x_3 + 10x_4 &= 0 \\ 4x_1 + 5x_2 + 6x_3 + x_4^2 &= 0 \\ 7x_1 + 8x_2 + 9x_3 + x_4^3 &= 0 \end{cases}$.

Show that this system implicitly defines x_1, x_2, x_4 as functions of x_3 near $(0,0,0,0)$; compute $x_1'(0)$, $x_2'(0)$, $x_4'(0)$.

7. (35 points)
Let V be the region of space of equation $x^2+y^2 \leq z \leq 2-(x^2+y^2)$. Denote by S the boundary of V, oriented by the outer normal. Define a vector field $\vec{F}$ by the formula $\vec{F}(x,y,z) = (x^3-y^3, x^2y, 0)$. Compute in two different ways the integral

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma \ .$$

8. (30 points)

(a) Let D denote the set of all x, y such that $y^2 - 2x \leq 0$, $x^2 - 2y \leq 0$. Compute $\displaystyle\iint_D e^{\frac{x^3+y^3}{xy}}\,dxdy$.
(Use the change of variables $x = u^2 v$ and $y = uv^2$)

(b) Let R be the set of all (x, y, z) such that $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$. Compute $\displaystyle\iiint_R x^2 y^2 z\,dxdydz$.