University of Illinois at Urbana-Champaign — Fall 2006
Math 380 — Group G1

Final Exam.
Friday, December 15.
Correction Key.

You are allowed to use your textbook, but no other kind of documentation.
Calculators, mobile phones and other electronic devices are prohibited.

NAME ____

SIGNATURE ____

1. (20 points)
Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous partial derivatives. Define a function $g\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$g(x) = f\big(x+1, \ln(1+x^2)\big) \ .$$

Explain why g is differentiable and give a formula for $g'(x)$ (the formula in question should involve the partial derivatives of f).

Answer. The function g is differentiable because it is a composition of differentiable functions; the Chain Rule gives us that

$$g'(x) = 1.\frac{\partial f}{\partial x}\big(x+1, \ln(1+x^2)\big) + \frac{2x}{1+x^2}\frac{\partial f}{\partial y}\big(x+1, \ln(1+x^2)\big) \ .$$

2. (20 points)
Recall that polar coordinates (r, θ) are linked to cartesian coordinates (x, y) by the formulas $x = r\cos(\theta)$, $y = r\sin(\theta)$. Let now γ be a curve parameterized (in polar coordinates) by the formula $r(\theta) = e^{\theta}$, for $0 \leq \theta \leq 1$.
(a) Find a parametrization for γ in cartesian coordinates.
(b) Compute the length of γ.

Answer. (a) Using θ as a parameter, one obtains $x(\theta) = r(\theta)\cos(\theta) = e^(\theta)\cos(\theta))$; $y(\theta) = r(\theta)\sin(\theta) = e^{\theta}\sin(\theta)$; $0 \leq \theta \leq 1$.
(b) We know that $ds = \sqrt{(x'(\theta))^2 + (y'(\theta))^2} d\theta$. We have $x'(\theta) = e^{\theta}(\cos(\theta) - \sin(\theta))$, $y'(\theta) = e^{\theta}(\sin(\theta) + \cos(\theta)$. Thus $(x'(\theta))^2 + (y'(\theta))^2 = e^{2\theta}(2\cos^2(\theta) + 2\sin^2(\theta)) = 2e^{2\theta}$. Plugging this in the integral, we obtain that the length of the curve is

$$\int_{\theta=0}^{1} \sqrt{2}e^{\theta}\, d\theta = \sqrt{2}(e-1)\ .$$

3. (20 points)

For $x \geq 0$, set $F(x) = \displaystyle\int_0^1 \ln(1 + xe^t)dt$. Give an expression for $F'(x)$ that doesn't involve an integral.

Answer. Using Leibnitz's rule (which we can do because the function inside the integral has continuous partial derivatives), we obtain

$$F'(x) = \int_0^1 \frac{\partial}{\partial x}(\ln(1 + xe^t))dt = \int_0^1 \frac{e^t}{1 + xe^t}dt = \left[\frac{1}{x}\ln(1 + xe^t)\right]_{t=0}^{x} = \frac{1}{x}\big(\ln(1 + xe) - \ln(1 + x)\big)\ .$$

4. (20 points)
Let γ be the path parameterized by $x(t) = 1 + \sin(2\pi t)e^{2t+3}$, $y(t) = \ln(1+8t)$, $z(t) = t^2 + 4t + 4$, $0 \le t \le 1$. Assume γ is oriented in the direction of increasing t; compute

$$\int_\gamma (3x^2 + 2x\sqrt{z}e^y)dx + (2y + x^2\sqrt{z}e^y)dy + (\frac{x^2e^y}{2\sqrt{z}})dz \ .$$

Answer. This line integral looks unbelievably ugly, so we guess that there must be some path-independence in there. Indeed, letting $f(x,y,z) = x^3 + x^2\sqrt{z}e^y + y^2$, we see that the line integral is equal to $\int_\gamma df$. Thus it only depends on the end-points of γ, which are $(x(1), y(1), z(1)) = (1, \ln(9), 9)$ and $(x(0), y(0), z(0)) = (1, 0, 4)$. Thus we obtain

$$I = \int_\gamma (3x^2 + 2x\sqrt{z}e^y)dx + (2y + x^2\sqrt{z}e^y + 1)dy + (\frac{x^2e^y}{2\sqrt{z}})dz = f(1, \ln(9), 9) - f(1, 0, 4) \ .$$

$$I = 1 + 3e^{\ln(9)} + (\ln(9))^2 - (1 + 2 + 0) = 1 + 27 + 4(\ln(3))^2 - 3 = 25 + 4(\ln(3))^2 \ .$$

5. (25 points)
Let S be the cone of equation $x^2 + y^2 = z^2$, $0 \leq z \leq H$ (viewed as a closed surface). Compute

$$\iint_S (1 - z^2) y^2 \, d\sigma$$

Answer. On the top of the surface we have $x^2 + y^2 \leq H^2$, $z = H$. Using polar coordinates, the integral on the top of the surface is

$$I_{\text{Top}} = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (1 - H^2) r^2 \sin^2(\theta) \, r d\theta dr = (1 - H^2)\pi \int_{r=0}^{1} r^3 \, dr = \frac{\pi(1 - H^2)}{4} \ .$$

On the side of the surface, we can use the parametrization $x = z\cos(\theta)$, $y = z\sin(\theta)$, $z = z$. We have already used this parametrization in exercise 5, so we can use the result obtained there to get

$$\frac{\partial P}{\partial z} \times \frac{\partial P}{\partial \theta} = \begin{pmatrix} -z\cos(\theta) \\ -z\sin(\theta) \\ z \end{pmatrix} \ .$$

The magnitude of this vector is $z\sqrt{2}$, so $d\sigma = z\sqrt{2} d\theta dz$. The formula for surface integrals gives that the integral on the side of the surface is

$$I_{\text{Side}} = \int_{z=0}^{H} \int_{\theta=0}^{2\pi} (1 - z^2) z^2 \sin^2(\theta) z\sqrt{2} d\theta dz = \pi\sqrt{2} \int_{z=0}^{H} (z^3 - z^5) \, dz = \pi\sqrt{2}(\frac{H^4}{4} - \frac{H^6}{6}) \ .$$

Overall, we get that the surface integral is equal to

$$\frac{\pi(1 - H^2)}{4} + \pi\sqrt{2}(\frac{H^4}{4} - \frac{H^6}{6}) \ .$$

6. (30 points)

Consider the system of equations $\begin{cases} x_1 + 2x_2 + 3x_3 + 10x_4 & = 0 \\ 4x_1 + 5x_2 + 6x_3 + x_4^2 & = 0 \\ 7x_1 + 8x_2 + 9x_3 + x_4^3 & = 0 \end{cases}$.

Show that this system implicitly defines x_1, x_2, x_4 as functions of x_3 near $(0, 0, 0, 0)$; compute $x_1'(0)$, $x_2'(0)$, $x_4'(0)$.

Answer. To check that the system implicitly defines x_1, x_2, x_4 as functions of x_3 near $(0, 0, 0, 0)$, we need to check that the determinant of the matrix $\begin{pmatrix} 1 & 2 & 10 \\ 4 & 5 & 0 \\ 7 & 8 & 0 \end{pmatrix}$ is different from 0. This determinant is equal to $10.(4.8 - 7.5) = -30$, so the implicit function theorem ensures that the system defines implicitly x_1, x_2, x_4 as functions of x_3 near $(0, 0, 0, 0)$. Implicit differentiation yields the relations

$$\begin{cases} dx_1 + 2dx_2 + 3dx_3 + 10dx_4 & = 0 \\ 4dx_1 + 5dx_2 + 6dx_3 + 8x_4dx_4 & = 0 \\ 7dx_1 + 8dx_2 + 9dx_3 + 3x_4^2 & = 0 \end{cases}$$

At the point $(0, 0, 0, 0)$ this gives us (considering x_1, x_2, x_3 as functions of x_3)

$$\begin{cases} x_1'(0) + 2x_2'(0) + 3 + 10x_4'(0) & = 0 \\ 4x_1'(0) + 5x_2'(0) + 6 + 0 & = 0 \\ 7x_1'(0) + 8x_2'(0) + 9 + 0 & = 0 \end{cases}$$

Linear combinations of the last two lines give $x_1'(0) = 1$, $x_2'(0) = -2$, and the first line then yields that $x_4'(0) = 0$.

7. (35 points)
Let V be the region of space of equation $x^2+y^2 \leq z \leq 2-(x^2+y^2)$. Denote by S the boundary of V, oriented by the outer normal. Define a vector field $\vec{F}$ by the formula $\vec{F}(x,y,z) = (x^3-y^3, x^2y, 0)$. Compute in two different ways the integral

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma \ .$$

Answer. Let us first compute the flow of $\vec{F}$ through S using the Divergence Theorem. We have $\operatorname{div}(\vec{F}) = 3x^2+x^2 = 4x^2$, so we have to compute the integral $I = \iint_V 4x\, dxdydz$. Given the equation of V, it is a good idea here to use cylindrical coordinates. This gives

$$I = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{z=r^2}^{2-r^2} 4r^2\cos^2(\theta)\, r dz d\theta dr = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} 4r^3(2-2r^2)\cos^2(\theta) d\theta) dr = 8\pi \int_{r=0}^{1} (r^3-r^5)\, dr \ .$$

This eventually yields $I = 8\pi(\frac{1}{4}-\frac{1}{6}) = \frac{2\pi}{3}$. Now we have to compute the integral using the definition of a surface integral ; when $0 \leq z \leq 1$ the surface S is given by $x^2+y^2=z$, and for $1 \leq z \leq 2$ the surface is given by $x^2+y^2 = 2-z$. It is a good idea to use parametrizations here ;

So the two different computations give the same result, which was to be expected but is nevertheless always a pleasure.

8. (30 points)

(a) Let D denote the set of all x, y such that $y^2 - 2x \leq 0$, $x^2 - 2y \leq 0$. Compute $\iint_D e^{\frac{x^3+y^3}{xy}}\, dxdy$.
(Use the change of variables $x = u^2v$ and $y = uv^2$)

(b) Let R be the set of all (x, y, z) such that $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$. Compute $\iiint_R x^2y^2z dxdydz.$

Answer. (a) The Jacobian matrix of this change of variables is

$$\begin{pmatrix} 2uv & u^2 \\ v^2 & 2uv \end{pmatrix} .$$

The determinant of this matrix is $3u^2v^2$. Next, we need to find the domain for (u, v); given the conditions on x, y one must have $u, v > 0$ and $u^2v^4 \leq 2u^2v$, $u^4v^2 \leq 2uv^2$. These conditions are the same as $u, v > 0$ and $u^3 \leq 2$, $v^3 \leq 2$. We still need to check that our change of variables is one-to-one, and this is obviously the case since one has $u^3 = \frac{x^2}{y}$ and $v^3 = \frac{y^2}{x}$ Eventually, our integral turns out to be equal to

$$\int_{u=0}^{2^{1/3}} \int_{v=0}^{2^{1/3}} e^{u^3+v^3} 3u^2v^2\, dudv = \int_{u=0}^{2^{1/3}} u^2 e^{u^3} e^2\, du = \frac{e^4}{3} .$$

(We used the fact that $e^{u^3+v^3} = e^{u^3}e^{v^3}$ and $3v^2e^{v^3}$ integrates as e^{v^3})

(b) Let us use spherical coordinates $x = r\cos(\theta)\sin(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$, $z = r\cos(\varphi)$. We need to compute the Jacobian determinant of this change of variables; of course it would be easier to just know it by heart, but recovering it is not that bad : the Jacobian matrix of the change of variables is

$$\begin{pmatrix} \cos(\theta)\sin(\varphi) & -r\sin(\theta)\sin(\varphi) & r\cos(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ \cos(\varphi) & 0 & -r\sin(\varphi) \end{pmatrix} .$$

Using an expansion with respect to the last row, we obtain that the Jacobian determinant D of this change of variables is

$$D = \cos(\varphi)\big(-r^2\sin^2(\theta)\sin(\varphi)\cos(\varphi) - r^2\cos^2(\theta)\sin(\varphi)\cos(\varphi)\big) - r\sin(\varphi)\big(r\cos^2(\theta)\sin^2(\varphi) + r\sin^2(\theta)\sin^2(\varphi)\big)$$

$$D = -r^2\cos^2(\varphi)\sin(\varphi) - r^2\sin(\varphi)\sin^2(\varphi) = -r^2\sin(\varphi) .$$

Remembering that we need to use the absolute value of the determinant in the integral, we obtain

$$I = \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} r^5\cos^2(\theta)\sin^2(\theta)\sin^4(\varphi)\cos(\varphi)r^2\sin(\varphi)\, d\varphi d\theta dr$$

$$I = \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} r^7\cos^2(\theta)\sin^2(\theta)\sin^5(\varphi)\cos(\varphi)\, d\varphi d\theta dr = \int_{r=0}^{2} \int_{\theta=0}^{2\pi} r^7 \frac{\sin^2(2\theta)}{4} \left[\frac{\sin^6(\varphi)}{6}\right]_{\varphi=0}^{\pi/2}$$

$$I = \int_{r=0}^{2} \int_{\theta=0}^{2\pi} r^7 \frac{\sin^2(2\theta)}{24} d\theta dr = \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \frac{r^7}{24} \frac{1-\cos(4\theta)}{2} d\theta dr = \pi \int_{r=0}^{2} \frac{r^7}{24} dr = \pi \frac{2^8}{8.24} = \frac{4\pi}{3} .$$