UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN — Fall 2006
Math 380 — Group G1

Graded Homework II

Due Friday, September 15.

1. Let $z = z(u, v)$ where $u = u(s, t)$ and $v = v(s, t)$. Give an expression of the differential dz in terms of du and dv, then in terms of ds and dt, using the first order derivatives $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.
Obtain from this formula the values of $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ in the following cases :
- $z = ue^v$, $u = t^2 + s$, $v = s - t$;
- $z = \cos(uv)$, $u = ts$, $v = \sin(t + s)$.

Correction.
By definition, we have $dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$, and the Chain Rule gives us

$$dz = (\frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s})ds + (\frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t})dt.$$

- In the first case, this yields

$dz = (e^v.1 + ue^v.1)ds + (e^v.2t + ue^v.(-1))dt = (e^{s-t} + (t^2 + s)e^{s-t})ds + (2te^{s-t} - (t^2 + s)e^{s-t})dt.$
The definition of the partial derivatives implies that $\frac{\partial z}{\partial s} = e^{s-t}(1 + t^2 + s)$ and $\frac{\partial z}{\partial t} = e^{s-t}(-t^2 + 2t - s)$.

- In the second case, we get $dz = (-v\sin(uv).t - u\sin(uv)\cos(t + s))ds + (-v\sin(uv).s - u\sin(uv).\cos(t + s))dt$, so

$dz = (-\sin(t + s)\sin(ts\sin(t + s))s - ts\sin(ts\sin(t + s))\cos(t + s))ds$
$\quad -(\sin(t + s)\sin(ts\sin(t + s))s - ts\sin(ts\sin(t + s))\cos(t + s))dt.$
Therefore, we have this time the equalities $\frac{\partial z}{\partial s} = -\sin(t + s)\sin(ts\sin(t + s))s - ts\sin(ts\sin(t + s))\cos(t + s)$,
and $\frac{\partial z}{\partial t} = -(\sin(t + s)\sin(ts\sin(t + s))s - ts\sin(ts\sin(t + s))\cos(t + s))$.

2. The temperature on the surface of a heated disk of radius a is given by the formula $T(r, \theta) = T_0 + T_1(1 - \frac{r^2}{a^2})$ (where T_0, T_1 are constants and r, θ are polar coordinates). You are on this disk, at point $(c, 0)$ (where $0 < c < a$) and start moving parallel to the y axis at constant speed v_0. What is the instantaneous rate of change of temperature that you feel at a given time t (assuming that you haven't yet fallen from the disk) ?
Explain the value obtained at $t = 0$.

Correction. The rate of change of temperature is $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$. Since $T = T_0 + T_1(1 - \frac{x^2 + y^2}{a^2})$, and $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = v_0$, so the formula above becomes $\frac{dT}{dt} = -\frac{2T_1yy'(t)}{a^2} = -\frac{2T_1v_0^2t}{a^2}$.
At $t = 0$ the instantaneous rate of change of temperature is 0 ; the reason is that the gradient of T is radial, so at $t = 0$ you are moving orthogonally to the gradient. Since the gradient is orthogonal to the level curves, this means that at that instant you don't feel the temperature changing at all (because you are tangent to a curve T = constant).

3. Let $z(x, y) = f(xy)$, where $f : \mathbb{R} \to \mathbb{R}$ is a function two times continuously differentiable. Give formulas for the first and second-order partial derivatives of z ; check that in that case both mixed derivatives $\frac{\partial^2 z}{\partial x\partial y}$ and $\frac{\partial^2 z}{\partial y\partial x}$ are equal.

Correction. By definition of the partial derivatives, we have : $\frac{\partial z}{\partial x}(x,y) = yf'(xy)$; $\frac{\partial z}{\partial y}(x,y) = xf'(xy)$.
The rule for the derivative of a product gives us that $\frac{\partial^2 z}{\partial x^2}(x,y) = y^2 f''(xy)$, $\frac{\partial^2 z}{\partial y \partial x}(x,y) = f'(xy) + yxf''(xy)$, and
$\frac{\partial^2 z}{\partial y^2}(x,y) = x^2 f''(xy)$, $\frac{\partial^2 z}{\partial x \partial y}(x,y) = f'(xy) + xyf''(xy)$. Thus we check that in that case we do have, as we should, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$.

4. Compute a normal vector to the surface S at the point P, and an equation for the tangent plane to S at P, in the following cases :
• S is defined by the equation $z = xy - x + y + 2$, and $P = (0,2,4)$;
• S is defined by the equation $z = \sin(\pi xy)$ and $P = (-\sqrt{2}, \sqrt{2}, 0)$;
• S is defined by the equation $x^2 + 4y^2 + xyz = 0$ and $P = (1,-1,5)$.
Correction.
• The surface equation is of the form $F(x,y,z) = 0$, where $F(x,y,z) = xy - x + y + 2 - z$, so that $\nabla F(0,2,4)$, if not 0, will be a normal vector to the surface (Actually, we know before computing it that ∇F is everywhere different from 0 because the surface is of the form $z = f(x,y)$). The computation yields $\nabla F(x,y,z) = (y-1, x+1, -1)$, so $\nabla F(0,2,4) = (1,1,-1)$ is a normal vector to P.
The equation of the tangent plane at $(0,2,4)$ is given by $\nabla F(0,2,4).(x-0, y-2, z-4) = 0$ (this is the plane orthogonal to $\nabla F(0,2,4)$ and going through $(0,2,4)$), which gives us $x + (y-2) - (z-4) = 0$, or $x + y - z = -2$.
• In the same fashion, one obtains that a normal vector to S at a point with coordinates $(x, y, \sin(\pi xy))$ is $(\pi y \cos(\pi xy), \pi x \cos(\pi xy), -1)$, so a normal vector at $(-\sqrt{2}, \sqrt{2}, 0)$ is $(0,0,-1)$. The equation of the normal plane to S at P is given by $(0,0,-1).(x+\sqrt{2}, y-\sqrt{2}, z) = 0$, in other words $z = 0$.
• This time S is the surface $F(x,y,z) = 0$ for $F(x,y,z) = x^2 + 4y^2 + xyz$. One obtains $\nabla F(x,y,z) = (2x + yz, 8y + xz, xy)$, so a normal vector at $(1,-1,5)$ is $(-3,-3,-1)$; to simplify the next computation, we pick as a normal vector the vector $(3,3,1)$. Then the equation of the tangent plane to S at $(1,-1,5)$ is $(3,3,1).(x-1, y+1, z-5) = 0$, or $3x + 3y + z = 5$.

5. You are on a mountain of equation $z = 24 - x^2 - 2y^2$, at the point $P = (3,2,7)$, and want to go down as quickly as possible. In which direction (in 2-dimensional space) should you turn at first ? What direction will you follow in 3-dimensional space ?
Correction. The direction of steepest ascent is the direction in which $z = f(x,y)$ increases most, so it is given by the gradient of f at $(3,2)$. One has $\nabla f(3,2) = (-6,-8)$, so the 2-dimensional direction of quickest descent is given by $(6,8)$. To obtain the direction in 3-dimensional space (i.e, to remain on the mountain while we walk...), we therefore need to follow the direction of the vector tangent to the mountain whose first coordinate is 6, and second coordinate is 8. The tangent plane to S at P is parallel to the plane of equation $z + 6x + 8y = 0$, so the 3-dimensional direction you will follow is given by $(6,8,-100)$.
Beware : the gradient is a bound vector, meaning that it should be thought of as a vector whose tail is at the point P. It works similarly when you're looking for what direction to follow while starting at P ; in other words, to decide in which direction you should move, the thing that matters is that the vector indicating your direction has to belong to the *vector* direction of the tangent plane to S at P, not to the *affine* tangent plane to S at P.