University of Illinois at Urbana-Champaign — Fall 2006
Math 380 — Group G1

Graded Homework III
Due Friday, September 29.

1. Find the directional derivative of the mapping f defined by $f(x,y) = xy + \ln(x^2+1)$ in the direction of $u = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

Correction. One has $\nabla f(x,y) = (y + \frac{2x}{x^2+1}, x)$; since u is a unit vector, the derivative of f in the direction of u is simply $\nabla f.u$, in other words

$$\nabla_u f(x,y) = \frac{\sqrt{3}}{2} y + \frac{\sqrt{3}x}{x^2+1} + \frac{x}{2} \ .$$

2. Given unit vectors $u = (u_x, u_y)$ and $v = (v_x, v_y)$, and a function $z = z(x,y)$ with continuous second-order partial derivatives, find a formula for the mixed second order directional derivative $\nabla_u \nabla_v z$.
By definition, we have $\nabla_v z = \nabla z.v = \frac{\partial z}{\partial x} v_x + \frac{\partial z}{\partial y} v_y$.
Applying the same definition, we obtain $\nabla_u \nabla_v z = \nabla(\frac{\partial z}{\partial x} v_x + \frac{\partial z}{\partial y} v_y).u$. Therefore,

$$\nabla_u \nabla_v z = (\frac{\partial^2 z}{\partial x^2} v_x + \frac{\partial^2 z}{\partial x \partial y} v_y, \frac{\partial^2 z}{\partial y \partial x} v_x + \frac{\partial^2 z}{\partial y^2} v_y).(u_x, u_y) = \frac{\partial^2 z}{\partial x^2} v_x u_x + \frac{\partial^2 z}{\partial x \partial y} v_y u_x + \frac{\partial^2 z}{\partial y \partial x} v_x u_y + \frac{\partial^2 z}{\partial y^2} v_y u_y \ .$$

3. Show that y is defined implicitly as a function of x in the neighborhood of the point P in the following equations :
• $x\cos(xy) = 0$, $P = (1, \frac{\pi}{2})$;
• $xy + \log(xy) = 1$, $P = (1,1)$ (log stands for the natural logarithm).
Use implicit differentiation to compute $y'(1)$ and $y''(1)$ in the first case, and $y'(1)$ in the second case.
Correction. The implicit function theorem tells us that, if $F \colon \mathbb{R}^2 \to \mathbb{R}$ and $\lambda \in \mathbb{R}$, then the equation $F(x,y) = \lambda$ defines implicitly y as a function of x in a neighborhood of a solution P as soon as $\frac{\partial F}{\partial y}$ is continuous and different from 0 at P.
• In the first case we have $F(x,y) = x\cos(xy)$, so that $\frac{\partial F}{\partial y} = -x\sin(xy)$, which implies that $\frac{\partial F}{\partial y}(1, \frac{\pi}{2}) = -1$. Therefore the hypothesis of the theorem is satisfied, and y is defined implicitly as a function of x near P.
Implicit differentiation yields $(\cos(xy) - y\sin(xy))dx - x\sin(xy)dy$=0. Since we are looking for $y'(1)$, we may notice that close to $(1, \frac{\pi}{2})$ the equation $x\cos(xy) = 0$ means that $\cos(xy) = 0$, so that implicit differentiation simply gives $-y\sin(xy)dx - x\sin(xy)dy = 0$, or $\frac{dy}{dx} = y'(x) = -\frac{y}{x}$. Thus $y''(x) = -\frac{y'}{x} + \frac{y}{x^2} = 2\frac{y}{x^2}$ close to P, and this gives $y'(1) = -\frac{\pi}{2}$, and $y''(1) = \pi$. (Notice that, in that case, one can actually solve the equation : in a neighborhood of P, one simply has $y = \frac{\pi}{2x}$; differentiating this equality gives for $y'(1)$ and $y''(1)$ the values we obtained earlier).
• In this case $F(x,y) = xy + \log(xy)$, so $\frac{\partial F}{\partial y} = x + \frac{1}{y}$. Therefore, $\frac{\partial F}{\partial y}(1,1) = 2 \neq 0$, so the equation defines implicitly y as a function of x near P.

Implicit differentiation gives this time $y'(x) = -\dfrac{y + \frac{1}{x}}{x + \frac{1}{y}}$, or $y'(x) = \dfrac{y^2(x)x + y(x)}{y(x)x^2 + x}$. This shows that $y'(1) = 1$.
(Bonus, just to show how atrocious iterated implicit differentiations can get) If we were asked to compute also $y''(1)$, then we would have to take a deep breath, then write that

$$y'' = \frac{(2yy'x + y^2 + y')(yx^2 + x) - (y^2x + y)(x^2y' + 2xy + 1)}{(yx^2 + x)^2}, \text{ so } y''(1) = \frac{(2+1+1)(1+1) - (1+1)(1+2+1)}{(1+1)^2} = 0\,.$$

4. Show that $2xy - z + 2xz^3 = 5$ can be solved implicitly for z as a function of x near $(1, 2, 1)$.
Compute the first-order partial derivatives of z at $(1, 2)$, as well as $\dfrac{\partial^2 z}{\partial y^2}(1, 2)$.
Correction. We have this time an equation of the type $F(x, y, z) = \lambda$; for this to define z implicitly near $(1, 2, 1)$, it is enough that $\dfrac{\partial F}{\partial z}$ be continuous and different from 0 at that point. A direct computation yields that $\dfrac{\partial F}{\partial z} = -1 + 6xz^2$, so that $\dfrac{\partial F}{\partial z}(1, 2, 1) = 5 \neq 0$. Therefore the equation defines z implicitly as a function of (x, y) near $(1, 2, 1)$. Implicit differentiation leads to $2ydx + 2xdy - dz + 2z^3dx + 6xz^2dz = 0$, so that $(2y + 2z^3)dx + 2xdy = (1 - 6xz^2)dz$, or $dz = \dfrac{(2y + 2z^3)dx + 2xdy}{1 - 6xz^2}$. This enables us to obtain

$$\frac{\partial z}{\partial x} = \frac{2y + 2z^3}{1 - 6xz^2}, \quad \frac{\partial z}{\partial y} = \frac{2x}{1 - 6xz^2} \quad (*)$$

From (*) we obtain that $\dfrac{\partial z}{\partial x}(1, 2) = -\dfrac{6}{5}$ and $\dfrac{\partial z}{\partial y}(1, 2) = -\dfrac{2}{5}$. We also get

$$\frac{\partial^2 z}{\partial y^2} = 2x \frac{12xz\frac{\partial z}{\partial y}}{(1 - 6xz^2)^2}\ .$$

Given the values computed above, this yields $\dfrac{\partial^2 z}{\partial y^2}(1, 2) = 2\dfrac{12(-\frac{2}{5})}{25} = -\dfrac{48}{125}$.

5. Determine whether the function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x, y, z) = (e^x \cos(y) + z, x \sin(y) \sin(z), xz \cos(y))$, admits a differentiable inverse g near $(1, \frac{\pi}{2}, \pi)$. If so, give the value of the Jacobian determinant of g at the point $f\big((1, \frac{\pi}{2}, \pi)\big) = (\pi, 0, 0)$.
Correction. The consequence of the implicit function theorem seen in class (called the local inversion theorem) implies that we only need to check whether the Jacobian determinant of f vanishes or not at $(1, \frac{\pi}{2}, \pi)$. The Jacobian matrix of f at (x, y, z) is

$$Jf_{(x,y,z)} = \begin{pmatrix} e^x \cos(y) & -e^x \sin(y) & 1 \\ \sin(y)\sin(z) & x\cos(y)\sin(z) & -x\sin(y)\cos(z) \\ z\cos(y) & -xz\sin(y) & x\cos(y) \end{pmatrix}$$

This implies that $Jf_{(1,\frac{\pi}{2},\pi)} = \begin{pmatrix} 0 & -e & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The determinant of this matrix is 0, which proves that f does not admit a differentiable inverse g near $f(1, \frac{\pi}{2}, \pi) = (\pi, 0, 0)$.