University of Illinois at Urbana-Champaign — Fall 2006
Math 380 — Group G1

Graded Homework V

Correction.

1. Compute the derivative of the function $x \mapsto \tan^{-1}(x) = \arctan(x)$; use it to compute $\displaystyle\int_a^b \frac{dx}{x^2+1}$, where $a, b \in \mathbb{R}$ (in terms of $\arctan(a), \arctan(b)$), then to compute $\displaystyle\int_0^1 \frac{dx}{x^2+x+1}$.
With a change of variable, compute the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos(x)dx}{2-\cos^2(x)+\sin(x)}$.

Correction. A direct computation shows that $\tan'(x) = 1+\tan^2(x)$. The fact that $\tan(\arctan(x)) = x$, and the Chain Rule for functions of one real variable, yields $\tan'(\arctan(x)).\arctan'(x) = 1$, so that

$$\arctan'(x) = \frac{1}{\tan'(\arctan(x))} = \frac{1}{1+(\tan(\arctan(x)))^2} = \frac{1}{1+x^2}\ .$$

This immediately yields $\displaystyle\int_a^b \frac{dx}{x^2+1} = \arctan(b) - \arctan(a)$.
One has

$$\int_0^1 \frac{dx}{x^2+x+1} = \int_0^1 \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dy}{y^2+\frac{3}{4}} = \frac{4}{3}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dy}{1+(\frac{2}{\sqrt{3}}y)^2} = \frac{4}{3}\left[\frac{\sqrt{3}}{2}\arctan(\frac{2}{\sqrt{3}}y)\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

Now we may use the fact that $\arctan(\sqrt{3}) = \frac{\pi}{3}$ (why ?) and $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$ to finally obtain that

$$\int_0^1 \frac{dx}{x^2+x+1} = \frac{2}{\sqrt{3}}\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}\ .$$

Finally, setting $u = \sin(x)$, we have $du = \cos(x)dx$, and $x \mapsto u(x)$ is a bijection from $[0, \frac{\pi}{2}]$ onto $[0,1]$, so the theorem of change of variables yields

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)dx}{2-\cos^2(x)+\sin(x)} = \int_0^1 \frac{du}{2-(1-u^2)+u} = \int_0^1 \frac{du}{u^2+u+1} = \frac{\pi}{3\sqrt{3}}\ .$$

2. Compute the area of the domain D in the two following cases :
(a) D is in the quarter-plane $x \geq 0$, $y \geq 0$ and is delimited by the curves $y^2 = x^3$, $y = x$.
(b) D is the set of all $x, y \geq 0$ such that $x^{2/3} + y^{2/3} \leq 1$.
For the second one, you may begin with the change of coordinates $u = x^{\frac{1}{3}}$, $v = y^{1/3}$; you may also use the fact that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos^2(\theta)d\theta = \frac{\pi}{16}$ (Proving this equality will give some extra credit on the homework).

Correction. (a) Looking at a picture, we see that this domain is contained in the square $0 \leq x \leq 1$, $0 \leq y \leq 1$; in that domain one has $x^{3/2} \leq x$. Thus, the area of D is

$$\iint_D dxdy = \int_{x=0}^1 \left(\int_{y=x^{3/2}}^x dy\right)dx = \int_{x=0}^1 (x - x^{3/2})dx = \left[\frac{x^2}{2} - \frac{2}{5}x^{5/2}\right]_0^1 = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}\ .$$

(b) We use changes of coordinates to simplify the computation : set $u = x^{1/3}$ and $v = y^{1/3}$. Then correspondence $(x,y) \mapsto (u,v)$ is bijective (from $\mathbb{R}^2$ onto $\mathbb{R}^2$), and the Jacobian determinant $\frac{\partial(u,v)}{\partial(x,y)}$is equal to $\frac{1}{9}x^{-2/3}y^{-2/3} = \frac{1}{9u^2v^2}$. The domain of all (u,v) corresponding to R is the top-right quarter of the unit circle (call it C^+) ; thus the change of variables theorem gives

$$\text{Area}(D) = \iint_D dxdy = \iint_{C^+} |\frac{\partial(x,y)}{\partial(u,v)}|dudv = \iint_{C^+} 9u^2v^2 dudv \ .$$

To compute this integral, we again use a change of variables, going this time to polar coordinates : $u = \rho\cos(\theta)$, $v = \rho\sin(\theta)$. This yields (don"t forget the Jacobian determinant...)

$$\text{Area}(F) = \int_{\theta=0}^{\frac{\pi}{2}} \left(\int_{\rho=0}^{1} 9\rho^4 \sin^2(\theta)\cos^2(\theta)\rho d\rho \right) d\theta = \frac{9}{6}\int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos^2(\theta)d\theta \ .$$

To compute this last integral, we use some trigonometry to obtain that $\sin^2(\theta)\cos^2(\theta) = \frac{\sin^2(2\theta)}{4} = \frac{-\cos(4\theta)+1}{8}$. We then obtain that $\int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos^2(\theta)d\theta = \frac{\pi}{16}$. Putting all this together, we have $\text{Area}(D) = \frac{3\pi}{32}$.

3. Compute the integral $\iint_D f(x,y)dxdy$ in the following cases :
(a) $f(x,y) = e^{x+y}$ and $D = \{(x,y) \in \mathbb{R}^2 \colon |x-y| \leq 1, |x+y| < 1\}$.
(b) $f(x,y) = x^2 - 2y$, D is the interior of the ellipse of equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
(c) $f(x,y) = x^2 + y^2 - 2y$, D is the circle of center $(1,1)$ and radius 1.
(d) $f(x,y) = xy$, D is the domain of all (x,y) such that $x, y \geq 0$ and $x^2 + \frac{y^2}{4} \leq 1$.
(If you have difficulties with some of these integrals, suitable changes of variables would be a good idea!)
Correction. For (a), it is more or less clear that one should set $u = x+y$, $v = x-y$. Then the corresponding domain for (u,v) is $-1 < u < 1$, $1 \leq v \leq 1$, and the correspondence $(x,y) \mapsto (u(x,y), v(x,y))$ is one to one because one has $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$ (so the inverse functions exist). We need to find $|\frac{\partial(x,y)}{\partial(u,v)}|$; the expressions of x and y in terms of (u,v) show that it is equal to $|\frac{1}{2}.\frac{-1}{2} - \frac{1}{2}\frac{1}{2}| = \frac{1}{2}$ (don't forget the absolute value...). Thus, we get that

$$\iint_D f(x,y)dxdy = \int_{u=-1}^{1}\int_{v=-1}^{1} e^u \frac{1}{2} dudv = \int_{u=-1}^{1} e^u du = e - \frac{1}{e} \ .$$

For (b), one uses the usual change of variables for an ellipse, setting $x = ar\cos(\theta)$, $y = br\sin(\theta)$. A picture shows that this change of variable is one-to-one (when r varies between 0 and 1, and θ varies between 0 and 2π), and one has

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} a\cos(\theta) & -ar\sin(\theta) \\ b\sin(\theta) & br\cos(\theta) \end{pmatrix} = abr\cos^2(\theta) + abr\sin^2(\theta) = abr \ .$$

We may assume that a and b are positive (only their squares appear in the equation of the ellipse), so the change of variables theorem gives

$$\iint_D f(x,y)dxdy = \int_{\theta=0}^{2\pi}\left(\int_{r=0}^{1} (abr)(a^2r^2\cos^2(\theta) - 2br\sin(\theta))dr\right)d\theta = \int_{\theta=0}^{2\pi} (\frac{a^3b}{4}\cos^2(\theta) - \frac{2ab^2}{3}\sin(\theta))d\theta \ .$$

The second part of the integral is easily seen to be equal to 0 ; to compute the first part, one may use the formula $\cos^2(\theta) = \frac{2\cos(\theta)+1}{2}$ and proceed from there, or notice that the integral is equal to the same one

where one replaces $\cos^2(\theta)$ by $\sin^2(\theta)$ (prove this), and deduce from this that it is worth $\frac{a^3 b\pi}{4}$ (remember that $\cos^2(\theta) + \sin^2(\theta) = 1$...)
(c) This time it is natural to set $x - 1 = r\cos(\theta)$, $y - 1 = r\sin(\theta)$. This is again one-to-one ($0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$) ; the Jacobian determinant is the same as above, so we get

$$\iint_D f(x,y)dxdy = \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{1} (r^2 + 2r\cos(\theta))rd\theta \right) dr = \int_{\theta=0}^{2\pi} \left(\frac{1}{4} + \frac{2}{3}\cos(\theta) \right) d\theta = \frac{\pi}{2} \ .$$

(d) Here, iterated integrals work for once :

$$\iint_D f(x,y)dxdy = \int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{4-4x^2}} xydy \right) dx = \int_{x=0}^{1} x \left[\frac{y^2}{2} \right]_0^{\sqrt{4-4x^2}} dx = \int_0^1 \frac{x(4-4x^2)}{2} dx = 1 - \frac{1}{2} = \frac{1}{2} \ .$$