University of Illinois at Urbana-Champaign — Fall 2006
Math 380 — Group G1

Graded Homework VI : Correction

1. Compute the following integrals :
(a) $\iiint_D xy\,dxdydz$, where $D=\{(x,y,z)\colon 0\le x, 0\le y, 0\le z\le 1, x^2+y^2\le z^2\}$.
(b) $\iiint_D y\,dxdydz$, where $D=\{(x,y,z)\colon 0\le x, 0\le y, x^2+y^2\le z\le 1\}$.
(c) $\iiint_D \sqrt{x^2+y^2+z^2}\,dxdydz$, $D=\{(x,y,z)\colon 1\le x^2+y^2+z^2\le 2\}$
(Use a mix of iterated integration and change of variables !).
Correction. (a) One can begin by using an iterated integral :

$$I=\iiint_D xy\,dxdydz=\int_{z=0}^1\Big(\iint_{x^2+y^2\le z^2} xy\,dxdy\Big)dz$$

Then, the simplest thing to do is to use the change of variables $x=r\cos(\theta), y=r\sin(\theta)$,to obtain that (denoting by D_z the domain $0\le r\le z$, $0\le\theta\le\frac{\pi}{2}$) :

$$I=\int_{z=0}^1\Big(\iint_{D_z} r^3\cos(\theta)\sin(\theta)drd\theta\Big)dz=\int_{z=0}^1\Big(\Big[\frac{r^4}{4}\Big]_{r=0}^z\int_{\theta=0}^{\pi/2}\frac{\sin(2\theta)}{2}d\theta\Big)dz=\int_{z=0}^1\frac{z^4}{4}.\frac{1}{2}dz=\frac{1}{40}\ .$$

(b) Once again, it is natural to begin with an iterated integral :

$$J=\iiint_D y\,dxdydz=\int_{z=0}^1\Big(\iint_{x^2+y^2\le z} y\,dxdy\Big)dz$$

. Then, with the same notations as in (a), one uses a change of variable to compute the double integral :

$$J=\int_{z=0}^1\Big(\iint_{D_z} r\sin(\theta)rdrd\theta\Big)dz=\int_{z=0}^1\Big[\frac{r^3}{3}\Big]_{r=0}^z\Big[-\cos(\theta)\Big]_{\theta=0}^{\pi/2}dz=\int_{z=0}^1\frac{z^3}{3}.1dz=\frac{1}{12}\ .$$

(c) This time we begin by going to spherical coordinates (r,θ,φ) ; the Jacobian determinant of this transformation is $r^2\sin(\varphi)$, so (letting $D'=\{r,\theta,\varphi)\colon 1\le r\le\sqrt{2}, 0\le\theta\le 2\pi, 0\le\varphi\le\pi\}$) :

$$K=\iiint_D\sqrt{x^2+y^2+z^2}\,dxdydz=\iiint_{D'} r^3\sin(\varphi)drd\theta d\varphi=4\pi\int_1^{\sqrt{2}} r^3dr=4\pi(1-\frac{1}{4})=3\pi\ .$$

2. Compute the coordinates (x_G,y_G) of the center of gravity G of the plane domain D of equation $x^2\le 2y\le x+2$ (recall that x_G is the average value of x in D, and y_G is the average value of y in D).
Note : The term "center of gravity" is used to denote the center of mass of a *homogeneous* solid, i.e one in which density is the same everywhere.
Correction. First, we need to find a usable equation for D : the inequality $x^2\le x+2$ is equivalent to $-1\le x\le 2$. Then we need to compute the area of D :

$$\text{Area}(D)=\iint_D dxdy=\int_{x=-1}^2\Big(\int_{y=x^2/2}^{x/2+1}dy\Big)dx=\int_1^2(\frac{x}{2}+1-\frac{x^2}{2})dx=\Big[\frac{x^2}{4}+x-\frac{x^3}{6}\Big]_{-1}^2=\frac{9}{4}\ .$$

Then, we have to compute $\iint_D x\,dxdy$, for which we have

$$\iint_D x\,dxdy=\int_{x=-1}^2\Big(\int_{y=x^2/2}^{x/2+1}x\,dy\Big)dx=\int_{x=-1}^2 x(\frac{x^2}{2}+1-\frac{x^2}{2})dx=\Big[\frac{x^3}{6}+\frac{x^2}{2}-\frac{x^4}{8}\Big]_{-1}^2=\frac{9}{8}\ .$$

The last integral we have to find is

$$\iint_D y\,dxdy = \int_{x=-1}^{2} \left(\int_{y=x^2/2}^{x/2+1} y\,dy \right) dx = \int_{x=-1}^{2} \left[\frac{y^2}{2} \right]_{y=x^2/2}^{x/2+1} dx = \int_{-1}^{2} \frac{(\frac{x}{2}+1)^2 - \frac{x^4}{4}}{2} dx = \frac{37}{20} \,.$$

Thus, we obtain $x_G = \dfrac{\iint_D x\,dxdy}{\iint_D dxdy} = \dfrac{1}{2}$, and $y_G = \dfrac{\iint_D y\,dxdy}{\iint_D dxdy} = \dfrac{37}{45}$.

3. Define, for $x \geq 0$, $H(x) = \displaystyle\int_0^x e^{-t^2} dt$, $G(x) = H(x)^2$ and $F(x) = \displaystyle\int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt$.
(a) Compute $G'(x)$; $F'(x)$. Show that the function $F+G$ is constant.
(b) Find the value of $(F+G)(0)$.
(c) (optional) Show that $0 \leq F(x) \leq e^{-x^2}$ for all $x \in \mathbb{R}$.
(d) (optional) Find $\displaystyle\lim_{x\to\infty} \int_0^x e^{-t^2} dt$.

Correction. (a) We have $G'(x) = 2H(x)H'(x) = 2e^{-x^2} \displaystyle\int_0^x e^{-t^2} dt$. Using the theorem for derivatives of integrals with a parameter, we get $F'(x) = \displaystyle\int_0^1 \frac{-2x(1+t^2)e^{-x^2(1+t^2)}}{1+t^2} dt = e^{-x^2} \int_0^1 -2xe^{-x^2t^2} dt$. Setting $u = tx$ in the integral, we finally obtain $F'(x) = -e^{-x^2} \displaystyle\int_0^x e^{-u^2} du$, which means that $F'(x) = -G'(x)$. This may also be stated as $(F+G)'(x) = 0$ for all x, which yields that $F+G$ is constant.
(b) $G(0) = H(0)^2 = 0$; $F(0) = \displaystyle\int_{t=0}^1 \frac{dt}{1+t^2} = \arctan(1) = \frac{\pi}{4}$. Hence $(F+G)(0) = \dfrac{\pi}{4}$.
(c) Since $1+t^2 \geq 1$, one has $e^{-x^2(1+t^2)} \leq e^{-x^2}$ for all t, thus $\dfrac{e^{-x^2(1+t^2)}}{1+t^2} \leq e^{-x^2}$ for all t. This gives $F(x) \leq \displaystyle\int_0^1 e^{-x^2} dt = e^{-x^2}$.
(d) Question (c) yields $\displaystyle\lim_{x\to+\infty} F(x) = 0$; since $F(x)+G(x) = \dfrac{\pi}{4}$ for all x, this gives $\displaystyle\lim_{x\to+\infty} G(x) = \frac{\pi}{4}$. This eventually gives $\displaystyle\lim_{x\to+\infty} \int_0^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

4.. (a) Find the length of the arc of helix of equation $x(t) = \cos(t)$, $y(t) = \sin(t)$, $z(t) = t$, where $0 \leq t \leq 2\pi$.
(b) A *hypocycloid* is a closed curve in plane of equation $x^{2/3} + y^{2/3} = a^{2/3}$, where a is some positive constant. Find the length of an hypocycloid (first find the length of the part of the curve that is in the first quadrant, then use symmetries of the curve).
Correction. (a) By definition, the length of the arc of helix is

$$l = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2 + 1} dt = \int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t) + 1} dt = \int_0^{2\pi} \sqrt{2} dt = 2\pi\sqrt{2} \,.$$

(b)Using x as a parameter for the hypocycloid, one has to find y as a function of x ; one has $y^{2/3} = a^{2/3} - x^{2/3}$, so in the first quadrant the equation is (since $y \geq 0$) $y = (a^{2/3} - x^{2/3})^{3/2}$. This gives $y'(x) = \dfrac{3}{2}(-\dfrac{2}{3}x^{-1/3})(a^{2/3} - x^{2/3})^{1/2}$. We are actually interested by $y'(x)^2$, the value of which is $x^{-2/3}(a^{2/3} - x^{2/3}) = (\dfrac{a}{x})^{2/3} - 1$. Thus, the length of the part of the hypocycloid that lies in the first quadrant is

$$\int_{x=0}^{a} \sqrt{1 + (\frac{a}{x})^{2/3} - 1} dx = \int_{x=0}^{a} (\frac{a}{x})^{1/3} dx = \frac{3a^{1/3}}{2} \left[x^{2/3} \right]_{x=0}^{a} = \frac{3a}{2} \,.$$

Thus, the total length of the hypocycloid is $6a$.