University of Illinois at Urbana-Champaign — Fall 2006
Math 380 — Group G1

Graded Homework VII .
Correction.

1. (a) Compute $\int_\Gamma x ds$, where Γ is the arc of the parabola $y=x^2+1$ joining $(0,1)$ and $(1,2)$ oriented following the decreasing y's.
(b) Compute $\int_\Gamma (x^2+y^2+z^2)ds$, where Γ is the triangle (in $\mathbb{R}^3$) with edges $(a,0,0)$, $(0,a,0)$ and $(0,0,a)$ (oriented in that order).

Correction.(a) Using x as a parameter, we get

$$\int_\Gamma x\,ds = -\int_{x=0}^1 x\sqrt{1+4x^2}\,dx = -\left[\frac{1}{12}(1+4x^2)^{3/2}\right]_{x=0}^1 = \frac{1-5\sqrt{5}}{12}\ .$$

(b) There are three "parts" to this curve , so one has to use three different parametrizations : the first one is $x=a-t$, $y=t$, $z=0$, where $0\le t\le a$. On this part the line integral is

$$\int_{t=0}^a \big((a-t)^2+t^2\big)\sqrt{1+1}\,dt = 2a^3\sqrt{2}\ .$$

Similarly, the second part of the curve is $x=0$, $y=a-t$ and $z=t$, where $0\le t\le a$; given the symmetries, the second line integral is equal to the first one.
And the third part is again equal to the first one (check this, the parameterization being this time $x=t$, $y=0$, $z=a-t$), so in the end one gets $\int_\Gamma (x^2+y^2+z^2)ds = 2a^3\sqrt{2}$.

2. Compute $\int_\Gamma (x^2+y^2)dx+(x^2-y^2)dy$, where Γ is the boundary of the domain $D=\{(x,y)\in\mathbb{R}^2\colon x\ge 0,\ y\ge 0,\ x+y\le 1\}$, oriented clockwise.
Correction. Looking at Γ, we see that there are three line integrals to compute (picture). The first part of Γ may be parameterized by setting $x=0$, $y=t$, where $0\le t\le 1$; the corresponding line integral is $\int_{t=0}^1 \big(0+(0-t^2)\big)dt = -\frac{1}{3}$. The second part may be parameterized by $x=t$, $y=1-t$, where $0\le t\le 1$; the corresponding integral is $\int_{t=0}^1 \big(t^2+(1-t)^2+(t^2-(1-t)^2)(-1)\big)dt = \int_{t=0}^1 2(1-t)^2dt = \frac{2}{3}$.
Finally, the last part can be parameterized by $x=1-t$, $y=0$ (pay attention to the direction here), so this time we get the integral $\int_{t=0}^1 \big(((1-t)^2+0)(-1)+0\big)dt = \int_{t=0}^1 -(1-t)^2dt = -\frac{1}{3}$. Eventually, we obtain that $\int_\Gamma (x^2+y^2)dx+(x^2-y^2)dy = -\frac{1}{3}+\frac{2}{3}-\frac{1}{3} = 0$.

3. Let V be the vector field of coordinates $V(x,y,z)=\begin{pmatrix}x+z & y & x\end{pmatrix}$. Compute the circulation of V along the path with parametric equation $x(t)=\cos(t)$, $y(t)=\sin(t)$, $z(t)=t$, where $0\le t\le 4\pi$.
Correction. We have to compute $\int_\Gamma (x+z)dx+ydy+xdz$, where Γ stands for the path along which we are computing the circulation. Given that this path is already parameterized, the computation is straightforward and one gets

$$\int_\Gamma (x+z)dx+y^2dy+xdz = \int_{t=0}^{4\pi} \big((\cos(t)+t)(-\sin(t))+\sin(t)\cos(t)+\cos(t)\cdot 1\big)dt =$$

$$-\int_{t=0}^{4\pi} t\sin(t)dt = -\Big[-t\cos(t)\Big]_{t=0}^{4\pi} - \int_0^{4\pi}\cos(t)\,dt = 4\pi\ .$$

4. Consider a particle moving in a force field of equation $F(x,y,z) = \begin{pmatrix} x-y-z & x^2+y & z-y \end{pmatrix}$. Compute the work of that force field in the following cases :
(a) The particle moves on a straight line from $(0,0,0)$ to $(1,2,4)$;
(b) The particle moves first on a straight line from $(0,0,0)$ to $(1,2,2)$, then on a straight line from $(1,2,2)$ to $(1,2,4)$.
Correction. (a) One can parameterize the line from $(0,0,0)$ to $(1,2,4)$ by setting $x=t$, $y=2t$ and $z=4t$ (This is not the arc-length parameterization !), where $0 \le t \le 1$. The work of the force field is then

$$\int_\Gamma (x-y-z)dx+(x^2+y)dy+(z-y)dz = \int_{t=0}^1 \big((t-2t-4t).1+(t^2+2t).2+(4t-2t).4\big)dt = \int_0^1 (2t^2+7t)dt = \frac{2}{3}+\frac{7}{2} = \frac{25}{6}$$

(b) Using again x as a parameter, the first part of the curve is described by the equation $x=t$, $y=2t$, $z=2t$, where $0 \le t \le 1$; using z as a parameter, the second part is parameterized by $x=1$, $y=2$ and $z=2+t$, where $0 \le t \le 2$. The work when the particle moves along that curve is thus

$$\int_{t=0}^1 \big((t-2t-2t).1+(t^2+2t).2+0.4\big)dt+\int_{t=0}^2 \big(0+0+(2+t-2)\big)dt = \int_0^1 (t+2t^2)dt+\int_0^2 tdt = \frac{1}{2}+\frac{2}{3}+2 = \frac{19}{6}\ .$$

5. Compute $I = \int_\Gamma x^2(y+1)dx + xy(2a-y)dy$ using two different methods (remember Green's theorem), where Γ is the boundary of the upper-half of the disk of center $(0,0)$ and radius $a>0$, oriented counterclockwise. For one of these methods, it might be useful to use trigonometric formulae such as $\sin(x)\cos(x) = \dfrac{\sin(2x)}{2}$ and $\sin^2(2x) = \dfrac{1-\cos(4x)}{2}$. (do you know how to recover these equalities ?)
Correction. Denote by S the interior of the upper-half of the disk. Using Green's theorem, one gets $I = -\iint_S y(2a-y) - x^2 dxdy$. Going to polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, where $0 \le r \le a$ and $0 \le \theta \le \pi$, this integral becomes

$$I = \int_{r=0}^a \left(\int_{\theta=0}^\pi (-r^2+2ar\sin(\theta))rd\theta\right)dr = \int_{r=0}^a (-\pi r^3+4ar^2)dr = -\pi\frac{a^4}{4}+\frac{4}{3}a^4 = a^4\left(\frac{4}{3}-\frac{\pi}{4}\right)\ .$$

Using the definition of a line integral, one has to look carefully at the boundary (a picture) : it is made up of the line segment between $(-a,0)$ and $(0,a)$, and the upper half of the circle of radius a. The line segment may be parameterized as $x=t$, $y=0, -a \le t \le a$, so the line integral on that part of Γ is $\int_{-a}^a \big(t^2(0+1).1+0\big)dt = \int_{-a}^a t^2dt = \dfrac{2a^3}{3}$.
The upper-half of the circle is best parameterized as $x=a\cos(t)$, $y=a\sin(t)$, $0 \le t \le \pi$, so the line integral I' on that part of Γ is $I' = \int_{t=0}^\pi \big(a^2\cos^2(t)(a\sin(t)+1)(-a\sin(t)) + a^2\cos(t)\sin(t)(2a-a\sin(t))a\cos(t)\big)dt$, so

$$I' = \int_0^\pi \big(-2a^4\cos^2(t)\sin^2(t) + (2a^4-a^3)\cos^2(t)\sin(t)\big)dt = \int_0^\pi -\frac{a^4\sin^2(2t)}{2}dt + \left[\frac{(-2a^4+a^3)\cos^3(t)}{3}\right]_0^\pi$$

The only nontrivial part remaining is to compute $\int_0^\pi \sin^2(t)dt$; for this, we use the formula $\sin^2(2t) = \dfrac{1-\cos(4t)}{2}$, and this yields $\int_0^\pi \sin^2(2t)dt = \dfrac{\pi}{2}$. Putting all this together, we obtain
$I' = -\dfrac{a^4}{2}\dfrac{\pi}{2} + (-2a^4+a^3)\dfrac{-2}{3} = a^4\left(\dfrac{4}{3}-\dfrac{\pi}{4}\right) - \dfrac{2a^3}{3}$. Adding this to the integral on the line segment that we computed earlier, we finally get that $I = \int_\Gamma x^2(y+1)dx + xy(2a-y)dy = a^4\left(\dfrac{4}{3}-\dfrac{\pi}{4}\right)$ (which is the same value that we obtained above, surprisingly enough).
From this exercise, you should remember the following facts : first, it is useful to know some trigonometry if one wants to compute line integrals. More importantly, when computing a line integral on the boundary of a domain one has to be careful when determining exactly what that boundary is (here it was easy to forget the

line segment) ; and Green's theorem gives the value of a line integral when the curve is oriented *counterclockwise.*
One final remark, or rather, a question : can you explain (using Green's theorem) why the a^3's above cancel out ?