Note. I'll add pictures as soon as my laptop functions again properly (or until I figure how to make pictures using the Linux computer in my office), so this is a temporary version.

1 : Computation of the divergence in cylindrical coordinates.
We use a surface adapted to our system of coordinates, in other words one in which one has r constant on two faces, θ constant on two faces, and z constant on the two remaining faces. We assume this surface is centered at the point with cylindrical oordinates (r_0, θ_0, z_0).
The volume of the domain enclosed by this surface is approximately $\Delta V \approx r_0 \Delta r \Delta\theta \Delta z$ (when the volume shrinks to a point, the quotient of the actual volume by this aproximation has limit 1, so for our purposes this approximation is sufficient). On the faces where r is constant one has $r = r_0 \pm \frac{\Delta r}{2}$, and $\vec{n} = \pm\vec{e_r}$. The area of the face where $\vec{n} = -\vec{e_r}$ is $(r_0 - \frac{\Delta r}{2})\Delta\theta\Delta z$, and the area of the face where $\vec{n} = \vec{e_r}$ is $(r_0 + \frac{\Delta r}{2})\Delta\theta\Delta z$.
Since our cylinder is small, we can pretend that on each face $\vec{F}$ is constant, equal to its value at the middle of the face. Thus the contribution to the flow of the two faces on which r is constant is (in first approximation)

$$F_r(r_0 + \frac{\Delta r}{2})(r_0 + \frac{\Delta r}{2})\Delta\theta\Delta z - F_r(r_0 - \frac{\Delta r}{2})\Delta\theta\Delta z \approx \frac{\partial}{\partial r}(rF_r)(r_0, \theta_0, z_0)\Delta_r\Delta_\theta\Delta z \ .$$

(To justify this approximation one would need to use the mean value theorem and the fact that the first-order partial derivatives of F_r are continuous).
Similarly, for θ one has the two faces $\theta = \theta_0 \pm \frac{\Delta\theta}{2}$, and the contribution of these surfaces to the total flow of $\vec{F}$ is more or less

$$F_\theta(r_0, \theta_0 + \frac{\Delta\theta}{2}, z)\Delta r\Delta z - F_\theta(r_0, \theta_0 - \frac{\Delta\theta}{2}, z)\Delta r\Delta z \approx \frac{\partial F_\theta}{\partial\theta}(r_0, \theta_0, z_0)\Delta r\Delta\theta\Delta z \ .$$

Using the same method, one finds that the contribution to the flow of the two horizontal faces is

$$\approx \frac{\partial F_z}{\partial z}(r_0, \theta_0, \varphi_0) r_0 \Delta r\Delta\theta\Delta_z \ .$$

(the area of both these faces is $r_0\Delta\theta\Delta r$)
Eventually, we obtain that the total flow of F is approximately

$$\iint_S \vec{F}\cdot\vec{n}d\sigma \approx \frac{\partial}{\partial r}(rF_r)(r_0, \theta_0, z_0)\Delta r\Delta\theta\Delta z + \frac{\partial F_\theta}{\partial\theta}(r_0, \theta_0, z_0)\Delta r\Delta\theta\Delta z + \frac{\partial F_z}{\partial z} r_0\Delta r\Delta\theta\Delta z \ .$$

Dividing by $\Delta V \approx r_0\Delta r\Delta\theta\Delta z$, we obtain (hoping that in the limit our $\approx$ becomes an $=$)

$$\mathrm{div}(\vec{F}) = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z} \ .$$

Since I'm typing this anyway, let's see how one computes the curl in spherical coordinates using a similar method.

2 : Computation of the curl in spherical coordinates. We have the intrinsic formula

$$\mathrm{curl}(\vec{F})(P)\cdot\vec{n} = \lim \frac{1}{\Delta S}\int_C (\vec{F}\cdot\vec{t})ds \ ,$$

where S is a small surface enclosing the point P with unit normal at P equal to $\vec{n}$ and C is the boundary of S, oriented according to the choice of $\vec{n}$ (right-hand rule).
Hence to compute the coordinate of $\mathrm{curl}(\vec{F})$ along $\vec{e_r}$, we need to compute circulations along small curves where the normal to the surface enclosed is in the direction of $\vec{e_r}$. The simplest such curve is made up of portions on which θ is constant and portions on which φ is constant, and centered at the point with spherical coordinates (r, θ, φ) (you should make a picture; I will put one online as soon as my computer decides to work normally

again).
The area of the surface enclosed by our curve is

$$r^2\Delta\theta\big(\cos(\varphi+\frac{\Delta\varphi}{2})-\cos(\varphi-\frac{\Delta\varphi}{2})\big)\approx r^2\sin(\varphi)\Delta\theta\Delta\varphi\ .$$

To obtain this, one can use a surface integral (can you recover this result ?) ; again $\approx$ means that the limit as the surface shrinks to a point of the quotient of the quantity on the left by the quantity on the right is 1, so that we can safely pretend that the area of the surface enclosed by our curve is $r^2\sin(\varphi)\Delta\theta\Delta\varphi$ (can you write down what we just did in terms of Jacobian matrices and Jacobian determinants ?).
The circulation along our curve is the sum of the circulations along each of the small circle arcs ; again pretending that $\vec{F}$ is constant of each arc, equal to its value at the middle of the arc, one gets that the sum of the circulations on the parts on which θ is constant is

$$F_\theta(r,\theta,\varphi+\frac{\Delta\varphi}{2})\cdot r\sin(\varphi+\frac{\Delta\varphi}{2})\Delta\theta-F_\theta(r,\theta,\varphi-\frac{\Delta\varphi}{2})\cdot r\sin(\varphi-\frac{\Delta\varphi}{2})\approx r\Delta\theta\Delta\varphi\frac{\partial}{\partial\varphi}(\sin(\varphi)F_\theta)\ .$$

To obtain this, we used the fact that along these arcs one has $\vec{t}=\pm\vec{e_\theta}$, so on these arcs $\vec{F}\cdot\vec{t}=\pm F_\theta$. Hence, if we pretend that F_θ is constant on the (small) arc (equal to its value at the midpoint of the curve), we get that the circulation of $\vec{F}$ along each of these arcs is $\pm F_\theta(r,\theta,\varphi\pm\frac{\Delta\varphi}{2})$ multiplied by the length of the curve (the $\pm$ sign depends on the orientation of our curve and this is to see which sign to choose that you need to make a picture).
Similarly, the sum of the circulations of $\vec{F}$ along the two curves on which φ is constant (and on which, consequently, $\vec{n}=\pm\vec{e_\varphi}$) is approximately

$$F_\varphi(r,\theta-\frac{\Delta\theta}{2},\varphi)r\Delta\varphi-F_\varphi(r,\theta+\frac{\Delta\theta}{2},\varphi)r\Delta\varphi\approx-\frac{\partial F_\varphi}{\partial\theta}r\Delta\theta\Delta\varphi\ .$$

(pay attention to the signs !)
Given that the area of the surface enclosed by our curve is $\approx r^2\sin(\varphi)\Delta\theta\Delta\varphi$, we obtain that the component of $\mathrm{curl}(\vec{F})$ along $\vec{e_r}$ is

$$\frac{1}{r\sin(\varphi)}\frac{\partial}{\partial\varphi}(\sin(\varphi)F_\theta)-\frac{1}{r\sin(\varphi)}\frac{\partial F_\varphi}{\partial\theta}\ .$$

Using a similar method for $\vec{e_\theta},\vec{e_\varphi}$ (it is **very** instructive to do so), one obtains

$$\mathrm{curl}(\vec{F})=\big(\frac{1}{r\sin(\varphi)}\frac{\partial}{\partial\varphi}(\sin(\varphi)F_\theta)-\frac{1}{r\sin(\varphi)}\frac{\partial F_\varphi}{\partial\theta}\big)\vec{e_r}+\big(\frac{1}{r}\frac{\partial}{\partial r}(rF_\varphi)-\frac{1}{r}\frac{\partial F_r}{\partial\varphi}\big)\vec{e_\theta}+\big(\frac{1}{r\sin(\varphi)}\frac{\partial F_r}{\partial\theta}-\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta)\big)\vec{e_\varphi}\ .$$

Of course, this might seem ugly ; but if one is dealing with a problem with spherical symmetry the expression simplifies considerably (since in that case everything only depends on r). It is immportant that you understand what it means to write down a vector field in a new system of coordinates.