UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN — Fall 2006
Math 380 — Group G1

Midterm III.
Monday, December 4.
50 minutes

You are not allowed to use your lecture notes, textbook, or any other kind of documentation. Calculators, mobile phones and other electronic devices are also prohibited.

1.(10 points)
Let γ be the curve of equation $x(t) = t$, $y(t) = e^{\sin(2\pi t)}$, $z(t) = \ln(1+t^2)$, for $0 \leq t \leq 1$, oriented in the direction of increasing t. Compute $\displaystyle\int_\gamma (3x^2 + 3yz + e^z)dx + (3xz + 3y^2)dy + (3xy + xe^z)dz$.

Correction. Looking at the curve, and the line integral, one can guess that there is a trick here; indeed, if one lets $f(x,y,z) = x^3 + y^3 + 3xyz + xe^z$ then the integral is equal to $\displaystyle\int_\gamma \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$. Hence the integral is equal to $f(B) - f(A)$, where A, B are the endpoints of the curve (B is the point corresponding to $t = 1$, A correspond to $t = 0$). This yields

$$\int_\gamma (3x^2 + 3yz + e^z)dx + (3xz + 3y^2)dy + (3xy + xe^z)dz = (1 + 1 + 3\ln(2) + 2) - (0 + 1 + 0 + 0) = 3 + 3\ln(2)\ .$$

2. (10 points)
Let S be the quarter-cylinder of equation $x \geq 0$, $y \geq 0$, $x^2 + y^2 = 1$, $0 \leq z \leq 2$ (viewed as a closed surface) and let $\vec{F}(x, y, z) = (x^3 + \sin(yz) + e^z, x + z^2, 3zy^2 - x)$. Compute $\displaystyle\iint_S \vec{F} \cdot \vec{n}\, d\sigma$.

Correction. Looking at the integral, applying the divergence theorem seems like a good idea ; indeed, one has

$$\operatorname{div}(\vec{F}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 3x^2 + 0 + 3y^2 \ .$$

Given that the domain of integration is a cylinder, cylindrical coordinates are the way to go. Since $x^2 + y^2 = r^2$, we obtain (don't forget the Jacobian determinant !) :

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \int_{z=0}^{2} \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} 3r^2 \cdot r\, dr d\theta dz = \frac{\pi}{2} \int_{z=0}^{2} \int_{r=0}^{1} 3r^3\, dr dz = \frac{\pi}{2} \cdot 2 \cdot \frac{3}{4} = \frac{3\pi}{4} \ .$$

3. (15 points)

Let S be the upper hemisphere of radius 1, i.e the set of points (x, y, z) such that $x^2 + y^2 + z^2 = 1$ and $z \geq 0$. Compute $\iint_S (2xy + z) d\sigma$.

Correction. Let us use the usual system of spherical coordinates : $x = \sin(\varphi)\cos(\theta)$, $y = \sin(\varphi)\sin(\theta)$, $z = \cos(\varphi)$; the domain is $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \frac{\pi}{2}$. We now need to determine $d\sigma$; for this, we compute the cross-product

$$\frac{\partial P}{\partial \theta} \times \frac{\partial P}{\partial \varphi} = \begin{pmatrix} -\sin(\theta)\sin(\varphi) \\ \cos(\theta)\sin(\varphi) \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos(\theta)\cos(\varphi) \\ \sin(\theta)\cos(\varphi) \\ -\sin(\varphi) \end{pmatrix} = \begin{pmatrix} \cos(\theta)\sin^2(\varphi) \\ \sin(\theta)\sin^2(\varphi) \\ \cos(\varphi)\sin(\varphi) \end{pmatrix}$$

The magnitude of this vector is $\sin(\varphi)$, so the integral is

$$I = \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \left(2\sin^3(\varphi)\cos(\theta)\sin(\theta) + \sin(\varphi)\cos(\varphi)\right) d\theta d\varphi = \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \left(\sin(2\theta)\sin^3(\varphi) + \frac{\sin(2\varphi)}{2}\right) d\theta d\varphi$$

$$I = 2\pi \int_{\varphi=0}^{\pi/2} \frac{\sin(2\varphi)}{2} d\varphi = \pi \ .$$

4. (15 points).
Let S be the surface of equation $x+y^3+z=1$, $x \geq 0$, $y \geq 0$, $z \geq 0$, oriented with normal vector pointing away from the origin, and $\vec{F}$ be the vector field defined by $\vec{F}(x,y,z) = (x+y^2+z, y-1, z)$. Compute $\iint_S \vec{F} \cdot \vec{n}\, d\sigma$.

Correction. One has $z = 1-x-y^3$, so it is natural here to use the cartesian representation ; the domain for x, y is given by $0 \leq x$, $0 \leq y$ and $x+y^3 \leq 1$. One obtains

$$\vec{F} \cdot \vec{n}\, d\sigma = \pm(x+y^2+z, y-1, z) \cdot (1, 3y^2, 1) dxdy = (x+y^2+z+3y^3-3y^2+z) dxdy = (-x-2y^2+y^3+2) dxdy$$

The choice of orientation above comes from the fact that, given our surface, the normal pointing away from the origin is the one that is going **up**. To see this, one doesn't really need to make a sketch (which would be hard in that case) : simply, our surface is an equipotential of the function $f(x,y,z) = x+y^3+z$, so it is normal to the gradient ; at any point on S, when z increases (with x, y) f increases too, which means that the gradient of $f$is pointing up. Also, on a straight line from the origin to a point on S, the function f is continually increasing. Hence the normal pointing away from the origin is the one given by the gradient (because for this surface going away from the origin is the same as having f increase). We'll discuss it in class.
Eventually, we obtain that our integral is

$$I = \int_{y=0}^{1} \int_{x=0}^{1-y^3} \big(-x-2y^2+y^3+2 \big) dxdy = \int_{y=0}^{1} \big(-\frac{(1-y^3)^2}{2} - 2y^2(1-y^3) + y^3(1-y^3) + 2(1-y^3) \big) dy$$

$$I = \int_{y=0}^{1} \big(\frac{3}{2} - \frac{3}{2}y^6 - 2y^2 + 2y^5 \big) dy = \frac{3}{2} - \frac{3}{14} - \frac{2}{3} + \frac{1}{3} = \frac{20}{21} .$$