University of Illinois at Urbana-Champaign Fall 2006

Math 444 Group E13

Graded Homework X.
Due Friday, November 17.

1.(a) Define a function f on $\mathbb{R}$ by setting $f(x) = \begin{cases} f(x) = x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$. Is this function continuous on $\mathbb{R}$? (you may use without proof the fact the the function $x \mapsto \sin(x)$ is continuous).

(b) Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1-x & \text{else} \end{cases}$. At which points in $\mathbb{R}$ is g continuous ?

Correction. (a) Since we are told that $x \mapsto \sin(x)$ is continuous on $\mathbb{R}$, we see immediately that f is continuous on $(-\infty,0)\cup(0,+\infty)$: indeed, on these two intervals f is a composition of continuous functions. Thus, our only problem is at $x=0$; we need to see whether f has a limit at 0, and whether that limit is equal to $f(0)=0$. As usual with sin, the important thing to remember is that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$; thus, for all $x \neq 0$ we see that $|f(x)| \leq |x|$. This is enough to ensure that $\lim\limits_{x\to 0} f(x) = 0$ (Squeeze theorem). Thus we have proved that $\lim\limits_{x\to 0} f(x) = f(0)$, so f is continuous at 0. Overall, we see that f is continuous on the whole real line.

(b) For all $x \in \mathbb{R}$ there exists a sequence of rational numbers (q_n) and a sequence of irrational numbers (α_n) such that $x = \lim(q_n) = \lim(\alpha_n)$. If g were continuous at x, then we would have $g(x) = \lim(f(q_n)) = \lim(q_n) = x$, and also $g(x) = \lim(f(\alpha_n)) = \lim(1-\alpha_n) = 1-x$. This shows that g can only be continuous at x if $x = 1-x$, in other words if $x = \frac{1}{2}$.

Now we strongly suspect that g is actually continuous at $\frac{1}{2}$; to show it, it is best not to use sequences, but come back to the ε,δ definition of continuity. Indeed, pick $\varepsilon > 0$; notice that for any real number x, we have $|x - \frac{1}{2}| = |g(x) - \frac{1}{2}|$ (check this in the two cases when x is rational, irrational). Thus we see that whenever $0 < |x - \frac{1}{2}| \leq \varepsilon$ we have $|g(x) - \frac{1}{2}| \leq \varepsilon$, and this proves that $\lim\limits_{x\to 1/2} g(x) = \frac{1}{2} = f(\frac{1}{2})$. Thus f is continuous exactly at the point $\frac{1}{2}$.

2. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function ($a<b$) such that $|f(x) - f(x')| < |x - x'|$ for all $x \neq x' \in [a,b]$.
(a) Using the ε,δ definition of continuity, show that f is continuous on $[a,b]$.
(b) Prove that there exists a unique point $x \in [a,b]$ such that $f(x) = x$ (introduce a suitable auxiiary function).

Correction. (a) The assumption on the function f implies that it is a Lipschitz function, so it must be (uniformly) continuous on $[a,b]$: indeed, pick some point $x \in [a,b]$ and $\varepsilon > 0$. Then, for any $y \in [a,b]$ such that $|x-y| \leq \varepsilon$, one has $|f(x) - f(y)| < |x-y| \leq |x-y|$. Thus, setting $\delta = \varepsilon$ works in this case (beware : it is not true in general that setting $\delta = \varepsilon$ works ; it just happens to be true here because of our assumption on f) to show that f is continuous at x. Since this is true for any $x \in I$, we have proved that f is continuous on I.
(b) Define $g(x) = f(x) - x$. Then one has $g(a) = f(a) - a \geq 0$ (because $a \leq g(x) \leq b$ for all $x \in [a,b]$) and $g(b) = f(b) - b \leq 0$. Thus, either $g(a) = 0$, or $g(b) = 0$, or g takes a strictly positive value at a and a strictly negative one at b ; in the third case, the mean value theorem tells us that there is $x \in [a,b]$ such that $g(x) = 0$, and this is obviouslyy true in the first two cases too. So we know that there exists a *fixed point* of f in $[a,b]$, i.e a point $x \in [a,b]$ such that $f(c) = x$ ($g(x) = 0$ if, and only if, $f(x) = x$). To show that this fixed point is unique, proceed by contradiction and assume there are two points $x \neq x' \in [a,b]$ such that $f(x) = x$ and $f(x') = x'$. Then by assumption on f we have $|f(x) - f(x')| < |x - x'|$, and this contradicts the fact that

$f(x) = x$, $f(x) = x'$.

3. Let $f\colon [0,1] \to [0,1]$ be a continuous function such that $f(0) = f(1)$.
Show that for all $n \in \mathbb{N}$ there exists $x \in [0, 1 - \frac{1}{n}]$ such that $f(x) = f(x + \frac{1}{n})$.
(Hint : is it possible that $f((k+1)/n) - f(k/n)$ keeps a constant sign for all $k = 0, \dots, n-1$?)
Correction. Notice first that one may assume $n \geq 2$ (the case $n = 1$ is trivial, since $f(0) = f(1)$).
If $f((k+1)/n) = f(k/n)$ for some $k = 0, 1, \dots, n-1$, then we have found $x \in [0, 1 - \frac{1}{n}]$ such that $f(x) = f(x + \frac{1}{n})$ (k/n works). So we may assume that $f((k+1)/n) \neq f(k/n)$ for all $k = 1, \dots, n-1$. We claim that there must then exist some $k \in 0, \dots n-2$ such that $f((k+1))/n - f(k/n)$ and $f((k+2)/n) - f(k+1/n)$ are of opposite signs; indeed, assume this is not true, and suppose for instance that $f(1/n) > f(0)$. Then $f(2/n) > f(1/n)$, $f(3/n) > f(2/n)$, etc., and one eventually obtains $f(1) = f(n/n) > f((n-1)/n) > \dots f(0)$, which contradicts our assumption that $f(1) = f(0)$. One obtains a similar contradiction if $f(1/n) < f(0)$, so there must exist k such that $f((k+1))/n - f(k/n)$ and $f((k+2)/n) - f(k+1/n)$ are of opposite signs. Set then $g(x) = f(x + 1/n) - f(x)$; we just proved that g must change signs on the interval $[k/n, (k+1)/n]$ for some $k = 0, \dots, n-2$. The mean value theorem then ensures that g has a 0 in that interval. Since a 0 of g is the same thing as a point x such that $f(x) = f(x + 1/n)$, we see that under the assumptions on f there must indeed exist a point $x \in [0, 1 - 1/n]$ such that $f(x) = f(x + 1/n)$.

4.(a) Show that if f is a continuous function on a closed bounded interval $[a, b]$ such that $f(x) > 0$ for all $x \in [a, b]$ then there exists $m > 0$ such that $f(x) \geq m$ for all $x \in [a, b]$.
In the following, we pick two continuous functions f, g from $[0,1] \to [0,1]$ such that $f(x) < g(x)$ for all $x \in [0,1]$.
(b) Show that there exists $m > 0$ such that $f(x) + m < g(x)$ for all $x \in [0,1]$.
(c) Show that there exists $M > 1$ such that $Mf(x) < g(x)$ for all $x \in [0,1]$.
Correction. (a) Since f is continuous function on the closed bounded interval $[a, b]$, we know that it admits a global minimum on $[a, b]$, i.e that there exists $c \in [a, b]$ such that $f(c) \leq f(x)$ for all $x \in [a, b]$. Setting $m = f(c)$, we have $m > 0$ because f only takes positive values, and $f(x) \geq m$ for all $x \in [a, b]$
(b) The function $g - f$ is continuous on $[0,1]$ taking only positive values, so by question (a) there exists $m' > 0$ such that $g(x) - f(x) \geq m'$ for all $x \in [0,1]$; but then $m = \frac{m'}{2}$ is such that $m > 0$ and $g(x) - f(x) > m$ for all $x \in [0,1]$, in other words $f(x) + m < g(x)$ for all $x \in [0,1]$ (we had to introduce this m' and then divide it by 2 because question (a) only gave us a *large* inequality, whereas we had to establish a *strict* inequality).
(c) First, notice that one has $0 \leq f(x) < g(x)$ for all $x \in [0,1]$, so g only takes positive values. Thus the function $h = \frac{f}{g}$ is continuous on $[0,1]$, and $h(x) < 1$ for all $x \in [0,1]$. Thus, there exists $m' > 0$ such that $0 \leq h(x) \leq 1 - m'$ for all $x \in [0,1]$ (apply a to $x \mapsto 1 - h(x)$, or directly use the fact that h admits a maximum on $[0,1]$). Again, setting $m = \frac{m'}{2}$ enables us to go from a large inequality, and obtain $h(x) < 1 - m'$ for some m' such that $0 < m' < \frac{1}{2}$ and all $x \in [0,1]$. Given the definition of h, and the fact that $1 - m' > 0$, $g(x) > 0$, this is equivalent to $\frac{1}{1-m'} f(x) < g(x)$ for all $x \in [0,1]$. Setting $M = \frac{1}{1-m'}$, we see that $M > 1$ (because $0 < m' < 1$)) and $Mf(x) < g(x)$ for all $x \in [0,1]$.
Remark. The idea behind this exercise is that, if one has an inequality that is true for all x and involves continuous functions on a closed bounded interval then one can use the fact that a continuous function on a closed bounded interval reaches its bounds to show that a strengthening of the inequality is also true on that interval; you should convince yourself that all the assumptions in this exercise are important (f must be **continuous**, and the interval must be **closed** and **bounded**).

5. Let f be a **continuous** function from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Recall that we proved in the first midterm that one has then $f(q) = qf(1)$ for all $q \in \mathbb{Q}$. Use this to show that $f(x) = xf(1)$ for all $x \in \mathbb{R}$.
Correction. We know that $f(q) = qf(1)$ for any rational q. Let now x be any real number; by the density theorem, we know that there exists a sequence (q_n) of rationals such that $\lim(q_n) = x$. Since f is conti-

nuous, this implies $\lim(f(q_n)) = f(x)$. But $f(q_n) = q_n f(1)$, so the algebraic theorems about limits give $\lim(f(q_n)) = f(1)\lim(q_n) = xf(1)$. Putting these two equalities together, we indeed obtain that $f(x) = xf(1)$ for all $x \in \mathbb{R}$.

Important note. In the homework assignment, the assumption that f is continuous was forgotten ; without this assumption the result is false in general (one can produce counterexamples using the Axiom of Choice). Hence this exercise will *not* be graded.