University of Illinois at Urbana-Champaign Fall 2006

Math 444 Group E13

Graded Homework XI.
Correction.

1. (a) Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is not constant and satisfies $f(x) = f(x^2)$ for all $x \in \mathbb{R}$.
(b) Assume now that f is continuous at 0 and 1 and $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Show that f must be constant.
Hint : assume that $|x| < 1$; then what is the limit of the sequence (x_n) defined by $x_1 = x$, $x_2 = x^2, \ldots, x_{n+1} = x_n^2$? How about the sequence $(f(x_n))$? Can you use a similar idea when $|x| > 1$?
Correction. (a) There are many possible examples, one of them being the function f defined by $f(0) = 0$, and $f(x) = 1$ for all $x \in \mathbb{R}$.
(b) Following the indication, pick first $x \in \mathbb{R}$ such that $|x| < 1$. Then define a sequence (x_n) by setting $x_1 = x$, $x_2 = x^2, \ldots x_{n+1} = x_n^2, \ldots$. Then this sequences converges to 0, and one has $f(x_{n+1}) = f(x_n^2) = f(x_n)$ for all $n \in \mathbb{N}$; thus an easy induction proof yields $f(x_n) = f(x)$ for all $n \in \mathbb{N}$. Since (x_n) is convergent to 0 and f is continuous at 0, one must also have $\lim f(x_n) = f(0)$, hence we obtain $f(x) = f(0)$ for all $x \in (-1, 1)$.
Since f is assumed to be continuous at 1, we also obtain that $f(1) = f(0)$. Pick now $x \in \mathbb{R}$ such that $x > 1$, and define this time a sequence (y_n) by setting $y_1 = x, y_2 = \sqrt{x}, \ldots y_{n+1} = \sqrt{y_n}, \ldots$. Then one has $y_n = y_1^{1/2^n}$, and thus $\lim(y_n) = 1$. Thus $\lim(f(y_n)) = f(1) = f(0)$. But the sequence (y_n) was defined in such a way that $f(y_n) = f(y_{n+1}^2) = f(y_{n+1})$ for all $n \in \mathbb{N}$, which again means that $f(y_n) = f(y_1) = f(x)$ for all $n \in \mathbb{N}$. Putting the pieces together, we obtain $f(x) = f(0)$ for all $x > 1$. Since f is clearly an even function (i.e $f(x) = f(-x)$, because $f(x) = f(x^2) = f((-x)^2))$ this also implies that $f(x) = f(0)$ for all $x < -1$. We have finally managed to prove that $f(x) = f(0)$ for all $x \in \mathbb{R}$, in other words f is constant on $\mathbb{R}$.
Remark. This proof is typical of how one uses continuity to describe the class of functions that satisfy a certain property (here, $f(x) = f(x^2)$) : first, try to understand what kind of information that property yields (here, it yields sequences (x_n), (y_n) converging to 0, 1 and such that $f(x_n)$, $f(y_n)$ are constant). Then use this, added to continuity, to obtain additional information on what the function looks like (here, it is constant).

2. Let $f\colon [0,1] \to [0,1]$ be a continuous function such that $f \circ f = f$ $(*)$. Set

$$E_f = \{x \in [0,1] \colon f(x) = x\} \ .$$

Show that E_f is nonempty, then that it is an interval.
Hint : what is the link between E_f and $f([0,1])$?
Can you describe (accurately and using as few words as possible) the functions that satisfy $(*)$?
Correction. Following the hint, let us prove that $E_f = f([0,1])$. Pick $x \in E_f$; then $x = f(x)$, so $x \in f([0,1])$, and this proves that $E_f \subset f([0,1])$. Conversely, pick $x \in f([0,1]$, i.e $x = f(y)$ for some $y \in [0,1]$. Then $f(x) = f(f(y)) = f(y) = x$ by the assumption $(*)$; this means that $f([0,1]) \subset E_f$. Both inclusions mean that $f([0,1]) = E_f$. Since f is continuous, the image of $[0,1]$ is a closed bounded interval, hence E_f is a closed bounded interval $[a,b]$, with $0 \le a \le b \le 1$. This means that f is a continuous mapping such that :
• f maps $[0,a]$ to $[a,b]$, and $f(a) = a$;
• f maps $[b,1]$ to $[a,b]$, and $f(b) = b$;
• for all $x \in [a,b]$ one has $f(x) = x$.
Conversely, if f satisfies the three conditions above for some $a \le b \in [0,1]$, then f satisfies $(*)$, so we have given a description of these functions.

3. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin(\frac{1}{x}) & \text{else} \end{cases}$. Prove that f is continuous, and even differentiable, on $\mathbb{R}$, but that f' is not continuous at 0.

(b) Is it true that any function satisfying the conclusion of the intermediate value theorem must be continuous ?
Correction. To see that f is continuous, see the preceding homework ($f(x) = xg(x)$, where g is a function that we proved continuous in HW 10 ; anyway, continuity is implied by differentiability, which we establish below). It is clear that f is differentiable both on $(-\infty, 0)$ and on $(0, +\infty)$ and that on these intervals one has $f'(x) = x\sin(\dfrac{1}{x}) - \sin(\dfrac{1}{x})$. Thus we see that f' doesn't have a limit at 0, for instance because $f'(\dfrac{1}{2\pi n}) = 0$ and $f(\dfrac{1}{2\pi n + \pi/2}) = \dfrac{1}{2\pi n + \pi/2} - 1$, which converges to -1. Hence, there are two sequences (x_n), (y_n) which both converge to 0 but are such that $f(x_n)$, $f(y_n)$ converge to different limits, and this means that f' doesn't have a limit at 0. You would probably expect it to mean that f' doesn't exist at 0 ; this is not the case, however, as we will see shortly. All it means is that even if f' exists at 0 it won't be continuous at that point.
To check that $f'(0)$ exists, one simply comes back to the definition and writes $\dfrac{f(x) - f(0)}{x - 0} = x\sin(\dfrac{1}{x})$, which converges to 0 at $x = 0$ (see last week's homework). Thus $f'(0)$ exists and is equal to 0.
(b) By the theorem of Darboux all derivatives satisfy the conclusion of the intermediate value theorem ; since the function above is a derivative and is not continuous, we see that it is **not** true that any function satisfying the conclusion of the intermediate value theorem must be continuous.

4. Determine $a, b \in \mathbb{R}$ such that the function $f\colon [0, +\infty) \to \mathbb{R}$ defined by $f(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1 \\ ax^2 + bx + 1 & \text{else} \end{cases}$ is differentiable on $(0, +\infty)$.
Correction. First, notice that f is differentiable on $[0, 1)$ and $(1, +\infty)$, and that $f'(x) = \frac{1}{2\sqrt{x}}$ on $(0, 1)$, $f'(x) = 2ax + b$ on $(0, 1)$. So the only problem is at $x = 1$. First we need to ensure that f is continuous ; for that it needs to satisfy $\lim_{x\to 1} f(x) = f(1)$. This implies that $a + b + 1 = 1$, in other words $a + b = 0$ (to see that, we equate the left-hand and right-hand limits of f at 1). Conversely, if $a + b = 0$, then one sees that f is continuous at 1. Now we need it to be differentiable at 1 ; for this, one must have $a + b = 0$, and $\frac{f(x)-f(1)}{x-1}$ must have a limit at 1. If $x < 1$, then the mean value theorem gives us $\frac{f(x)-f(1)}{x-1} = f'(c) = \frac{1}{2\sqrt{c}}$ for some $c \in (x, 1)$, so we see that the left-hand limit of $\frac{f(x)-f(1)}{x-1}$ is $\frac{1}{2}$. Using the same method, we see that the right hand-limit of $\frac{f(x)-f(1)}{x-1}$ at 1 is $2a + b$, thus these two limits are equal if and only if $2a + b = \frac{1}{2}$. Hence f is differentiable at 1 if, and only if, $a + b = 0$ and $2a + b = \frac{1}{2}$; this is equivalent to $a = \frac{1}{2}$ and $b = -\frac{1}{2}$.
Remark. In this exercise we used the important fact that a function f has a limit at a point $x \in \mathbb{R}$ if, and only if, it has both a left-hand limit and a right-hand limit at x and they are equal.

5. Show that a polynomial function of the form $f(x) = x^n + ax + b$ has at most three distinct real roots (here a, b are reals, and n is a natural integer).
Hint : How many zeros can f' have ? What must happen to f' between any two zeros of f ?
Correction. Between any two zeros of f there must be a zero of f', because of Rolle's theorem. This means that if f' doesn't have many zeros then f cannot have many zeros either. Here $f'(x) = nx^{n-1} + a$, so $f'(x) = 0 \Leftrightarrow x^{n-1} = -\dfrac{a}{n}$, and this has at most two solutions. Thus $f(x) = 0$ cannot have four distinct real roots or more : if $a_1 < a_2 < a_3 < a_4$ were distinct roots then there would have to be $b_1 \in (a_1, a_2)$ such that $f'(b_1) = 0$, $b_2 \in (a_2, a_3)$ such that $f'(b_2) = 0$, and $b_3 \in (a_3, a_4)$ such that $f'(b_3) = 0$. Thus b_1, b_2, b_3 would be three distinct solutions of the equation $f'(x) = 0$, and we saw that this is impossible.

6. Pick a function $f\colon \mathbb{R}^+ = [0, +\infty) \to \mathbb{R}$, and $l \in \mathbb{R}$. One says that f *has limit l at* $+\infty$, and one writes $\lim_{x\to+\infty} f(x) = l$, if for any $\varepsilon > 0$ there exists $M \in \mathbb{R}^+$ such that $x \ge M \Rightarrow |f(x) - l| \le \varepsilon$.
(a) Show that, for any continuous function f, one has the following implication : if $f\colon \mathbb{R}^+ \to \mathbb{R}$ has a limit at $+\infty$ then f is bounded on $\mathbb{R}^+$. What is the converse of this assertion ? Is it true ?
(b) Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be such that $f(0) = 1$ and $\lim_{x\to+\infty} f(x) = 0$. Show that f admits a global maximum on $\mathbb{R}^+$. Must it also admit a global minimum on $\mathbb{R}^+$?
(c) Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be differentiable on $\mathbb{R}^+$, and suppose that $\lim_{x\to+\infty} f'(x) = l$, where l is some real number.

Using the mean value theorem, show that $\lim_{x\to+\infty} \frac{f(x)}{x} = l$.

Hint : First prove that for any $\varepsilon > 0$, there exists $a > 0$ such that for any $x > a$ one has $\left|\frac{f(x)-f(a)}{x-a} - l\right| \leq \varepsilon$.

How can you prove this ? Why does this help ?

Correction. (a) Call l the limit of f at $+\infty$. Then there exists M such that $x \geq M \Rightarrow l-1 \leq f(x) \leq l+1$. We also know that, since f is continuous on $[0, M]$ it is bounded on this interval (which is closed and bounded), which means that there exist A, B such that $A \leq f(x) \leq B$ for all $x \in [0, M]$. Define now $C = \min(A, l-1)$ and $D = \max(B, l+1)$: then $C \leq f(x) \leq D$ for all $x \in \mathbb{R}^+$.

The converse of the assertion would be : "any continuous bounded function on $\mathbb{R}^+$ has a limit at $+\infty$" ; it is not true, as $f(x) = \sin(x)$ shows. Indeed, this function if bounded but doesn't have a limit at $+\infty$ (to see it, simply notice that $f(2\pi n) = 0$ and $f(2\pi n + \frac{\pi}{2}) = 1$ for all $n \in \mathbb{N}$).

(b) Let us apply the same reasoning as in the preceding question : this time $l = 0$, and $B \geq 1 = l+1$, so the reasoning before shows that $f(x) \leq B$ for all $x \in \mathbb{R}$. But B is defined as being the supremum of f over the closed bounded interval $[0, M]$, so it is actually the maximum of f on that interval, hence there exists $c \in [0, M]$ such that $f(c) = B$. We then have $f(c) \geq f(x)$ for all $x \in \mathbb{R}^+$, and this proves that f admits a global maximum on $\mathbb{R}^+$.

The idea here is that, since "at $+\infty$" f is below some value (here $f(0)$) the supremum of $f(\mathbb{R}^+)$ cannot be "attained at $+\infty$", hence it is actually the supremum of f on a closed bounded interval, hence it is actually a maximum.

In general, f doesn't have a global minimum on $\mathbb{R}^+$; to see this, simply consider the function $f(x) = \frac{1}{x+1}$: $f([0, +\infty)) = (0, 1]$, hence the infimum of f on $[0, +\infty)$ (0) is not a minimum.

(c) Following the hint, begin by picking $\varepsilon > 0$ and find a such that $x \geq a \Rightarrow |f'(x) - l| \leq \varepsilon$. Then pick $x > a$; by the mean value theorem applied to f on $[a, x]$, there exists $c \in (a, x)$ such that $\frac{fx)-f(a)}{x-a} = f'(c)$. We thus obtain that $|\frac{f(x)-f(a)}{x-a} - l| \leq \varepsilon$ for all $x > a$. One has $\frac{f(x)-f(a)}{x-a} = \frac{f(x)}{x-a} - \frac{f(a)}{x-a}$. If $M \geq a$ is big enough, one has $x \geq M \Rightarrow \frac{f(a)}{x-a} \leq \varepsilon$; thus, $x \geq M \rightarrow \left|\frac{f(x)}{x-a} - l\right| \leq 2\varepsilon$. We want to have $\frac{f(x)}{x}$ instead of $\frac{f(x)}{x-a}$. Notice that

$$\frac{f(x)}{x-a} = \frac{f(x)}{x}\frac{x}{x-a} = \frac{f(x)}{x} + a\frac{f(x)}{x(x-a)} .$$

Hence (for all $x \geq M$)

$$\left|\frac{f(x)}{x} - l\right| \leq \left|\frac{f(x)}{x-a} - l\right| + \frac{1}{x}\left|a\frac{f(x)}{x-a}\right| \leq 2\varepsilon + \frac{1}{x}\left|a(l+2\varepsilon)\right| .$$

This enables us to see that there exists $M' \geq M$ such that $x \geq M' \Rightarrow \left|\frac{f(x)}{x} - l\right| \leq 3\varepsilon$, and this concludes the proof.