University of Illinois at Urbana-Champaign — Fall 2006
Math 444 — Group E13

**Graded Homework I**
Due Friday, Sept. 8 .

1. Prove by induction that for all $n \geq 1$ one has

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} < 2 \; .$$

(*Hint :* it is actually simpler to prove that the quantity on the left is $\leq 2 - 1/n$...)

2. Let us prove by induction that all that the pencils in the world are the same color : denote by $P(n)$ the property "in any group of $n$ pencils,all the pencils are the same color". Then $P(1)$ is true..
Let us now assume that $P(n)$ is true, and try to prove that $P(n+1)$ is also true. Given a group of $n+1$ pencils, take one of them away : by the induction hypothesis, the $n$ pencils remaining are all the same color. Put that pencil back, and take away another one : the $n$ pencils remaining are again the same color ; consequently, all the $n+1$ pencils are the same color, so $P(n+1)$ is true.
Therefore $P(n)$ is true for all $n$, and we are done with the proof.
*What is the problem with the proof above ?*

3. Let $X, Y, Z$ be three sets and $f \colon Y \to Z$, $g \colon X \to Y$ two bijective maps.
- What is the domain of $f \circ g$ ? What is its range ?
- Same questions for $(f \circ g)^{-1}$.
- Prove that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

4. Let $X, Y, Z$ be three sets and $f \colon X \to Y$, $g \colon Y \to Z$ be functions . Prove the following assertions :
- $(g \circ f$ one-to-one$) \Rightarrow (f$ one-to-one$)$
- $(g \circ f$ onto$) \Rightarrow (g$ onto$)$ .

Are the converse assertions true in general ?

5. Let $g, h \colon \mathbb{R} \to \mathbb{R}$ be the functions defined by $g(x) = x^2$, $h(x) = x + 2$. Let $h = g \circ f$.
- Determine $h(\mathbb{R})$ and $h(E)$, where $E = \{x \in \mathbb{R} \colon 0 \leq x \leq 1\}$.
- Determine $h^{-1}(F)$ and $h^{-1}(G)$, where $F = (1, +\infty)$ and $G = [0, 4]$.

6. Let $f \colon \mathbb{N} \to \mathbb{N}$ and $g \colon \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\forall k \in \mathbb{N} \; f(k) = 2k \text{ and } g(k) = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even} \\ \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases} \; .$$

- Determine if $f$ is one-to-one, onto ; same questions for $g$.
- Compute $f \circ g$ and $g \circ f$ ; determine whether they are one-to-one, onto.