University of Illinois at Urbana-Champaign — Fall 2006
Math 444 — Group E13

Graded Homework IV
Due Friday, October 6.

1. Compute, if they exist, $\sup(A)$ and $\inf(A)$ in the following cases. In each case, state whether A admits a maximal element, and do the same for minimal elements.
$A = \{\frac{n - \frac{1}{n}}{n + \frac{1}{n}} : n \in \mathbb{N}\}$; $A = \{\frac{p}{pq+1} : q, p \in \mathbb{N}\}$.

2. Consider the set A of all $x \in \mathbb{R}$ such that there exist two natural integers p, q satisfying $p < q$ and $x = \frac{2p^2 - 3q}{p^2 + q}$.
(a) Prove that -3 is a lower bound of A, and 2 is an upper bound.
(b) Compute $\inf(A)$ and $\sup(A)$ (for $\sup(A)$, look at what happens when $q = p + 1$, divide by p and try to use the Archimedean property of the reals)

3 (a). Prove that, for any $x \in \mathbb{R}$, $E(x) = \sup(\{n \in \mathbb{Z} : n \leq x\})$ exists, and that it is the unique integer n such that $n \leq x < n + 1$ (we more or less saw this in class). Use this characterization of $E(x)$ to solve the questions (b), (c), (d) and (e) below.
(b) Show that, for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$, one has $E(x + n) = E(x) + n$.
(c) Prove that, for all $x, y \in \mathbb{R}$ one has $E(x) + E(y) \leq E(x + y) \leq E(x) + E(y) + 1$.
(d) Given $x \in \mathbb{R}$, what is the value of $E(x) + E(-x)$? (Hint : distinguish the cases $x \in \mathbb{Z}$ and $x \notin \mathbb{Z}$).
(e) Show that, for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$, one has $E(x) = E(\frac{E(nx)}{n})$.

4. Let $\{a_i : i \in \mathbb{N}\}$ and $\{b_i : i \in \mathbb{N}\}$ be two bounded countable subsets of $\mathbb{R}$.
Prove that $\{|a_i - b_i| : i \in \mathbb{N}\}$ is bounded, and that $|\sup(a_i) - \sup(b_i)| \leq \sup(|a_i - b_i|)$.