University of Illinois at Urbana-Champaign — Fall 2006
Math 444 — Group E13

Graded Homework VII
Correction.

1. Let u_n be the sequence defined by $u_1 = \sqrt{2}$, $u_2 = \sqrt{2+\sqrt{2}}$, $u_n = \sqrt{2+\sqrt{2+\dots\sqrt{2}}}$.
(a) Give an induction formula $u_{n+1} = f(u_n)$ defining u_{n+1} as a function of u_n.
(b) Prove that (u_n) is convergent and compute its limit (hint : show that (u_n) is increasing and bounded above by 2).
Correction. (a) One has $u_{n+1} = \sqrt{2+u_n}$.
(b) Let us prove by induction that $u_n \leq u_{n+1}$: one has $u_2 = \sqrt{2+\sqrt{2}} \geq \sqrt{2} = u_1$, so this is true if $n = 1$. Assuming that $u_n \leq u_{n+1}$, one has $u_{n+2} = \sqrt{2+u_{n+1}} \geq \sqrt{2+u_n} = u_{n+1}$, so the statement $u_{n+1} \geq u_n$ is true for all $n \in \mathbb{N}$.
Now, let us prove by induction that $u_n \leq 2$ for all $n \in \mathbb{N}$: this is true for $n = 1$, and if $u_n \leq 2$ then $u_{n+1} = \sqrt{2+u_n} \leq \sqrt{2+2} = 2$. This shows that $u_n \leq 2$ for all $n \in \mathbb{N}$.
We proved that (u_n) is an increasing, bounded above sequence : therefore (u_n) is convergent.
Let $l = \lim(u_n)$; then $(\sqrt{2+u_n})$ is convergent and has limit $\sqrt{2+l}$ (using an easy result proved in the textbook), so we obtain that $l = \sqrt{2+l}$. Thus $l^2 = 2 + l$, and this implies that $l = 2$ or $l = -1$. Yet, since $u_n \geq 0$ for all n, so $\lim(u_n)$ must be ≥ 0; thus it is impossible that $l = -1$. This means thatg $l = 2$, so $\lim(u_n) = 2$.
(Read the correction of this exercise carefully : we prove first that (u_n) converges. Then, using the definition of the sequence, we get that the only possible limits are 2 and -1 ; using the fact that the sequence is nonnegative we remark that -1 is not a possible limit. Since (u_n) has a limit, and only one limit is possible, we finally get $\lim(u_n) = 2$.

2. Show that the sequences defined by the formulas $u_n = \dfrac{1}{n} + \cos(\dfrac{n\pi}{3})$ and $v_n = \dfrac{(-1)^n n^2 + n}{3n^2 + n}$ are not convergent.
Correction. One has $u_{6n} = \dfrac{1}{6n} + \cos(2n\pi) = \dfrac{1}{6n} + 1$, so (u_{6n}) converges to 1. Similarly, one obtains $u_{6n+3} = \dfrac{1}{3n+3} - 1$, so (u_{6n+3}) converges to -1. Thus (u_n) has two subsequences which converge to different limits, and this proves that (u_n) is not convergent.
One has $v_{2n} = \dfrac{(2n)^2 + 2n}{3(2n)^2 + 2n} = \dfrac{1 + \frac{1}{2n}}{3 + \frac{1}{2n}}$, so (v_{2n}) converges to $\dfrac{1}{3}$. A similar computation yields that (v_{2n+1}) converges to $-\dfrac{1}{3}$; as above, this is enough to show that (v_n) is not convergent.

3. Recall that we saw in class that if (u_n) is a sequence of real numbers such that (u_{2n+1}) and (u_{2n}) converge to the same limit l then (u_n) is convergent and $\lim(u_n) = l$.
(a) Use the same method to show that if (u_n) is a sequence of real numbers such that (u_{3n}), (u_{3n+1}) and (u_{3n+2}) converge to the same limit l, then (u_n) is convergent and $\lim(u_n) = l$.
(b) Let (u_n) be a sequence of real numbers such that (u_{2n}), (u_{2n+1}) and (u_{3n}) are all convergent ; show that (u_n) is convergent (Hint : use the fact that (u_{3n}) converges to prove that (u_{2n}) and (u_{2n+1}) converge to the same limit, then use the result seen in class).
Correction. (a) Call l the common limit of the three sequences (u_{3n}), (u_{3n+1}) and (u_{3n+3}). Pick $\varepsilon > 0$; we know that there exist M_1, M_2, M_3 such that $n \geq M_1 \Rightarrow |u_{3n} - l| \leq \varepsilon$, $n \geq M_2 \Rightarrow |u_{3n+1} - l| \leq \varepsilon$ and $n \geq M_3 \Rightarrow |u_{3n+2} - l| \leq \varepsilon$. Let then $M = 3M_1 + 3M_2 + 3M_3$, and pick $n \geq M$. We either have $n = 3k$ with $k \geq M_1$, or $n = 3k+1$ with $k \geq M_2$, or $3k+2$ with $k \geq M_3$; in any case we get $|u_n - l| \leq \varepsilon$. This proves that (u_n) converges to l.
(b) Let l, l', l'' denote the limits of u_{2n}, u_{2n+1} and u_{3n} (in that order). Then $u_{6n} = u_{2(3n)}$, so (u_{6n}) is a

subsequence of (u_{2n}), which proves that (u_{6n}) converges to l. But (u_{6n}) is also a subsequence of (u_{3n}) (why ?), so it converges to l''. Since the limit of a sequence is unique, this yields $l = l'$. In the same way, one sees that (u_{6n+3}) is a subsequence of both (u_{3n}) and (u_{2n+1}), so their limits are equal and $l' = l''$.
Putting all this together, we obtain $l = l'$, so (u_{2n}) and (u_{2n+1}) converge to the same limit, and we saw in class that this implies that (u_n) is convergent.

4. Given a sequence (u_n) of reals numbers, define another sequence s_n by the formula $\dfrac{u_1 + u_2 + \ldots + u_n}{n}$.
(a) Here we assume that (u_n) is convergent and $\lim(u_n) = 0$. Show that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ one has

$$|s_n| \leq \frac{|u_1 + u_2 + \ldots u_N|}{n} + \frac{\varepsilon}{2} .$$

Use this to prove that $\lim(s_n) = 0$.
(HInt : for the inequality, pick some suitable N and then cut the sum in two parts ; use the triangle inequality, and the fact that the sum of $n - N$ reals having each an absolute value less than $\frac{\varepsilon}{2}$ has an absolute value less that $(n-N)\frac{\varepsilon}{2}$)
(b) Show that if (u_n) is convergent and $\lim(u_n) = l$ then (s_n) is convergent and $\lim(s_n) = l$.
(Hint : apply the result of question (a) to the sequence $(u_n - l)$)
(c) Show that the converse of this assertion is not true (look at what happens if $(u_n) = (-1)^n$ for instance).
Correction. (a) Pick $\varepsilon > 0$; since $\lim(u_n) = 0$, we know that there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |u_n| \leq \frac{\varepsilon}{2}$ (apply the definition of convergence with $\varepsilon' = \frac{\varepsilon}{2}$. Then, for $n \geq N$, one has

$$|s_n| = \left|\frac{u_1 + u_2 + \ldots + u_n}{n}\right| \leq \left|\frac{u_1 + u_2 + \ldots + u_N}{n}\right| + \frac{|u_{N+1}| + \ldots + |u_n|}{n} \leq \frac{|u_1 + u_2 + \ldots u_N|}{n} + \frac{(n-N)}{n}\frac{\varepsilon}{2} .$$

Thus, we finally obtain that for $n \geq N$ one has $|s_n| \leq \frac{1}{n}\frac{|u_1 + u_2 + \ldots u_N|}{n} + \frac{\varepsilon}{2}$.
Now pick $\varepsilon > 0$ and find $N \in \mathbb{N}$ as above ; the sequence $\frac{1}{n}(|u_1 + u_2 + \ldots u_N|)$ converges to 0, so for some $M \in \mathbb{N}$ big enough one has $n \geq M \Rightarrow \frac{1}{n}(|u_1| + |u_2| + \ldots |u_N|) \leq \frac{\varepsilon}{2}$. But then, the inequality above shows that, for any $n \geq \max(M, N)$ one has $|s_n| \leq \varepsilon$.
This proves that (s_n) converges to 0 if (u_n) converges to 0.
(b) If (u_n) converges to l, then $(u_n - l)$ converges to 0 ; therefore the result of the preceding question tells us that $\dfrac{(u_1) - l + (u_2 - l) + \ldots (u_n - l)}{n}$ converges to 0. But one has

$$\frac{(u_1 - l) + (u_2 - l) + \ldots (u_n - l)}{n} = \frac{u_1 + u_2 + \ldots + u_n - nl}{n} = \frac{u_1 + u_2 + \ldots u_n}{n} - l = s_n - l .$$

So, the result of question (a) tells us that $(s_n - l)$ converges to 0, in other words that (s_n) is convergent and $\lim(s_n) = l$
(c) If $u_n = (-1)^n$, then $s_n = \dfrac{(-1)^n - 1}{2n}$ (proved by induction) so (s_n) converges to 0, whereas (u_n) does not converge. This means that the converse of the assertion in (b) is not true.

5. Show that a subsequence of a Cauchy sequence is also a Cauchy sequence.
Correction. Let $\varphi \colon \mathbb{N} \to \mathbb{N}$ be strictly increasing, and let (u_n) be a Cauchy sequence. Pick $\varepsilon > 0$; we know that there exists $N \in \mathbb{N}$ such that for any $n, m \in \mathbb{N}$, $n, m \geq N \Rightarrow |u_n - u_m| \leq \varepsilon$. Since $\varphi(n) \geq n$ for all $n \in \mathbb{N}$, we get that

$$n, m \geq N \Rightarrow \varphi(n), \varphi(m) \geq N \Rightarrow |u_{\varphi(n)} - u_{\varphi(m)}| \leq \varepsilon .$$

This proves that $(u_{\varphi(n)})$ is a Cauchy sequence.