University of Illinois at Urbana-Champaign — Fall 2006
Math 444 — Group E13

Graded Homework VIII
Due Monday, November 6.

1. Let q be an integer larger than or equal to 2. For all $n \in \mathbb{N}$, define u_n by the formula $u_n = \cos(\frac{2n\pi}{q})$. Compute u_{nq}, u_{nq+1} ; is the sequence (u_n) convergent ?
Correction. One obtains $u_{nq} = \cos(\frac{2nq\pi}{q}) = \cos(2n\pi) = 1$; similarly, one has $u_{nq+1} = \cos(\frac{(2nq\pi + 2\pi}{q}) = \cos(2n\pi + \frac{2\pi}{q}) = \cos(\frac{2\pi}{q}) \neq 1$ because $q \geq 2$. This shows that the subsequences (u_{nq}) and (u_{nq+1}) have different limits, hence the sequence (u_n) is not convergent.

2. Let $A \subset \mathbb{R}$. A function $f \colon A \to A$ is said to be *increasing* if $x \leq y \Rightarrow f(x) \leq f(y)$ for all $x, y \in A$. Similarly, one may define what a *decreasing function* is.
1. Prove that if f is decreasing then $f \circ f$ is increasing.
2. Let now (u_n) be a sequence such that $u_{n+1} = f(u_n)$, where $u_1 \in [0,1]$ and f is a function from $[0,1]$ to $[0,1]$.
2.a. Prove that if f is increasing then (u_n) is monotone.
2.b. Prove that if f is decreasing then (u_{2n}) and (u_{2n+1}) are monotone.
Correction. 1. Assume that f is decreasing, and pick $x \leq y$, with $x, y \in A$; since $x \leq y$, one gets $f(x) \geq f(y)$ because f is decreasing. But then $f(f(x)) \leq f(f(y))$ for the same reason ; this is what we wanted to prove.
2..a. Assume that $u_1 \leq u_2$; let us then prove by induction that $u_n \leq u_{n+1}$ for all $n \in \mathbb{N}$. This is true for $n = 1$ by our assumption, so assume it holds for some $n \in \mathbb{N}$, i.e $u_n \leq u_{n+1}$; then one has $f(u_n) \leq f(u_{n+1})$, which is the same as saying that $u_{n+1} \leq u_{n+2}$. Thus in that case the sequence (u_n) is increasing.
Similarly, if $u_2 \leq u_1$, one sees that $u_n \leq u_{n+1}$ for all $n \in \mathbb{N}$, in other words the sequence is decreasing in that case. In both cases, we see that the sequence (u_n) is indeed monotone.
2.b. Let $v_n = u_{2n}$; one has $v_{n+1} = u_{2(n+1)} = u_{2n+2} = f(u_{2n+1}) = f(f(u_{2n})) = (f \circ f)(v_n)$. Thus, since $f \circ f$ is increasing, the preceding question enables us to assert that (v_n) is monotone. Similarly, if one lets $w_n = u_{2n+1}$ one also obtains that $w_{n+1} = (f \circ f)(w_n)$; so (w_n) is also monotone.

3.Prove that a subset A of $\mathbb{R}$ is dense if, and only if, for any real number x there exists a sequence (a_n) of elements of A such that $\lim(a_n) = x$.
Correction. Assume that $A \subset \mathbb{R}$ is dense, and let $x \in \mathbb{R}$. Since A is dense, we know that for any $n \in \mathbb{N}$ there exists $a_n \in A$ such that $a_n \in [x - \frac{1}{n}, x + \frac{1}{n}]$. This particular sequence of elements (a_n) is then such that $\lim(a_n) = x$ (because of the Squeeze Theorem, for instance). Conversely, assume that $A \subset \mathbb{R}$ has the property that for any $x \in \mathbb{R}$ there is a sequence (a_n) of elements of A such that $\lim(a_n) = x$, and pick $x < y$. Define $z = \frac{x+y}{2}$; there exists a sequence (a_n) of elements of A such that $\lim(a_n) = z$. Intuitively, since this sequence converges to the middle of the interval $]x, y[$, it has to enter this interval at some point ; to prove it, pick $\varepsilon = \frac{y-x}{4}$. Then $\varepsilon > 0$, so by definition of a convergent sequence there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - z| \leq \varepsilon$. Given the definition of z and ε, the inequality $|a_n - z| \leq \varepsilon$ is the same as $x + \frac{y-x}{4} \leq a_n \leq y - \frac{y-x}{4}$. In particular, this means that any a_n for $n \geq N$ is an element of A which also belongs to the interval $]x, y[$; given that x, y were arbitrary, this proves that A is dense in $\mathbb{R}$.

4. Given a sequence of real numbers (x_n), we say that $\lim(u_n) = +\infty$ if, and only if, for any $M \in \mathbb{R}$ there exists a naturel number N such that for any $n \in \mathbb{N}$ one has $n \geq N \Rightarrow u_n \geq M$.

1.a. Prove that a if sequence (x_n) is such that $\lim(x_n) = +\infty$ then all of its subsequences are such that $\lim(x_{\varphi(n)} = +\infty$.
1.b. Prove that if (x_n) is a sequence of positive reals such that $\lim(x_n) = +\infty$ is not true then (x_n) has a bounded subsequence.
1.c. Prove that a sequence of positive reals (x_n) is such that $\lim(x_n) = +\infty$ if, and only if, it doesn't have a convergent subsequence.
2. We wish to prove that, if $\alpha > 0$ is an irrational number and (p_n), (q_n) are sequence of natural integers such that $\lim\left(\frac{p_n}{q_n}\right) = \alpha$ then $\lim(p_n) = +\infty$ and $\lim(q_n) = +\infty$.
2.a. Pick an irrational number $\alpha > 0$; explain why there exist sequences (p_n), (q_n) as above.
In the following questions we assume we have picked α, (p_n), (q_n) as above.
2.b Prove that if $\lim(q_n) = +\infty$ then $\lim(p_n) = +\infty$.
2.c. Prove that if (q_n) is not such that $\lim(q_n) = +\infty$ then (q_n) admits a constant subsequence $(q_{\psi(n)})$ (use 1.d; what can you tell about a convergent sequence of integers?).
2.d. Prove that $(p_{\psi(n)})$ is such that for n, m big enough one has $p_{\psi(n)} = p_{\psi(m)}$.
2.e. Conclude.
Correction. 1.a Let $(x_{\varphi(n)})$ be a subsequence of (x_n), and pick $M \in \mathbb{R}$; then we know that there exists N such that $n \geq N \Rightarrow x_n \geq M$. Since $\varphi\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing, we know that $\varphi(n) \geq n$ for any $n \in \mathbb{N}$; this yields in particular that

$$n \geq N \Rightarrow \varphi(n) \geq \varphi(N) \geq N \Rightarrow x_{\varphi(n)} \geq M\ .$$

This proves that $\lim(x_{\varphi(n)}) = +\infty$.
1.b. If (x_n) is a sequence of positive reals such that $\lim(x_n) = +\infty$ is not true, then there must exist some $M \in \mathbb{R}$ with the property that for any $N \in \mathbb{N}$ there exists $i \geq N$ such that $x_i \leq M$. But then, one can inductively build a strictly increasing sequence of integers (i_n) such that $x_{i_n} \leq M$ for all $n \in \mathbb{N}$ (if $x_{i_1}, \ldots, x_{i_n}$ have been obtained, apply the property from the preceding sentence with $N = n_i + 1$ to find n_{i+1}). Then, setting $\varphi(n) = i_n$, the subsequence $(x_{\varphi(n)})$ of (x_n) is bounded below by 0, and above by M.
1.c. If (x_n) is such that $\lim(x_n) = +\infty$ then question 1.a shows that (x_n) cannot have a convergent subsequence. To prove the converse, assume that (x_n) is such that $\lim(x_n) = +\infty$ is not true. Then question 1.b shows that (x_n) has a bounded subsequence $(x_{\varphi(n)})$; the Bolzano-Weierstrass theorem tells us that $(x_{\varphi(n)})$ must have a convergent subsequence $(x_{\varphi(\psi(n))})$, which is the desired convergent subsequence of (x_n).
2.a. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the fact that for any $\alpha \in \mathbb{R}$ there exists a sequence of rational numbers r_n such that $\lim(r_n) = \alpha$ is a consequence of exercise 3; and when $\alpha \geq 0$ one can also assume that $p_n, q_n \geq 0$ for all $n \in \mathbb{N}$.
2.b. Assume that $\lim(q_n) = +\infty$, and pick $M \in \mathbb{R}$. Since $\lim\left(\dfrac{p_n}{q_n}\right) = \alpha \geq 0$, there must exist some N_1 such that $n \geq N_1 \Rightarrow \dfrac{p_n}{q_n} \geq \dfrac{\alpha}{2}$. Since $\lim(q_n) = +\infty$, we know that there exists N_2 such that $n \geq N_2 \Rightarrow q_n \geq \dfrac{2M}{\alpha}$. Putting these two inequalities together, we obtain that for any $n \geq N = \max(N_1, N_2)$ one has $p_n \geq M$.
2.c. If (q_n) is not such that $\lim(q_n) = +\infty$, then question 1.d tells us that it admits a convergent subsequence $(q_{\varphi(n)})$; the sequence $(q_{\varphi(n)})$ is a convergent sequence of integers, so the Cauchy criterion ensures that for $N \in \mathbb{N}$ big enough $n, m \geq N \Rightarrow q_{\varphi(n)} = q_{\varphi(m)}$ (we saw this in class). Set then $\psi(n) = N + n$; ψ is stricly increasing, and the subsequence $(q_{\psi(n)})$ of (q_n) is constant, equal to $q \in \mathbb{N}$.
2.c. Since $\lim\left(\dfrac{p_n}{q_n}\right) = \alpha$, we know that $\lim\left(\dfrac{p_{\psi(n)}}{q_{\psi(n)}}\right) = \alpha$ (it is a subsequence of the sequence $\left(\dfrac{p_n}{q_n}\right)$). Thus, $(p_{\psi(n)})$ is convergent to αq. But then we know that there exists $N \in \mathbb{N}$ such that $n, m \geq N \Rightarrow p_{\psi(n)} = p_{\psi(m)} = p$ (again becasue $(p_{\psi(n)})$ is a convergent sequence of integers).
2.e The preceding questions imply that the sequence $\left(\frac{p_{\psi(n)}}{q_{\psi(n)}}\right)$ is eventually constant, equal to $\dfrac{p}{q}$; it is supposed to converge to α, so we get $\alpha = \dfrac{p}{q} \in \mathbb{Q}$, and this contradicts the fact that α is irrational. This contradiction can only come from our assumption that $\lim(q_n) = +\infty$ is not true; in other words, $\lim(q_n) = +\infty$ has to be true, and question 2.b implies that $\lim(p_n) = +\infty$ is also true.