UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN — Fall 2006

Math 444 — Group E13

Midterm I

Thursday, September 28
50 minutes

You are not allowed to use your lecture notes, textbook, or any other kind of documentation. Calculators, mobile phones and other electronic devices are also prohibited.

1. (25 points)
(a) State the Density Theorem.
(b) Let $A \subset \mathbb{R}$. Explain what the sentence "A is bounded" means.
(c) Let $A \subset \mathbb{R}$. Define the concept of a greatest lower bound of A.
(d) State the Completeness Property of the reals.
(e) Let $f\colon A \to B$ be a function and $C \subset A$, $H \subset B$. Define what is meant by the notations $f(C)$ and $f^{-1}(H)$.
Correction. Read your notes, or the textbook!

2. (10 points)
Let A, B be sets and $f\colon A \to B$ be a function. Prove that, for all $G, H \subset B$, one has $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ and $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.
Correction. For any $x \in A$, one has
$x \in f^{-1}(G \cup H) \Leftrightarrow f(x) \in G \cup H \Leftrightarrow (f(x) \in G)$ or $(f(x) \in H) \Leftrightarrow (x \in f^{-1}(G))$ or $(x \in f^{-1}(H)) \Leftrightarrow x \in f^{-1}(G) \cup f^{-1}(H)$. This proves that $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$.
Similarly, for any $x \in A$ one has
$x \in f^{-1}(G \cap H) \Leftrightarrow f(x) \in G \cap H \Leftrightarrow (f(x) \in G)$ and $(f(x) \in H) \Leftrightarrow (x \in f^{-1}(G))$ and $(x \in f^{-1}(H)) \Leftrightarrow x \in f^{-1}(G) \cap f^{-1}(H)$. This proves that $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.

3. (10 points)
Let $A, B \subset \mathbb{R}$ be nonempty sets such that $a < b$ for all $a \in A$ and all $b \in B$. Prove that $\sup(A)$ and $\inf(B)$ exist, and that $\sup(A) \le \inf(B)$.
(Hint : you may begin by proving that $\sup(A) - \varepsilon < b$ for all $b \in B$ and all $\varepsilon > 0$)
Under the same assumptions, is it true that $\sup(A) < \inf(B)$?
Correction. Pick some $b_0 \in B$ and some $a_0 \in A$. Then for all $a \in A$ one has $a \le b_0$, so b_0 is an upper bound of A; similarly, $b \ge a_0$ for all $b \in B$, so a_0 is a lower bound of B. Thus, A has an upper bound, and B has a lower bound; the Completeness Property of the reals implies that $\sup(A)$ and $\inf(B)$ exist.
Pick now some $\varepsilon > 0$. By definition of a least upper bound, there exists $a \in A$ such that $\sup(A) - \varepsilon < a$. Since $a < b$ for all $b \in B$, we get that $\sup(A) - \varepsilon < b$ for all $b \in B$ (transitivity of $<$). Thus, $\sup(A) - \varepsilon$ is a lower bound of B, which implies that $\sup(A) - \varepsilon \le \inf(B)$. Since this is true for all $\varepsilon > 0$, we finally obtain that $\sup(A) \le \inf(B)$.
It is not true in general that $\sup(A) < \inf(B)$: consider for instance $A = \{0\}$ and $B = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then A, B satisfy the hypothesis above, yet $\sup(A) = \inf(B) = 0$ (the fact that $\inf(B) = 0$ was seen in class and is a corollary of the archimedean property of the reals).

4. (10 points)
Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that, for all $x, y \in \mathbb{R}$, one has $f(x+y) = f(x) + f(y)$.
(a) (1.5 points) Prove that $f(0) = 0$, and that $f(-x) = -f(x)$.
(b) (3.5 points) Prove by induction on n that, for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, one has $f(nx) = nf(x)$.
(c) (1 point) Prove that, for all $n \in \mathbb{Z}$ and all $x \in \mathbb{R}$, one has $f(nx) = nf(x)$.
(d) (2 points) Prove that, for all $q \in \mathbb{Q}$ and all $x \in \mathbb{R}$ one has $f(qx) = qf(x)$, and in particular $f(q) = qf(1)$.
(e) (2 points) Assume now that f is increasing, i.e $x \leq y \Rightarrow f(x) \leq f(y)$. Prove that for all $x \in \mathbb{R}$ one has $f(x) = xf(1)$.
(Hint : use the density theorem to approximate x by rational numbers, and consider the cases $f(1) = 0$, $f(1) \neq 0$)

Correction.
(a) One must have $f(0+0) = f(0)+f(0)$, so $f(0) = 2f(0)$, which means that $f(0) = 0$. Also, since $x+(-x) = 0$ for all $x \in \mathbb{R}$, one has $f(0) = f(x + (-x)) = f(x) + f(-x)$, so that $f(x) + f(-x) = 0$, in other words $f(-x) = -f(x)$.
(b) Fix $x \in \mathbb{R}$, and let us prove by induction that $f(nx) = nf(x)$ for all $n \in \mathbb{N}$. This statement is true for $n = 1$ (it is then just the statement "$f(x) = f(x)$"). Assume that $f(nx) = nf(x)$ for some $n \in \mathbb{N}$. Then one has $f((n+1)x) = f(nx + x) = f(nx) + f(x) = nf(x) + f(x)$ (by the induction hypothesis). So we finally obtain $f((n+1)x) = (n+1)f(x)$. We have proved that the property "$f(nx) = nf(x)$" is true for $n = 1$, and is hereditary : therefore, by the Induction Theorem, it must be true for all $n \in \mathbb{N}$.
(c) Let $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. If $n = 0$ then $f(nx) = f(0x) = f(0)$ by question (a), so we do have $f(nx) = nf(x)$ in that case. If $n > 0$ then question (b) tells us that $f(nx) = nf(x)$. The remaining case is $n < 0$; in that case $-n > 0$, so that $f((-n)x) = -nf(x)$. Since $f((-n)x) = f(-nx)$, we know by (a) that $f((-n)x) = -f(nx)$. Putting these two things together, we obtain that $-f(nx) = -nf(x)$, in other words $f(nx) = nf(x)$.
(d) Let $x \in \mathbb{R}$ and $q = \dfrac{n}{m} \in \mathbb{Q}$, where $n \in \mathbb{Z}$ and $m > 0$. Then, by (c), we know that
$mf(\frac{n}{m}x) = f(m\frac{n}{m}x) = f(nx) = nf(x)$. This yields $\dfrac{n}{m}f(x) = f(\dfrac{n}{m}x)$, in other words $f(qx) = qf(x)$.
(e) Let now $x \in \mathbb{R}$, and fix $\varepsilon > 0$. Then, thanks to the Density Theorem, we know that there exist rationals q, q' such that $x - \varepsilon \leq q \leq x \leq q' \leq x + \varepsilon$. By our additional assumption on f, we have $f(q) \leq f(x) \leq f(q')$, i.e $qf(1) \leq f(x) \leq q'f(1)$. If $f(1) = 0$ then we get $f(x) = 0 = xf(1)$ for all x, and we are done. If $f(1) \neq 0$, notice first that $f(1) \geq f(0) = 0$ because f is increasing; so $f(1) > 0$. Then we get $q \leq \dfrac{f(x)}{f(1)} \leq q'$, so by our choice of q, q' we obtain $x - \varepsilon \leq \dfrac{f(x)}{f(1)} \leq x + \varepsilon$. This means that $|\dfrac{f(x)}{f(1)} - x| \leq \varepsilon$ for all $\varepsilon > 0$; we saw in class that this implies $\dfrac{f(x)}{f(1)} - x = 0$, in other words $f(x) = xf(1)$.