Midterm II

Thursday, November 2.
50 minutes

You are not allowed to use your lecture notes, textbook, or any other kind of documentation. Calculators, mobile phones and other electronic devices are also prohibited.

1. (20 points)
(a) State the Density Theorem (of rational numbers in the real line).
(c) State the Bolzano-Weierstrass theorem for sequences of real numbers.
(d) Define the notions of a bounded set of reals, of a bounded sequence.
(e) Define what a sequence of real numbers is, and what a subsequence of a sequence (x_n) is.

2. (5 points)
Show that the sequence (x_n) defined by the formula $x_n = 1 - (-1)^n + \frac{1}{n}$ is not convergent.
Correction. One has $x_{2n} = \frac{1}{2n}$ and $x_{2n+1} = 2 + \frac{1}{2n+1}$, so we see that (x_{2n}) converges to 0 and (x_{2n+1}) converges to 2. This is enough to show that (x_n) is not convergent : if it were, all of its subsequences would be convergent and have the same limit.

3 (15 points)
(a) Prove that $x \leq x^2$ for all $x \geq 1$; use this to show that $\sqrt{x} \leq x$ for all $x \geq 1$.
(b) Let $x_1 \geq 2$ and $x_{n+1} = 1+\sqrt{x_n - 1}$. Prove that (x_n) is decreasing and bounded below by 2. Is this sequence convergent ? If so, what is its limit ?
Correction. (a) For $x \geq 1$, one has $x^2 = x.x \geq x$. Also, since $\sqrt{x}.\sqrt{x} = x$, we recover that $\sqrt{x}$ has to be ≥ 1 if x is. Thus, we get $x = (\sqrt{x})^2 \geq \sqrt{x}$ and we are done.
(b) Let us prove by induction that $x_{n+1} \leq x_n$ and $x_n \geq 2$ for all $n \in \mathbb{N}$. For $n = 1$, we know that $x_1 \geq 2$, so we only need to check that $x_2 \leq x_1$; using question (a), we get $x_2 = 1 + \sqrt{x_1 - 1} \leq 1 + x_1 - 1 = x_1$ (because $x_1 - 1 \geq 1$). Assume now that $x_{n+1} \leq x_n$ and $x_n \geq 2$. Then $x_n - 1 \geq 1$, so $x_{n+1} = 1 + \sqrt{x_n - 1} \geq 1 + \sqrt{1} = 2$. But then we also have $x_{n+2} \leq 1 + x_{n+1} - 1 = x_{n+1}$ (because $x_{n+1} - 1 \geq 1$) so we get what we wanted to prove. We have shown that (x_n) is a decreasing sequence, and it is bounded below by 2 : thus, it has to be convergent to some real number $l \geq 2$. But then, (x_{n+1}) is also convergent to l; given the definition of (x_n), this yields $l = 1 + \sqrt{l-1}$. This implies that $(l-1)^2 = l - 1$, so $l - 1 = 0$ or 1. Given that $l \geq 2$, we obtain $l - 1 = 1$, or $l = 2$. So we have proven that $\lim(x_n) = 2$.

4. (15 points)
Let (x_n) be a bounded sequence; define two sequences (s_n), (t_n) by the formulas $s_n = \sup\{x_k \colon k \geq n\}$, $t_n = \inf\{x_k \colon k \geq n\}$.
(a) Explain why these sequences are well-defined.
(b) Prove that for all $n \in \mathbb{N}$, one has $s_n \geq t_n$, $s_n \geq s_{n+1}$, $t_n \leq t_{n+1}$. Use this to show that the sequences (s_n), (t_n) are convergent.
(c) Prove that if $\lim(s_n) = \lim(t_n) = l$ then (x_n) is convergent and $\lim(x_n) = l$.
(d) Show that there exist subsequences $(x_{\varphi(n)})$, $(x_{\psi(n)})$ such that $\lim(x_{\varphi(n)}) = \lim(s_n)$ and $\lim(x_{\psi(n)}) = \lim(t_n)$. Use this to obtain the converse of the result in question (c). Can you use the results in this exercise to obtain a new proof of the Bolzano-Weierstrass theorem ?
Correction. (a) Since (x_n) is bounded, there exist real numbers m and M such that $m \leq x_k \leq M$ for all

$k \in \mathbb{N}$. Then we see that m is a lower bound for $\{x_k\colon k \geq n\}$, and M is an upper bound for it (for any $n \in \mathbb{N}$). Thus, the completeness property of the reals implies that $\{x_k\colon k \geq n\}$ admits an infimum and a supremum for all $n \in \mathbb{N}$.

(b) Pick some natural integer n. Since t_n is a lower bound of a nonempty set of which s_n is an upper bound, it is clear that $s_n \geq t_n$. By definition, s_n is an upper bound for $\{x_k\colon k \geq n\}$, so it is also an upper bound for $\{x_k\colon k \geq n+1\}$. By definition of a supremum, this implies that $\sup\{x_k\colon k \geq n+1\} \leq s_n$, in other words $s_{n+1} \leq s_n$. Similarly, t_n is a lower bound for $\{x_k\colon k \geq n+1\}$, and this yields $t_{n+1} \geq t_n$.

Hence, the two sequences (s_n), (t_n) are monotone and bounded (by m, M of question (a)), so the monotone convergence theorem implies that they are convergent.

(c) The definition of s_n, t_n implies that $t_n \leq x_n \leq s_n$. Thus, the Squeeze theorem ensures that if $\lim(t_n) = \lim(s_n) = l$ then (x_n) is convergent and $\lim(x_n) = l$.

(d)Let us prove that there exists a subsequence $(x_{\varphi(n)})$ such that $\lim(x_{\varphi(n)}) = \lim(s_n)$. To that end, we claim that it is possible to build a strictly increasing sequence of integers (k_n) such that $s_{k_n+1} - \dfrac{1}{n} \leq x_{k_{n+1}}$ for all $n \in \mathbb{N}$; the idea here is just to find a condition which ensures that the (x_{k_n}) will be increasingly close to the limit of (s_n). To do so, begin by picking $k_1 = 1$. By definition of a supremum there must exist k_2 such that $x_{k_2} \geq \sup\{x_k\colon k \geq 2\} - 1$. This gives us the second term of our sequence. Assume now that $k_1, \ldots, k_n$ have been defined. By definition of s_{k_n+1}, there must exist $k \geq k_n + 1$ such that $x_k \geq s_{k_n+1} - \dfrac{1}{n}$; set $k_{n+1} = k$ and go on to the next step.

The reasoning above shows that an induction process enables us to define a suitable sequence of integers k_n; setting $\varphi(n) = k_n$, and using the fact that $x_n \leq s_n$ for all $n \in \mathbb{N}$, we get $s_{\varphi(n)+1} - \dfrac{1}{n} \leq x_{\varphi(n+1)} \leq s_{\varphi(n+1)}$. Since (s_n) is convergent, and subsequences of a convergent sequence converge to the same limit, the Squeeze theorem ensures that $(x_{\varphi(n)})$ converges and that $\lim(x_{\varphi(n)}) = \lim(s_n)$.

A similar proof would work to show that there exists a subsequence $(x_{\psi(n)})$ such that $\lim(x_{\psi(n)}) = \lim(t_n)$. This shows that, if (x_n) is convergent, then $\lim(t_n) = \lim(s_n)$, since both limits are equal to the limit of a subsequence of (x_n).

This also gives a new proof of the Bolzano-Weierstrass theorem : indeed, if (x_n) is a bounded subsequence then the sequence (s_n) defined at the beginning of the exercise is convergent, and there exists a subsequence of (x_n) which converges to $\lim(s_n)$. In particular, this proves that (x_n) has a convergent subsequence.