UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN — Fall 2006
Math 444 — Group E13

Midterm III

Thursday, November 30.
Correction.

1. Look up your notes, or the textbook!

2. (10 points)
Are the following statements true of false? Explain.
(a) The image of an open interval by a continuous function is an open interval.
(b) If a function f is continuous on $(0, 1)$ then it is bounded on $(0, 1)$.
(c) If $f: \mathbb{R} \to \mathbb{R}$ is continuous and takes only rational values then f is constant.
(d) There exists a differentiable function $f: \mathbb{R} \to \mathbb{R}$ such that $f'(x) = -1$ for all $x < 0$, $f'(0) = 0$ and $f'(x) = 1$ for all $x > 0$.

Correction. (a) This is false; for instance the function defined by $f(x) = 0$ is continuous on $\mathbb{R}$, and the image of the open interval $(0, 1)$ is $\{0\}$, which is not an open interval.
(b) Again, this is false; for instance $f: x \mapsto \frac{1}{x}$ is continuous on $(0, 1)$ but is not bounded ($f(\frac{1}{n}) = n$ for all $n \in \mathbb{N}$ and $\mathbb{N}$ is not bounded).
(c) This is true : indeed, if f is continuous on $\mathbb{R}$ then $f(\mathbb{R})$ is an interval. Thus, if f takes two values x, y with $x < y$, f must take all the values in the interval $[x, y]$; and this interval contains an irrational number because of the density of irrationals in the real line. Thus we proved that if f is continuous and not constant then it takes irrational values; this is the same as saying that if a continuous function f takes only rational values then it is constant.
(d) By Darboux's theorem, a derivative must satisfy the conclusion of the intermediate value theorem (i.e the image of an interval is an interval). The function here doesn't satisfy this property (the image of $\mathbb{R}$ is the three-element set $\{-1, 0, 1\}$, which is not an interval). Hence there cannot exist a function whose derivative is the function in the statement above (and thus (d) is false).

3. (10 points)
Let f be defined on $\mathbb{R}$ by $f(x) = \begin{cases} x^2 + ax + b & \text{if } x \geq 0 \\ \sin(x) & \text{if } x < 0 \end{cases}$.
Is it possible to find a, b such that f is differentiable on $\mathbb{R}$? If not, explain why; if yes, give the values of a, b.

Correction. On $(-\infty, 0)$ one has $f(x) = \sin(x)$, which is differentiable ; on $(0, +\infty)$ one has $f(x) = x^2 + a + b$ which is also differentiable for any $a, b \in \mathbb{R}$. Thus we only need to look at what happens at 0 ; f has a limit at 0 if, and only if, its left-hand limit and its right-hand limit at 0 are equal, thus f is continuous at 0 if, and only if, $b = 0$. Now assume $b = 0$; we need to look at which condition ensures that $\frac{f(x)}{x}$ has a limit at 0. The mean value theorem shows that if $x > 0$ one has $\frac{f(x)}{x} = 2c + a$ for some $c \in (0, x)$, and if $x < 0$ one has $\frac{f(x)}{x} = \cos(c)$ for some $c \in (x, 0)$. Thus the right-hand limit of $\frac{f(x)}{x}$ at 0 is a, and its left-hand limit is at 0 is 1. Hence f is differentiable at 0 if, and only if, $b = 0$ and $a = 1$. This shows that there indeed exist a, b such that f is differentiable on $\mathbb{R}$, and that one has $a = 1$, $b = 0$.

4. (10 points) Assume that $f \colon [0,1] \to \mathbb{R}$ is a continuous function such that for all $x \in [0,1]$ there exists $y \in [0,1]$ such that $|f(y)| \leq \dfrac{1}{2}|f(x)|$. Show that there exists $x \in [0,1]$ such that $f(x) = 0$.

Correction. Let us first solve this using general theorems about continuous functions : the function $|f|$ is continuous on $[0,1]$ since it is a composition of two continuous functions. Thus it must have a minimum on $[0,1]$, i.e there must exist x such that $|f(x)| \leq |f(y)|$ for all $y \in [0,1]$. But we know that there must exist y such that $|f(y)[\leq \dfrac{1}{2}|f(x)|$; this implies that $|f(x)| \leq \dfrac{1}{2}|f(x)|$, which is only possible if $f(x) = 0$.
One may also solve this exercise using the usual methods to prove results involving continuous functions on closed bounded intervals : produce a sequence, use the Bolzano-Weierstrass theorem to obtain a convergent subsequence, then use the continuity of the function to conclude. Here, we proceed as follows : pick any $x_1 \in [0,1]$, then use the property of the function to build inductively a sequence (x_n) of elements of $[0,1]$ such that $|f(x_{n+1})| \leq \frac{1}{2}|f(x_n)|$. By induction, one obtains $|f(x_n)| \leq \frac{1}{2^{n-1}}|f(x_1)|$. Hence $(f(x_n))$ converges to 0. Now we can use the Bolzano-Weierstrass theorem (the sequence (x_n) is bounded) to obtain that there exists a convergent subsequence $(x_{\varphi(n)})$ of (x_n), and its limit x must belong to $[0,1]$ because it is a closed interval. But then, since f is continuous, $f(x_{\varphi(n)})$ converges to $f(x)$, which yields $f(x) = 0$.