# A NOTE ON HJORTH'S OSCILLATION THEOREM.

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Abstract. We reformulate, in the context of continuous logic, an oscillation theorem proved by G. Hjorth and give a proof of the theorem in that setting which is similar to, but simpler than, Hjorth's original one. The point of view presented here clarifies the relation between Hjorth's theorem and first-order logic.

## 1. Introduction and definitions.

Recently, G. Hjorth obtained a nice "oscillation theorem" for actions of Polish groups by isometries (see [3]). In [4], V. Pestov points out the importance of this result, before noting that the original proof is rather intricate: "The proof of Hjorth at this stage looks highly technical, as they say, hard. As it is being slowly digested by the mathematical community, there is no doubt that it will lead to new concepts and insights into the theory of topological groups and eventually will come to be fully understood and made into a "soft" proof". This short note may be thought of as an attempt at "digesting" the proof of Hjorth's oscillation theorem.

Hjorth himself pointed out that his result is related to a first-order logic result, which he also proved in [3]; below we try to understand this connection, by proving an equivalent version of the oscillation theorem in the framework of *continuous logic*. This leads to a statement mirroring the first-order one; proving the theorem in this setting also enables one to simplify the original proof a bit. The underlying idea is that continous logic enables one to extend the techniques and combinatorics of first-order logic to the context of metric spaces. Intuitively, the equality symbol is replaced by the distance function.

We refer to [2] for background on Polish groups, and to [1] for information about continuous logic.

**Definition 1.** A *Polish metric space* is a separable metric space whose distance is complete. A *Polish group* is a separable topological group, the topology of which admits a compatible complete distance.

These spaces and groups are ubiquitous in analysis and geometry; Polish groups in particular have been an important point of interest for descriptive set theory over the past few decades. The following example is of particular importance: whenever $(X,d)$ is a Polish metric space, its isometry group $Isom(X,d)$, endowed with the pointwise convergence topology, is a Polish group.

**Convention.** Whenever $(X,d)$ is a metric space and $n \in \mathbb{N}$, we endow $X^n$ with the sup-metric, which we still denote by $d$. Explicitly,

$$d\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big) = \sup\{d(x_i,y_i)\colon 1 \leq i \leq n\} \ .$$

Below we will only deal with *relational metric structures*, which we define now, along with some concepts that will be needed later on.

**Definition 2.** A *relational metric structure* $\mathcal{M}$ is a complete metric space $(M,d)$ with $d$ bounded by 1, along with a family $(P_i)_{i\in I}$ of *predicates*, i.e uniformly continuous maps from $M^{k_i}$ to $[0,1]$ (where $k_i \in \mathbb{N}$). We always assume that the distance function $d\colon M^2 \to [0,1]$ is included in our list of predicates.
The structure is said to be *Polish* if the underlying metric space is, that is, if $M$ is separable.

Once we have this definition, we need to be able to say when two tuples look similar.

**Definition 3.** Let $\mathcal{M}$ be a relational metric structure. We say that two tuples $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ in $M^n$ *have the same quantifier-free type* if for all $\{j_1,\ldots,j_k\} \subseteq \{1,\ldots,n\}$ and all $i \in I$ with $k_i = k$ one has

$$P_i(a_{j_1},\ldots,a_{j_k}) = P_i(b_{j_1},\ldots,b_{j_k}) \ .$$

The natural mappings to consider in this setting are of course those who preserve the structure we put on the set $M$.

**Definition 4.** A *morphism* from a relational metric structure $\mathcal{M}$ into itself is a mapping from $M$ to $M$ that also preserves all the predicates. It is important to bear in mind that a morphism is always distance-preserving. A morphism is an *automorphism* if it is onto.

We endow the automorphism group $Aut(\mathcal{M})$ of a relational metric structure $\mathcal{M}$ with the pointwise convergence topology. Note that $Aut(\mathcal{M})$ is a closed subgroup of the isometry group of $(M,d)$ and so is a Polish group. Indeed, preserving a given predicate is a closed condition for the pointwise convergence topology since predicates are continuous.

Our next definition introduces a concept that extends the usual notion of ultrahomogeneous structure from first-order logic.

**Definition 5.** We say that a relational metric structure $\mathcal{M}$ is *approximately ultrahomogeneous* if for any $n$-tuples $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ with the same quantifier-free type and any $\varepsilon > 0$ there exists $g \in Aut(\mathcal{M})$ such that $d(g(a_i),b_i) \leq \varepsilon$ for all $i = 1,\ldots,n$.

Note that if $\mathcal{M}$ is separable and approximately ultrahomogeneous then any morphism is a pointwise limit of automorphisms. This is due to the fact that morphisms preserve quantifier-free type.

Whenever $d$ is the discrete distance on $X$, the definitions above coincide with the usual concepts of classical first-order logic. But there are of course many natural examples where the distance is not discrete; in some sense continuous logic complements classical combinatorics with analysis.

Now that we have dealt with the basic definitions of continuous logic, we come to Hjorth's theorem. To state it, we need to introduce yet another notion, which is important when one works with metrizable topological groups.

**Definition 6.** Let $G$ be a metrizable topological group with a compatible left-invariant distance $\delta$. The *left-completion* of $G$, denoted by $\hat{G}$, is simply the metric completion of $(G,\delta)$.

Note that $G$ naturally acts on $\hat{G}$ by isometries; however $\hat{G}$ is not in general a group but only a semigroup. An important feature of this definition, which is well explained in Hjorth's paper [3] and Gao's book [2], is that $\hat{G}$ does not depend on the choice of left-invariant metric $\delta$, in the sense that any two left-invariant metrics on $G$ (compatible with its topology, of course) will produce isomorphic $\hat{G}$. This happens because Cauchy sequences are the same for all left-invariant distances.

We will consider the left-completion of groups acting by isometries on Polish metric spaces; recall that if $(X,d)$ is such a space, its isometry group $Isom(X,d)$, endowed with the pointwise convergence topology, is a Polish group. A compatible left-invariant metric $\delta$ can be obtained as follows: fix a countable dense set $\{x_i\}_{i\in\mathbb{N}}$ and then define, for two isometries $\varphi,\psi$ of $X$,

$$\delta(\varphi,\psi)=\sum_{i=0}^{+\infty}2^{-i}\min\big(1,d(\varphi(x_i),\psi(x_i))\big)\ .$$

Beware, the metric $\delta$ above is in general not complete! If $G\le Isom(X,d)$, we identify $\hat{G}$ with the left-completion of $(G,\delta)$, which one can naturally see as a semigroup of isometric embeddings of $(X,d)$ into itself.

We are now ready to state Hjorth's oscillation theorem:

**Theorem 1.** (Hjorth)
*Let $(X,d)$ be a complete separable metric space, and $G\le Isom(X,d)$ a group of cardinality bigger than one. Then there exists $x_0,x_1\in X$ and uniformly continuous*

$$f\colon \overline{\{(\pi.x_0,\pi.x_1)\colon \pi\in G\}}\to[0,1]$$

*such that for any $\rho\in\hat{G}$ there exist*

$$(y_0,y_1),(z_0,z_1)\in\{\big(\rho(\pi(x_0)),\rho(\pi(x_1))\big)\colon \pi\in\hat{G}\}$$

*with $f(y_0,y_1)=0$ and $f(z_0,z_1)=1$.*

The goal of this note is to establish the following version of Hjorth's theorem:

**Theorem 2.**
*Let $\mathcal{M}$ be an approximately ultrahomogeneous relational Polish metric structure such that $|Aut(\mathcal{M})|>1$. Then there exist a uniformly continuous $f\colon M^2\to[0,1]$ and $(a_0,a_1)\in M^2$ such that for any morphism $\rho\colon\mathcal{M}\to\mathcal{M}$ one can find $(b_0,b_1)$ and $(c_0,c_1)$ in the image of $\rho^2$, with the same quantifier-free type as $(a_0,a_1)$ and such that $f(b_0,b_1)=1$, $f(c_0,c_1)=0$.*

This statement mirrors the first-order result proved by Hjorth in [3], corresponding to the case when $d$ only takes the values 0 and 1, and extends it to the context of metric structures. It is not immediately clear why theorems 1 and 2 are equivalent; we will first give a proof of Theorem 2 and then explain why both theorems are really saying the same thing, albeit in somewhat different languages.

Before we move on to the proof of Theorem 2 let us, for the sake of completeness, state Hjorth's first-order result that is mentioned above. In this theorem all notions should be understood in the usual first-order theoretic sense; it should be obvious that our Theorem 2 is just an extension of this theorem to the setting of continuous logic.

**Theorem 3.** (Hjorth)
*Let $\mathcal{M}$ be a ultrahomogeneous relational countable first-order structure such that*

$|Aut(\mathcal{M})| > 1$. *Then there exist a function* $f\colon M^2 \to \{0,1\}$ *and* $(a_0, a_1) \in M^2$ *such that for any morphism* $\rho\colon \mathcal{M} \to \mathcal{M}$ *one can find* $(b_0, b_1)$ *and* $(c_0, c_1)$ *in the image of* $\rho^2$*, with the same quantifier-free type as* $(a_0, a_1)$ *and such that* $f(b_0, b_1) = 1$ *while* $f(c_0, c_1) = 0$.

## 2. Proof of theorem 2.

Most ideas in the proof below are already present in Hjorth's paper; however the point of view adopted here limits the use of what he calls "messy approximations". We divide the proof in three subcases.

In the following we let $G = Aut(\mathcal{M})$. Recall that, because of the approximate ultrahomogeneity of $\mathcal{M}$, any morphism of $\mathcal{M}$ is a pointwise limit of elements of $G$.

**Case I.** Any $a \in M$ has a precompact orbit under $G$. Since any morphism induces an isometry of $\overline{G.a}$ into itself, and self-isometries of compact metric spaces are necessarily onto, we see that in this case any morphism is onto. Thus there is essentially nothing to prove in this case.

In what follows, we fix some $a$ such that $G.a$ is not precompact, and pick $\varepsilon > 0$ such that $G.a$ contains an infinite set $B$ satisfying $d(b, b') > 100\varepsilon$ whenever $b \neq b' \in B$. For any $\delta > 0$, we let $Stab_\delta(a) = \{g \in G\colon d(g.a, a) < \delta\}$, and

$$acl_\delta(a) = \{y \in G.a\colon Stab_\delta(a).y \text{ is covered by finitely many balls of radius } \varepsilon\} \ .$$

**Case II.** There exists $\delta > 0$ such that $acl_\delta(a)$ contains an infinite set $Z$ with $d(z, z') > 10\varepsilon$ whenever $z, z'$ are two distinct elements of $Z$.
From now on we fix such a $\delta$ and some countable dense set $\{a_i\}_{i\in\mathbb{N}}$ in $G.a$.

**Lemma 4.** *We can find sequences* $(d_i), (e_i)$ *such that* $d_i, e_i \in acl_\delta(a_i)$ *for all* $i \in \mathbb{N}$ *and*

$$d(Stab_{\delta/2}(a_i).d_i, Stab_{\delta/2}(a_j).e_j) \geq \epsilon \textit{ for any } i, j \in \mathbb{N} \ .$$

*Proof.* Assume that we have been able to define $d_i, e_i$ up to some $n$. One can find an infinite set $\{z_j\colon j \in \mathbb{N}\}$ contained in $acl_\delta(a_{n+1})$ and such that $d(z_i, z_j) > 10\varepsilon$ whenever $i \neq j$. What we want is to find some $j$ such that $Stab_{\delta/2}(a_{n+1}).z_j$ is at distance larger than $\varepsilon$ from the set

$$A = \bigcup_{i=0}^{n} Stab_{\delta/2}(a_i).e_i \ .$$

Since each $e_i$ is in $acl_\delta(a_i)$, the set $A$ is covered by a finite number $N$ of balls $B_1, \ldots, B_N$ of radius $\varepsilon$. If we cannot find a satisfactory $j$, there is $i_0 \in \{1, \ldots, N\}$ and an infinite $J \subset \mathbb{N}$ such that, for all $j \in J$, $z_j$ is mapped by some $g_j \in Stab_{\delta/2}(a_{n+1})$ at distance strictly less than $\varepsilon$ from $B_{i_0}$; so for $j, k \in J$ we get

$$d(g_j(z_j), g_k(z_k)) < 4\varepsilon \ .$$

Fix some $j \in J$; we have $d(g_k^{-1}g_j(z_j), z_k) < 4\varepsilon$ for all $k \in J$. Using the fact that $d(z_k, z_l) > 10\varepsilon$ whenever $l \neq k$, the triangle inequality yields, for any $l \neq k \in J$, that

$$d(g_k^{-1}g_j(z_j), g_l^{-1}g_j(z_j)) > 2\varepsilon \ .$$

Since each $g_k^{-1}g_j$ belongs to $Stab_\delta(a_{n+1})$, this contradicts the fact that $z_j \in acl_\delta(a_{n+1})$. Hence one can find some suitable $z_j$, and set $d_{n+1} = z_j$; the same line of reasoning works to obtain $e_{n+1}$. This concludes the proof of the lemma . □

Now we are ready to conclude the proof of case II: set $D = \bigcup Stab_{\delta/2}(a_i).d_i$, $E = \bigcup Stab_{\delta/2}(a_i).e_i$. From the lemma we get $d(D,E) \geq \varepsilon$, and so one can find a uniformly continuous map $f\colon M \to [0,1]$ such that $f(x) = 1$ whenever $d(x,D) < \varepsilon/10$ and $f(x) = 0$ whenever $d(x,E) < \varepsilon/10$.
Then for any morphism $\rho$ of $\mathcal{M}$ we may assume (up to multiplying $\rho$ on the right by some automorphism, which does not change the image of $\rho$) that there is some $i$ such that $d(a_i, \rho(a_i)) < \delta/2$. Since any morphism is a pointwise limit of automorphisms, this shows that $\rho(d_i) \in \overline{D}$ while $\rho(e_i) \in \overline{E}$, and so $f(\rho(d_i)) = 1$ while $f(\rho(e_i)) = 0$.
Note that in case II, as in case I, we obtain a function of *one* variable which oscillates on the image of any morphism of $\mathcal{M}$.

**Case III.** For any $\delta > 0$, $acl_\delta(a)$ is covered by finitely many closed balls of radius $10\varepsilon$.
We pick $(a_i)$ dense in $G.a$, and find a uniformly continuous $f\colon M^2 \to [0,1]$ such that, for all $n$, $f$ equals 1 on $C_1^n = B(a_n, \varepsilon/10) \times (X \setminus \bigcup_{m \leq n} B(a_m, \varepsilon/4))$ while $f$ equals 0 on $C_0^n = (X \setminus \bigcup_{m<n} B(a_m, \varepsilon/4)) \times B(a_n, \varepsilon/10)$.

To see that this is indeed possible, let $C_0$ (resp. $C_1$) denote the union of all the $C_0^n$ (resp. $C_1^n$). Then pick $c_0 \in C_0$, $c_1 \in C_1$; one has $c_0 \in C_0^n$ and $c_1 \in C_1^m$ for some $n, m$. If $m \geq n$, the second coordinates of $c_0, c_1$ ensure that $d(c_0, c_1) \geq \varepsilon/10$. If $m < n$, then the first coordinates of $c_0, c_1$ again ensure that $d(c_0, c_1) \geq \varepsilon/10$. Hence one has $d(c_0, c_1) \geq \varepsilon/10$ for any $c_0 \in C_0$, $c_1 \in C_1$ and so one can find $f$ as above.

**Lemma 5.** *For any $\delta > 0$ there exist $x_0, x_1 \in G.a$ such that $x_0 \notin acl_\delta(x_1)$ and $x_1 \notin acl_\delta(x_0)$.*

*Proof of Lemma 5.* Fix $\delta > 0$. There is some $N$ such that, for any $b \in G.a$, $acl_\delta(b)$ is covered by $N$ closed balls of radius $10\varepsilon$. By choice of $\varepsilon$, we can find $b_0, \dots, b_N \in G.a$ such that $d(b_i, b_j) > 100\varepsilon$ and then pick some $c \in G.a$ with $d(c, \bigcup acl_\delta(b_i)) > 10\varepsilon$. In particular, $c \notin acl_\delta(b_i)$ for all $i$. Since $acl_\delta(c)$ is covered by $N$ balls of radius $10\varepsilon$, there has to be some $i_o$ such that $b_{i_0} \notin acl_\delta(c)$.
Setting $x_0 = b_{i_0}$, $x_1 = c$, we are done. $\square$

Now pick $x_0$, $x_1$ as above for $\delta = \varepsilon/20$. We claim that $(f, x_0, x_1)$ satisfy the conclusion of the theorem. Pick a morphism $\rho$ of $\mathcal{M}$ ; we can find $a_{m_0}, a_{m_1}$ such that $d(\rho(x_0), a_{m_0}) < \delta$ and $d(\rho(x_1), a_{m_1}) < \delta$. Let $k = m_0 + m_1 + 1$. Since $x_1$ does not belong to $acl_\delta(x_0)$ and $x_0$ does not belong to $acl_\delta(x_1)$, we can find $(\pi_i)_{i=1,\dots,k} \in Stab_\delta(x_0)$ and $(\pi'_i)_{i=1,\dots,k} \in Stab_\delta(x_1)$ such that the balls $B(\pi_i(x_1), \varepsilon/2)$ are disjoint, and similarly for $B(\pi'_i(x_0), \varepsilon/2)$.
But then we obtain that $d(\rho \circ \pi_i(x_0), a_{m_0}) < \varepsilon/10$ for all $i = 1, \dots, k$ while $d(\rho \circ \pi_i(x_1), \rho \circ \pi_j(x_1)) \geq \varepsilon/2$ for any $i \neq j$. Hence any $a_j$, $j \leq k$, can only belong to one ball $B(\rho \circ \pi_i(x_1), \varepsilon/4)$, so there is some $i_0 < k$ such that no $a_j$, $j \leq k$, belongs to $B(\rho \circ \pi_{i_0}(x_1), \varepsilon/4)$. Looking at the definition of $f$, we obtain $f(\rho \circ \pi_{i_0}(x_0), \rho \circ \pi_{i_0}(x_1)) = 1$. Similarly one finds some $j_0$ such that $f(\rho \circ \pi'_{j_0}(x_0), \rho \circ \pi'_{j_0}(x_1)) = 0$, which concludes the proof of Theorem 2.

## 3. A few comments.

Let us begin this final section with a short comparison of the proof above with Hjorth's original proof. Both proofs are divided in subcases; case I is the same in both, but case II and case III above are different from the cases considered by Hjorth. The reason is fairly simple: we allow $\delta$ and $\varepsilon$ to vary independently, while Hjorth considers $\delta$ as a function of $\varepsilon$. Because of this he needs more complicated combinatorics to make the proof work. The ideas of the proof of case III (the definition of $f$ and the clever combinatorial argument) are taken from Hjorth's paper, while case II is much simpler since the only bit of combinatorics that it requires is the pigeon-hole principle. Here the point of view of continuous logic makes clear what the right notion of algrebraic closure is in our context, and this makes the proof flow more smoothly.

Now, we should explain precisely why Theorem 2 and Hjorth's oscillation theorem are indeed equivalent. It should be clear that Theorem 2 is a direct consequence of Hjorth's theorem, since when $\mathcal{M}$ is approximately ultrahomogeneous the left-completion of $Aut(\mathcal{M})$ coincides with the set of morphisms from $\mathcal{M}$ into itself, and morphisms preserve quantifier-free type. To see why the converse is true, we will prove the following result, which is of some independent interest.

**Theorem 6.** *Let $(X,d)$ be a Polish metric space with diameter less than 1 and $G \leq Isom(X,d)$ be a Polish group. Then there exists a family $(R_i)_{i\in I}$ of predicates such that $\mathcal{M} = (X,d,(R_i)_{i\in I})$ is an approximately ultrahomogeneous Polish metric structure with automorphism group $G$.*
*In particular, any Polish group is the automorphism group of some approximately ultrahomogeneous Polish metric structure.*

Theorem 6 is the continuous logic version of the well-known theorem that states that any closed subgroup of $\mathcal{S}_\infty$ is isomorphic (as a topological group) to the automorphism group of some countable first-order structure. Some variants of theorem 6 were already known (see for example theorem 2.4.5 in [2]). It is not clear, at least to the author, who this theorem should be attributed to. It may be folklore. Anyway, it provides the tool that enables one to deduce Hjorth's theorem from Theorem 2: note first that replacing the distance $d$ by the distance $d/(1+d)$ does not change either the isometry group of $X$ or the uniformly continuous maps from $X^2$ to $[0,1]$, so one may assume in Hjorth's theorem that $d$ is bounded by 1. It is also clear that one may assume that $G$ is closed in $Isom(X,d)$. Then it is not too hard to see that the combination of Theorem 6 and Theorem 2 yields Hjorth's theorem.

*Proof of Theorem 6.* Consider for any $n$ the closed equivalence relation $\sim_n$ coming from the diagonal action of $G$ on $X^n$:

$$x=(x_1,\dots,x_n)\sim_n y=(y_1,\dots,y_n)\Leftrightarrow x\in\overline{G.y}\ .$$

This is an equivalence relation because $G$ acts on $X^n$ by isometries. For any $\sim_n$-class $C$, add a predicate $R_C\colon M^n\to[0,1]$ defined by

$$R_C(x_1,\dots,x_n)=d\big((x_1,\dots,x_n),C\big)\ .$$

We claim that the metric structure $\mathcal{M}$ obtained by adding all those predicates to $X$ is approximately ultrahomogeneous and has $G$ as its automorphism group. It is immediate that any element of $G$ preserves all our predicates, and hence is an automorphism of $\mathcal{M}$; given the predicates we chose, we then see that $\mathcal{M}$ is approximately ultrahomogeneous.

To show that $G = Aut(\mathcal{M})$, let $\pi$ be an automorphism of $\mathcal{M}$. Then, for any $x_1, \ldots, x_n \in X$, $(x_1, \ldots, x_n)$ and $(\pi(x_1), \ldots, \pi(x_n))$ have the same quantifier-free type, and so for all $\varepsilon > 0$ there is $g \in G$ such that $d(g(x_i), \pi(x_i)) < \varepsilon$. This shows that $\pi$ is a pointwise limit of elements of $G$ and so belongs to $G$, for since $G$ is Polish it must be closed in $Isom(X, d)$.

Note that one could easily make the family of predicates above countable, by fixing some countable dense set $A = \{a_i\}_{i \in \mathbb{N}}$ and then only considering $\sim_n$ classes of tuples belonging to $A^n$. $\square$

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