Forcing nonuniversal Banach spaces

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Introduction

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Classical examples

- 2^ω is universal for the class of all compact metrizable spaces.
- $C[0,1]$ is isometrically universal for the class of all separable Banach spaces.

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Remarks:

- Given any compact space K, $C(K)$ is a Banach space of density equal to the weight of K.
- Given any Banach space X, B_{X^*} equipped with the weak* topology is a compact space of weight equal to the density of X.

compact spaces x Banach spaces

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- Dow, Hart:
  It is consistent that there is no universal for compact spaces of weight $\mathfrak{c}$.
- Shelah, Usvyatsov:
  It is consistent that there is no isometrically universal for Banach spaces of density $\mathfrak{c}$.
- B., Koszmider:
  It is consistent that there is no universal for Banach spaces of density $\mathfrak{c}$ (and of density ω_1).

UE compact spaces x UG Banach spaces

Example 2: uniform Eberlein compact spaces of weight κ (UE) and Banach spaces of density κ which have a uniformly Gâteaux differentiable renorming (UG).

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Theorem (Bell)

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- *It is consistent that there is no universal UE compact space of weight* ω_1.

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- *CH implies that there is a universal for UE compact spaces of weight $\omega_1 = \mathfrak{c}$ and that there is an isometrically universal for UG Banach spaces of density $\omega_1 = \mathfrak{c}$.*
- *It is consistent that there is no universal UE compact space of weight ω_1.*

Theorem (B., Koszmider)

It is consistent that there is no universal for UG Banach spaces of density ω_1 nor of density $\mathfrak{c}$.

Strategy

We will force the existence of ω_2-many UG Banach spaces of density ω_1 such that no Banach space of density ω_1 in the extension can contain isomorphic copies of all of these spaces.

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Define a forcing notion $\mathbb{P}$ which adds a c-algebra to the ground model V and prove that given a Banach space X in V, there is no isomorphic embedding $T : C(K) \to X$ in $V^{\mathbb{P}}$, where K is the Stone space of the generic c-algebra.

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Consider Σ the product of ω_2 copies of $\mathbb{P}$, with finite supports. Given any Banach space X of density ω_1 in V^{Σ}, it is already "determined" at an intermediate model V^{Σ_λ} for some $\lambda < \omega_2$.

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Consider Σ the product of ω_2 copies of $\mathbb{P}$, with finite supports. Given any Banach space X of density ω_1 in V^{Σ}, it is already "determined" at an intermediate model V^{Σ_λ} for some $\lambda < \omega_2$.Then, if K is the Stone space of the c-algebra corresponding to the λ copy of $\mathbb{P}$, $C(K)$ cannot be isomorphically embedded into X in V^{Σ}.

A Boolean algebra $\mathcal{B}$ is a **c-algebra** *if* $\mathcal{B} = \langle A_{\xi,n} : \xi < \omega_1, i \in \omega \rangle$ *where* $\{A_{\xi,i} : \xi < \omega_1\}$ *are pairwise disjoint antichains such that*

$$A_{\xi_1,i_1} \vee \cdots \vee A_{\xi_m,i_m} \neq 1 \text{ for distinct pairs } (\xi_1, i_1), \ldots, (\xi_m, i_m) \in \omega_1 \times \omega.$$

Theorem (Bell)

If $\mathcal{B}$ is a c-algebra, then its Stone space K is a uniform Eberlein compact space. Therefore, $C(K)$ is a UG Banach space.

The forcing notion

$p = (n_p, D_p, F_p) \in \mathbb{P}$ *if*

- $n_p \in \omega$,
- $D_p \in [\omega_1]^{<\omega}$,
- F_p *is a finite subset of* $Fn_{<\omega}(n_p, D_p)$,
- $[n_p \times D_p]^1 \subseteq F_p$.

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$p \leq q$ if

- $n_p \geq n_q$,
- $D_p \supseteq D_q$,
- $F_p \supseteq F_q$,
- $\forall f \in F_p \exists g \in F_q$ *such that* $f \cap (n_q \times D_q) \subseteq g$.

Given a model V and a $\mathbb{P}$-generic filter G over V, for each $(i, \xi) \in \omega \times \omega_1$, we define in $V[G]$ the following set:

$$A_{\xi,i} = \{f \in Fn_{<\omega}(\omega, \omega_1) : \exists p \in G \text{ such that } f \in F_p \text{ and } f(i) = \xi\}.$$

Let $\mathcal{B}$ be the subalgebra of the Boolean algebra $\wp(Fn_{<\omega}(\omega, \omega_1))$ generated by the sets $\{A_{\xi,i} : (i, \xi) \in \omega \times \omega_1\}$.

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In $V[G]$, we have that:

- *given distinct pairs* $(i_1, \xi_1), ..., (i_k, \xi_k), (i, \xi)$, $A_{\xi,i} \setminus (A_{\xi_1,i_1} \cup ... \cup A_{\xi_k,i_k}) \neq \emptyset$;

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Therefore, $\mathcal{B}$ is a c-algebra of cardinality ω_1 and its Stone space K is a uniform Eberlein compact space of weight ω_1.

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Theorem

Given any Banach space X in the ground model V, there is no isomorphic embedding $T : C(K) \to X$ in $V[G]$.

Given $p_1 = (n_1, D_1, F_1), p_2 = (n_2, D_2, F_2) \in \mathbb{P}$, *we say that they are isomorphic if* $n_1 = n_2$ *and there is an order-preserving bijection* $e : D_1 \to D_2$ *such that* $e|_{D_1 \cap D_2} = id$ *and for all* $f \in Fn_{<\omega}(\omega, \omega_1)$, $f \in F_1$ *if and only if* $e[f] \in F_2$, *where* $e[f](i) = e(f(i))$.

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Lemma

Let $p_k = (n, D_k, F_k)$ in $\mathbb{P}$, for $1 \leq k \leq m$, be pairwise isomorphic conditions such that $(D_k)_{1 \leq k \leq m}$ is a Δ-system with root D.

1. *There is $p \leq p_1, \ldots, p_m$ such that*

$$\forall \xi \in D_k \setminus D \quad \forall \xi' \in D_{k'} \setminus D \quad \forall i \neq i' \qquad p \Vdash \dot{A}_{\check{\xi}, i} \cap \dot{A}_{\check{\xi}', i'} = \emptyset.$$

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2. *Given $\xi_k \in D_k \setminus D$ and distinct $i_k < n$, there is $p \leq p_1, \ldots, p_m$ such that*

$$p \Vdash \dot{A}_{\check{\xi}_1, i_1} \cap \cdots \cap \dot{A}_{\check{\xi}_m, i_m} \neq \emptyset.$$

Definition

Σ is the product of ω_2 copies of $\mathbb{P}$, with finite supports.

Theorem

$V^\Sigma \models$ "$\mathfrak{c} = \omega_2$ and there is no Banach space X of density ω_1 such that for every uniform Eberlein compact space K of weight at most ω_1, $C(K)$ can be isomorphically embedded into X".

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