RESEARCH STATEMENT

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1. Theme and Motivation

Totally disconnected, locally compact topological groups appear throughout mathematics. Most immediately, such groups appear as automorphism groups of countable, first order structures with sufficient local finiteness; e.g. countable, locally finite graphs. Since they admit a Haar measure, these groups also arise in abstract harmonic analysis. They appear again in Lie theory as the quotient by the connected component of the identity. These groups are historically intractable. However, in the last twenty years there have been a number of breakthroughs: two of the most powerful are Willis' theory of tidy subgroups [3] and Krön and Möller's work in rough geometry [2]. These advances give novel methods to approach the theory of totally disconnected, locally compact groups and, moreover, a glimmer of hope for a structure theory of locally compact, totally disconnected groups.

My work is motivated by results of Kechris and Rosendal. In [1], Kechris and Rosendal give a model-theoretic characterization of both totally disconnected topological groups which have a dense conjugacy class and those with a generic conjugacy class. However, their work encompasses all closed subgroups of S_∞ without regard for local compactness. My research is an exploration of the locally compact case.

2. Results and Current Directions

Locally compact, totally disconnected groups with a dense conjugacy class are periodic - i.e. every element lies in a compact subgroup. This fact follows from a result of Willis in [3]: he proves the collection of periodic elements is closed. Understanding the notion of periodicity is then necessary to understanding groups with dense conjugacy classes. In regard to this, Professor Rosendal posed the following question to me:

QUESTION 2.1. *Say that G is a locally compact, totally disconnected topological group. For each $n > 0$ we may define a natural extension of the notion of periodicity:*

$$P_n(G) := \{\overline{g} \in G^{\times n} : cl(\langle \overline{g} \rangle) \text{ is compact}\}$$

Are the $P_n(G)$ closed in the product topology on $G^{\times n}$?

I have obtained positive answers in a number of special cases:

PROPOSITION 2.2. *The following totally disconnected, locally compact groups G are such that $P_n(G)$ is closed for each n.*

1. *$Aut(\mathcal{T})$ for $\mathcal{T}$ a locally finite tree.*
2. *G compactly generated with only thin rough ends.*
3. *G compactly generated with polynomial growth.*
4. *G compactly generated, amenable, and with more than one rough end.*
5. *G solvable.*

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I am currently working to expand the above list to all compactly generated, totally disconnected, and locally compact groups. If the result holds in the compactly generated case, it is an easy argument to extend to all totally disconnected, locally compact groups. I am further trying to understand how the sets $P_n(G)$ interact with the global structure of the group. One result I have in this direction is the following:

PROPOSITION 2.3. *Let G be locally compact and totally disconnected. Then $P_n(G)$ is dense in $G^{\times n}$ for all $n > 0$ if and only if G is an increasing union of compact, open subgroups.*

I am now working to produce weaker structure results and, moreover, trying to understand how such global structure results allow or disallow dense conjugacy classes.

References

[1] A. Kechris and C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures.* , **Proc. Lond. Math. Soc. (3)**, 94 (2007), no. 2, 302-350.
[2] B. Krön and R. Möller *Analogues of Cayley graphs for topological groups*, **Math. Z.**, 258 (2008), no. 3, 637-67.
[3] G. Willis, *The structure of totally disconnected, locally compact groups*, **Math. Ann.**, 300 (1994), no. 2, 341-363.

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