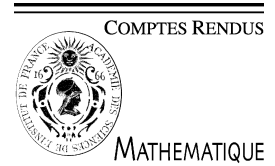


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Partial Differential Equations

Stability estimates on general scalar balance laws

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Abstract

Consider the general scalar balance law in N space dimensions $\partial_t u + \text{Div } f(t, x, u) = F(t, x, u)$. Under suitable assumptions on f and F , we provide bounds on the total variation of the solution. Based on this first result, we establish estimates on the dependence of the solutions from f and F . In the more particular cases considered in the literature, the present estimate reduces to the known ones. **To cite this article:** *R.M. Colombo et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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Résumé

Estimation de la variation totale et stabilité pour des lois de conservations scalaires généralisées. Nous considérons ici une loi de conservation généralisée en dimension N : $\partial_t u + \text{Div } f(t, x, u) = F(t, x, u)$. Sous des hypothèses adaptées pour f et F , nous obtenons une borne de la variation totale de la solution. À partir de ce résultat, il est alors possible de donner une estimation de la dépendance des solutions au flot f et au terme source F . Dans les cas particuliers déjà étudiés, notre résultat se réduit à ceux déjà connus. **Pour citer cet article :** *R.M. Colombo et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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1. Introduction

Let $f \in C^2(\bar{\mathbb{R}}_+ \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$, $F \in C^1(\bar{\mathbb{R}}_+ \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R})$ and $u_o \in L^\infty(\mathbb{R}^N; \mathbb{R})$. Then, the classical result [8, Theorem 5] ensures the well posedness of the Cauchy problem

$$\begin{cases} \partial_t u + \text{Div } f(t, x, u) = F(t, x, u), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0, x) = u_o(x), & x \in \mathbb{R}^N. \end{cases} \quad (1)$$

In the present Note, under suitable further assumptions on the flow f and on the source F , we state that the solution $u(t)$ to (1) is in $\mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ and provide bounds on its total variation.

This result allows us to obtain estimates on the dependence of the solution on f , F and u_o . Similar results were obtained in [2, Theorem 2.1] in the case of systems of conservation laws in 1 space dimension, with $f = f(u)$ and

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$F = 0$. In the case of a scalar equation with $f = f(u)$, $F = 0$ and with N space dimensions, the same problem was addressed by Bouchut and Perthame [3] who proved, among other results, the following estimate (that was already known, see [6,9]):

$$\|u(t) - v(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \leq \|u_o - v_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} + \text{CTV}(u_o) \mathbf{Lip}(f - g)t. \quad (2)$$

The estimate proved in Theorem 2.4 below reduces to (2) as soon as $f = f(u)$ and $F = 0$. In this context, we recall that [2, Theorem 2.6] provides a sharp estimate in the scalar 1D case with $f = f(u)$ and $F = 0$.

The case of x -dependent flows was considered in [4] and [7], where it is assumed that $f(x, u) = k(x)v(u)$. However, in both papers, the resulting estimate holds under the further assumption that the solutions have uniformly bounded total variation. Here, the bound on $\text{TV}(u(t))$ is not assumed, but proved.

All proofs, together with an application to a radiating gas model, are deferred to [5].

2. Main results

Introduce the notation: $\bar{\mathbb{R}}_+ = [0, +\infty[$, $\mathbb{R}_+ =]0, +\infty[$, N is a positive integer and $\Omega = \bar{\mathbb{R}}_+ \times \mathbb{R}^N \times \mathbb{R}$. For a vector valued function $f = f(t, x, u)$ with $u = u(t, x)$, $\text{Div } f$ stands for the total divergence while $\text{div } f$, respectively ∇f , denotes the partial divergence, respectively gradient, with respect to the space variables. ∂_u and ∂_t are the usual partial derivatives. Thus, $\text{Div } f = \text{div } f + \partial_u f \cdot \nabla u$.

Recall the definition of weak entropy solution to (1), see [8, Definition 1]:

Definition 2.1. A bounded measurable function $u : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a weak entropy solution to (1) if:

1. for any constant $k \in \mathbb{R}$ and any test function $\varphi \in \mathbf{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^N; \bar{\mathbb{R}}_+)$

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^N} [(u - k) \partial_t \varphi + [f(t, x, u) - f(t, x, k)] \nabla \varphi + [F(t, x, u) - \text{div } f(t, x, k)] \varphi] \text{sign}(u - k) \, dx \, dt \geq 0;$$

2. there exists a set \mathcal{E} of zero measure in $\bar{\mathbb{R}}_+$ such that for $t \in \bar{\mathbb{R}}_+ \setminus \mathcal{E}$ the function $u(t, x)$ is defined almost everywhere in \mathbb{R}^N and $\lim_{t \in \bar{\mathbb{R}}_+ \setminus \mathcal{E}, t \rightarrow 0} \int_{\|x\| \leq r} |u(t, x) - u_o(x)| \, dx = 0$ for any $r > 0$.

We refer to [1] as general references for the theory of **BV** functions. In particular, recall the following basic definition, see [1, Definition 3.4 and Theorem 3.6]:

$$\text{TV}(u) = \sup \left\{ \int_{\mathbb{R}^N} u \, \text{div } \psi \, dx : \psi \in \mathbf{C}_c^1(\mathbb{R}^N; \mathbb{R}^N) \text{ and } \|\psi\|_{\mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R}^N)} \leq 1 \right\},$$

$$\mathbf{BV}(\mathbb{R}^N; \mathbb{R}) = \{u \in \mathbf{L}_{\text{loc}}^1(\mathbb{R}^N; \mathbb{R}) : \text{TV}(u) < +\infty\}.$$

The following sets of assumptions will be of use below:

- $$\begin{aligned} \text{(H1): } & \begin{cases} f \in \mathbf{C}^2(\Omega; \mathbb{R}^N), & F \in \mathbf{C}^1(\Omega; \mathbb{R}), \\ \partial_u f \in \mathbf{L}^\infty(\Omega; \mathbb{R}^N), & \partial_u(F - \text{div } f) \in \mathbf{L}^\infty(\Omega; \mathbb{R}), \quad F - \text{div } f \in \mathbf{L}^\infty(\Omega; \mathbb{R}), \end{cases} \\ \text{(H2): } & \begin{cases} f \in \mathbf{C}^2(\Omega; \mathbb{R}^N), & F \in \mathbf{C}^1(\Omega; \mathbb{R}), \\ \partial_t \partial_u f \in \mathbf{L}^\infty(\Omega; \mathbb{R}^N), & \partial_t \text{div } f \in \mathbf{L}^\infty(\Omega; \mathbb{R}), \quad \partial_t F \in \mathbf{L}^\infty(\Omega; \mathbb{R}), \\ \nabla \partial_u f \in \mathbf{L}^\infty(\Omega; \mathbb{R}^{N \times N}), & \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \|\nabla(F - \text{div } f)(t, x, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^N)} \, dx \, dt < +\infty, \end{cases} \\ \text{(H3): } & \begin{cases} f \in \mathbf{C}^1(\Omega; \mathbb{R}^N), & F \in \mathbf{C}^0(\Omega; \mathbb{R}), \quad \partial_u F \in \mathbf{L}^\infty(\Omega; \mathbb{R}), \\ \partial_u f \in \mathbf{L}^\infty(\Omega; \mathbb{R}^N), & \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \|(F - \text{div } f)(t, x, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})} \, dx \, dt < +\infty. \end{cases} \end{aligned}$$

Assumption (H1) is sufficient for the classical results by Kruřkov [8, Theorem 1 and Theorem 5] to hold.

Theorem 2.2 (Kruřkov). *Let (H1) hold. For any $u_o \in \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$, there exists a unique right continuous weak entropy solution u to (1) in $\mathbf{L}^\infty(\bar{\mathbb{R}}_+; \mathbf{L}_{\text{loc}}^1(\mathbb{R}^N; \mathbb{R}))$. Moreover, if a sequence $u_o^n \in \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$ converges to u_o in $\mathbf{L}_{\text{loc}}^1$, then for all $t > 0$ the corresponding solutions $u^n(t)$ converge to $u(t)$ in $\mathbf{L}_{\text{loc}}^1$.*

The next result contains the estimate on the total variation, a key point in the stability proof below:

Theorem 2.3. Assume that (H1) and (H2) hold. Let $u_o \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$. Then, the weak entropy solution u of (1) satisfies $u(t) \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ for all $t > 0$. Moreover, let

$$\kappa_o = NW_N((2N+1)\|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}) \quad \text{and} \quad W_N = \int_0^{\pi/2} (\cos \theta)^N d\theta. \quad (3)$$

Then,

$$\mathrm{TV}(u(T)) \leq \mathrm{TV}(u_o)e^{\kappa_o T} + NW_N \int_0^T e^{\kappa_o(T-t)} \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt. \quad (4)$$

This estimate is optimal in the following senses:

- (i) If f is independent from x and $F = 0$, then $\kappa_o = 0$ and the integrand in the right hand side above vanishes. Hence, (4) reduces to the well known optimal bound $\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_o)$.
- (ii) In the 1D case, if f and F are both independent from t and u , then $\kappa_o = 0$ and (1) reduces to the ordinary differential equation $\partial_t u = F - \operatorname{div} f$. In this case, (4) becomes

$$\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_o) + t\mathrm{TV}(F - \operatorname{div} f).$$

- (iii) If $f = 0$ and $F = F(t)$ then, trivially, $\mathrm{TV}(u(t)) = \mathrm{TV}(u_o)$ and (4) is optimal.

Let now $(f, F), (g, G)$ verify (H1) and $u_o, v_o \in \mathbf{L}_{\text{loc}}^1(\mathbb{R}^N; \mathbb{R})$. We want to prove estimates for $u - v$ in terms of $f - g, F - G$ and $u_o - v_o$, u being the entropy solution of (1) and v being the entropy solution of

$$\begin{cases} \partial_t v + \operatorname{Div} g(t, x, v) = G(t, x, v), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ v(0, x) = v_o(x), & x \in \mathbb{R}^N. \end{cases}$$

Similar estimates were derived in [3] when f and g depend only on u . Here, we add the (t, x) -dependence.

Theorem 2.4. Let $(f, F), (g, G)$ verify (H1), (f, F) verify (H2) and $(f - g, F - G)$ verify (H3). Let $u_o, v_o \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$. We denote κ_o and W_N as in (3), $\kappa = 2N\|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^{N \times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \|\partial_u(F - G)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})}$ and $M = \|\partial_u g\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N)}$. Then, for any $T, R > 0, x_o \in \mathbb{R}^N$,

$$\begin{aligned} \int_{\|x-x_o\| \leq R} |u(T, x) - v(T, x)| dx &\leq e^{\kappa T} \int_{\|x-x_o\| \leq R+MT} |u_o(x) - v_o(x)| dx + \frac{e^{\kappa_o T} - e^{\kappa T}}{\kappa_o - \kappa} \mathrm{TV}(u_o) \|\partial_u(f - g)\|_{\mathbf{L}^\infty} \\ &+ NW_N \int_0^T \frac{e^{\kappa_o(T-t)} - e^{\kappa(T-t)}}{\kappa_o - \kappa} \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt \|\partial_u(f - g)\|_{\mathbf{L}^\infty} \\ &+ \int_0^T e^{\kappa(T-t)} \int_{\|x-x_o\| \leq R+M(T-t)} \|((F - G) - \operatorname{div}(f - g))(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt. \end{aligned} \quad (5)$$

Formally, the above inequality is undefined for $\kappa = \kappa_o$. However, as shown in [5], when $(\kappa - \kappa_o) \rightarrow 0$ the right hand side above has a finite limit which bounds the distance between solutions. Note that (4), as well as (5), does not depend on all second derivatives, hence the regularity requirements on f can be relaxed, see [5] for details. Besides, (5) is optimal in the cases considered before. Assume $u_o, v_o \in \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})$.

- (i) In the standard case of a conservation law, i.e. when $F = G = 0$ and f, g are independent of x , we have $\kappa_o = \kappa = 0$ and (5) becomes, see [2, Theorem 2.1],

$$\|u(T) - v(T)\|_{\mathbf{L}^1} \leq \|u_o - v_o\|_{\mathbf{L}^1} + TTV(u_o) \|\partial_u(f - g)\|_{\mathbf{L}^\infty}.$$

- (ii) If $\partial_u f = \partial_u g = 0$ and $\partial_u F = \partial_u G = 0$, then $\kappa_o = \kappa = 0$ and (5) now reads

$$\|u(T) - v(T)\|_{\mathbf{L}^1} \leq \|u_o - v_o\|_{\mathbf{L}^1} + \int_0^T \|((F - G) - \operatorname{div}(f - g))(t)\|_{\mathbf{L}^1} dt.$$

- (iii) If (f, F) and (g, G) are dependent only on x , then (5) reduces to

$$\|u(T) - v(T)\|_{\mathbf{L}^1} \leq \|u_o - v_o\|_{\mathbf{L}^1} + T \|(F - G) - \operatorname{div}(f - g)\|_{\mathbf{L}^1}.$$

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