Estimates of $\mathrm{li}(\theta(x)) - \pi(x)$ and the Riemann Hypothesis

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To Krishna Alladi for his sixtieth birthday

Abstract Let us denote by $\pi(x)$ the number of primes $\leqslant x$, by $\mathrm{li}(x)$ the logarithmic integral of x, by $\theta(x) = \sum_{p \leqslant x} \log p$ the Chebyshev function and let us set $A(x) = \mathrm{li}(\theta(x)) - \pi(x)$. Revisiting a result of Ramanujan, we prove that the assertion "$A(x) > 0$ for $x \geqslant 11$" is equivalent to the Riemann Hypothesis.

Keywords Chebyshev function · Riemann Hypothesis · Explicit formula

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1 Introduction

Let us denote by $\pi(x)$ the number of primes $\leqslant x$ and by $\mathrm{li}(x)$ the logarithmic integral of x (see, below, §2.2). It has been observed that, for small x, $\pi(x) < \mathrm{li}(x)$ holds, but Littlewood (cf. [7] or [5, chap. 5]) has proved that, for x tending to infinity, the difference $\pi(x) - \mathrm{li}(x)$ oscillates infinitely many often between positive and negative values.

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Let us set $\theta(x) = \sum_{p \leqslant x} \log p$, the Chebyshev function, and

$$A(x) = \mathrm{li}(\theta(x)) - \pi(x). \tag{1.1}$$

What is the behavior of $A(x)$? In [11, (220), (222), (227), and (228)], under the Riemann Hypothesis (RH), Ramanujan proved that

$$A(x) = \frac{2\sqrt{x} + \sum_\rho x^\rho/\rho^2}{\log^2(x)} + \mathcal{O}\left(\frac{\sqrt{x}}{\log^3(x)}\right) \tag{1.2}$$

where ρ runs over the nontrivial zeros of the Riemann ζ function. Moreover, in [11, (226)], Ramanujan writes under the Riemann Hypothesis

$$\begin{aligned}\left|\sum_\rho \frac{x^\rho}{\rho^2}\right| \leqslant \sum_\rho \left|\frac{x^\rho}{\rho^2}\right| = \sqrt{x}\sum_\rho \frac{1}{\rho(1-\rho)} = \sqrt{x}\sum_\rho\left(\frac{1}{\rho} + \frac{1}{1-\rho}\right)\\ = 2\sqrt{x}\sum_\rho \frac{1}{\rho} = \sqrt{x}(2+\gamma_0 - \log(4\pi)) = 0.046\ldots\sqrt{x}\end{aligned} \tag{1.3}$$

where γ_0 is the Euler constant and concludes

$$\text{under RH, } \exists\, x_0 \text{ such that, for } x \geqslant x_0,\ A(x) \text{ is positive.} \tag{1.4}$$

The aim of this paper is to make these results effective and, in particular, to show that Ramanujan's result (1.4) is true for $x_0 = 11$.

Let us set $\lambda = \sum_\rho \frac{1}{|\rho|^2}$. Under the Riemann Hypothesis, we have (see below (2.26))

$$\lambda = \sum_\rho \frac{1}{|\rho|^2} = \sum_\rho \frac{1}{\rho(1-\rho)} = 0.0461914179322420\ldots \tag{1.5}$$

We shall prove

Theorem 1.1. *Under the Riemann Hypothesis, we have*

$$\limsup_{x\to\infty} \frac{A(x)\log^2(x)}{\sqrt{x}} \leqslant 2+\lambda = 2.046\ldots, \tag{1.6}$$

$$\liminf_{x\to\infty} \frac{A(x)\log^2(x)}{\sqrt{x}} \geqslant 2-\lambda = 1.953\ldots, \tag{1.7}$$

$$A(x) \textit{ is positive for } \quad x \geqslant 11, \tag{1.8}$$

$$A(x) \geqslant (2-\lambda)\frac{\sqrt{x}}{\log^2(x)} \quad \textit{for } x \geqslant 37, \tag{1.9}$$

and

$$A(x) \leqslant M\frac{\sqrt{x}}{\log^2(x)} \quad \textit{for } x \geqslant 2, \tag{1.10}$$

where $M = A(3643)(\log^2 3643)/\sqrt{3643} = 5.0643569138\ldots$

Corollary 1.2. *Each of the five assertions* (1.6)–(1.10) *is equivalent to the Riemann Hypothesis.*

Proof. In 1984, Robin (cf. [10, Lemma 2 and (8)] has shown that, if the Riemann Hypothesis does not hold, there exists $b > 1/2$ such that

$$A(x) = \Omega_{\pm}(x^b), \quad \text{i.e. } \limsup_{x\to\infty} \frac{A(x)}{x^b} > 0 \text{ and } \liminf_{x\to\infty} \frac{A(x)}{x^b} < 0$$

and the five assertions of the theorem are no longer satisfied. □

1.1 *Notation*

$\pi(x) = \sum_{p\leqslant x} 1$ is the prime counting function.

$\Pi(x) = \sum_{p^k\leqslant x} \frac{1}{k} = \sum_{k=1}^{\kappa} \frac{\pi(x^{1/k})}{k}$ with $\kappa = \left\lfloor \frac{\log x}{\log 2} \right\rfloor$.

$\theta(x) = \sum_{p\leqslant x} \log p$ and $\psi(x) = \sum_{p^m\leqslant x} \log p = \sum_{k=1}^{\kappa} \theta(x^{1/k})$ are the Chebyshev functions.

$\Lambda(x) = \begin{cases} \log p & \text{if } x = p^k \\ 0 & \text{if not} \end{cases}$ is the von Mangoldt function.

$\tilde{\psi}(x) = \psi(x) - \frac{1}{2}\Lambda(x)$ and $\tilde{\Pi}(x) = \Pi(x) - \frac{\Lambda(x)}{2\log x}$.

$\mathrm{li}(x)$ denotes the logarithmic integral of x (cf. below §2.2).

$L_1(t) = \mathrm{li}(t) - \frac{t}{\log t}$, $L_2(t) = \mathrm{li}(t) - \frac{t}{\log t} - \frac{t}{\log^2 t}$,

$F_1(t) = \frac{L_1(t)}{t/\log^2 t}$, $F_2(t) = \frac{L_2(t)}{t/\log^3 t}$ $\quad (t > 1)$.

$\widetilde{F}_1(t)$ and $\widetilde{F}_2(t)$ are defined below in (3.16).

$\gamma_0 = 0.57721566\ldots$ is the Euler constant. λ is defined in (1.5), cf. also (2.26).

$\sum_\rho f(\rho) = \lim_{T\to\infty} \sum_{|\Im(\rho)|\leqslant T} f(\rho)$ where $f : \mathbb{C} \to \mathbb{C}$ is a complex function and ρ runs over the nontrivial zeros of the Riemann ζ function.

1.2 *Plan of the article*

In §2, we shall recall some definitions and prove some results that we shall use in the sequel, first, in §2.2, about the logarithmic integral, and, further, in §2.3, about the Riemann ζ function and explicit formulas of the theory of numbers.

In §3, the proof of Theorem 1.1 is given. First, we write $A(x) = A_1(x) + A_2(x)$ with

$$A_1(x) = \mathrm{li}(\psi(x)) - \Pi(x) \quad \text{and} \quad A_2(x) = \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x).$$

In §3.1, under the Riemann Hypothesis, an estimate of $A_1(x)$ is given, by applying the explicit formulas. In §3.2, it is shown that $A_2(x)$ depends on the quantity $B(y) = \pi(y) - \theta(y)/\log y$ which is carefully studied.

In §3.3 (resp. §3.4), an effective lower (resp. upper) estimate for $A(x)$ is given when $x \geqslant 10^8$.

In §3.5, for $x < 10^8$, estimates of $A(x)$ are given by numerical computation.

Finally, Theorem 1.1 is proved in two steps, depending on the cases $x \leqslant 10^8$ or $x > 10^8$.

The computations, both algebraic and numerical, have been carried out with Maple. On the website [13], one can find the code and a Maple sheet with the results.

We often implicitly use the following result : for u and v positive, the function

$$t \mapsto \frac{\log^u t}{t^v} \quad \text{is increasing for } 1 \leqslant t \leqslant e^{u/v} \text{and decreasing for } \quad t > e^{u/v}. \tag{1.11}$$

Moreover

$$\max_{t\geqslant 1} \frac{\log^u t}{t^v} = \left(\frac{u}{e\,v}\right)^u. \tag{1.12}$$

2 Preliminary results

2.1 *Effective estimates*

Without any hypothesis, Platt and Trudgian [9] have shown by computation that

$$\theta(x) < x \text{ for } 0 < x \leq 1.39 \times 10^{17} \tag{2.1}$$

so improving on results of Schoenfeld [12] and Dusart [3]. Under the Riemann Hypothesis, for $x \geqslant 599$, we shall use the upper bounds (cf. [12, (6.3)])

$$|\psi(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\log^2 x \quad \text{and} \quad |\theta(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\log^2 x. \tag{2.2}$$

2.2 *The logarithmic integral*

For x real > 1, we define $\mathrm{li}(x)$ as (cf. [1, p. 228])

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t} = \lim_{\varepsilon\to 0^+}\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \frac{dt}{\log t}\right) = \int_2^x \frac{dt}{\log t} + \mathrm{li}(2).$$

We have the following values:

x	1	1.45136...	2	3.8464...	8.3	599
$\mathrm{li}(x)$	$-\infty$	0	1.145163...	2.8552...	5.39671...	117.49...

(2.3)

From the definition of $\mathrm{li}(x)$, it follows that

$$\frac{d}{dx}\mathrm{li}(x) = \frac{1}{\log x} \quad \text{and} \quad \frac{d^2}{dx^2}\mathrm{li}(x) = -\frac{1}{x\log^2 x}. \tag{2.4}$$

We also have

$$\mathrm{li}(x) = \gamma_0 + \log(\log(x)) + \sum_{n=1}^{\infty}\frac{(\log x)^n}{n \times n!}$$

(where $\gamma_0 = 0.577\ldots$ is the Euler constant) which implies

$$\mathrm{li}(x) = \log(\log(x)) + \gamma_0 + o(1), \qquad x \to 1^+. \tag{2.5}$$

Let N be a positive integer. For $t > 1$, we have (cf. [13])

$$\int \frac{dt}{\log^N t} = \frac{1}{(N-1)!}\left(\operatorname{li}(t) - \sum_{k=1}^{N-1}(k-1)!\frac{t}{\log^k t}\right) \tag{2.6}$$

and, for $x \to \infty$,

$$\operatorname{li}(x) = \sum_{k=1}^{N} \frac{(k-1)!\,x}{(\log x)^k} + \mathcal{O}\left(\frac{x}{(\log x)^{N+1}}\right). \tag{2.7}$$

Lemma 2.1. *For $t > 1$, we have*

$$L_2(t) = \operatorname{li}(t) - \frac{t}{\log t} - \frac{t}{\log^2 t} = F_2(t)\left(\frac{t}{\log^3 t}\right) < 4.05\frac{t}{\log^3 t}. \tag{2.8}$$

For $t \geqslant t_0 \geqslant 381$, we have

$$L_2(t) < F_2(t_0)\,\frac{t}{\log^3 t}. \tag{2.9}$$

For $t > 29$, we have

$$L_2(t) > 2\frac{t}{\log^3 t}. \tag{2.10}$$

Proof. let us set (cf. the Maple sheet [13])

$$f_1(t) = (3 - \log t)\ \operatorname{li}(t) + t - \frac{2t}{\log t} - \frac{t}{\log^2 t} = \frac{t^2 F_2'(t)}{\log^2(t)},$$

$$f_2(t) = \frac{t}{\log t} + \frac{t}{\log^2 t} + 2\frac{t}{\log^3 t} - \operatorname{li}(t) = t f_1'(t)$$

and

$$f_3(t) = f_2'(t) = -\frac{6}{\log^4(t)}.$$

Since $f_2'(t) = f_3(t)$ is negative, $f_2(t)$ decreases and vanishes for

$$t_2 = 28.19524\ldots$$

It follows that $f_1'(t) = f_2(t)/t$ is positive for $1 < t < t_2$ and negative for $t > t_2$ so that $f_1(t)$ has a maximum for $t = t_2$,

$$f_1(t_2) = 4.54378\ldots.$$

and f_1 vanishes (and so does F_2') in two points

$$t_3 = 3.384879\ldots \qquad t_4 = 380.1544\ldots$$

From (2.5), we get $\lim_{t\to 1^+} F_2(t) = 0$ and the variation of F_2 is given in the following array:

t	1	3.38...		10.39...		380.15...		∞	
$F_2(t) = \frac{L_2(t)}{t/\log^3 t}$	0 ↘	$-1.369496\ldots$	↗	0	↗	$4.040415\ldots$	↘	2	(2.11)

The proof of (2.8) and (2.9) follows from Array 2.11 and also the proof of (2.10), after deducing from $f_2(t_2) = 0$ that $F_2(t_2) = 2$ holds. □

In the same way, it is possible to study the variation of the function

$$F_1(t) = \frac{L_1(t)}{(t/\log^2 t)} = \frac{\mathrm{li}(t) - \frac{t}{\log t}}{(t/\log^2 t)},$$

The details can be found on [13]. We have

t	1	1.85...		3.8464...		94.6...		∞	
$F_1(t) = L_1(t)\left(\frac{\log^2 t}{t}\right)$	0 ↘	$-0.448\ldots$	↗	0	↗	$1.784\ldots$	↘	1	(2.12)

Since $L_1(10.3973\ldots) = 1$, Array (2.12) yields

$$t > 10.4 \implies L_1(t) = \mathrm{li}(t) - \frac{t}{\log t} > \frac{t}{\log^2 t}. \tag{2.13}$$

The derivative of $\mathrm{li}(t)/t$ is $\frac{t/\log t - \mathrm{li}(t)}{t^2} = -\frac{F_1(t)}{t\log^2 t}$ which, from Array 2.12, is positive for $1 < t < 3.8464$ and negative for $t > 3.8465$. Therefore, we have

$$t > 1 \implies \mathrm{li}(t) \leqslant \frac{\mathrm{li}(3.8464\ldots)}{3.8464\ldots}\, t = 0.7423\ldots\, t < \frac{3t}{4}. \tag{2.14}$$

Lemma 2.2. *Let a and x be two real numbers satisfying* $\exp(1) \leqslant a < a^3 \leqslant x$. *Let κ_1 and κ_2 be two integers such that*

$$2 \leqslant \kappa_1 < \kappa_2 = \left\lfloor \frac{\log x}{\log a} \right\rfloor.$$

Then we have

$$\sum_{k=\kappa_1+1}^{\kappa_2} \frac{1}{k} L_1(x^{1/k}) \leqslant 1.785 \left(4.05 \frac{\kappa_1^3 x^{1/\kappa_1}}{\log^3 x} - L_2(a)\right). \tag{2.15}$$

Proof. Let us set

$$T = \sum_{k=\kappa_1+1}^{\kappa_2} \frac{1}{k} L_1(x^{1/k}).$$

It follows from Array 2.12 that, for $t > 1$, $L_1(t) = tF_1(t)/\log^2 t \leqslant 1.785 \frac{t}{\log^2 t}$ holds and therefore,

$$T \leqslant \frac{1.785}{\log^2 x} \sum_{k=\kappa_1+1}^{\kappa_2} k x^{1/k}.$$

Now, as $x \geqslant \exp(1) > 1$, the function $t \mapsto t x^{1/t}$ is positive and decreasing for $0 < t \leqslant \log x$ so that

$$T \leqslant \frac{1.785}{\log^2 x} \int_{\kappa_1}^{\kappa_2} t x^{1/t} dt \leqslant \frac{1.785}{\log^2 x} \int_{\kappa_1}^{\frac{\log x}{\log a}} t x^{1/t} dt = 1.785 \int_a^{x^{1/\kappa_1}} \frac{du}{\log^3 u}$$

by the change of variable $u = x^{1/t}$. Finally, by (2.6) and (2.8), we get

$$T \leqslant 1.785 \left(L_2(x^{1/\kappa_1}) - L_2(a)\right) \leqslant 1.785 \left(4.05 \frac{x^{1/\kappa_1}}{\log^3(x^{1/\kappa_1})} - L_2(a)\right)$$

which ends the proof of Lemma 2.2. □

Lemma 2.3. *Let $a \geqslant 2.11$ and $x \geqslant a^3$ be real numbers and $\kappa_2 = \left\lfloor \frac{\log x}{\log a} \right\rfloor$. Then we have*

$$\sum_{k=2}^{\kappa_2} \frac{1}{k} x^{1/(2k)} \leqslant \frac{5}{4} x^{1/4}. \tag{2.16}$$

Proof. Let us set

$$T = \sum_{k=2}^{\kappa_2} \frac{1}{k} x^{1/(2k)}.$$

Since $x \geqslant a^3 > 1$, the function $t \mapsto x^{1/(2t)}/t$ is positive and decreasing for $t > 0$ so that

$$T = \frac{1}{2} x^{1/4} + \sum_{k=3}^{\kappa_2} \frac{1}{k} x^{1/(2k)} \leqslant \frac{1}{2} x^{1/4} + \int_2^{\frac{\log x}{\log a}} \frac{x^{1/(2t)}}{t} dt = \frac{1}{2} x^{1/4} + \int_{\sqrt{a}}^{x^{1/4}} \frac{du}{\log u}$$

by the change of variable $u = x^{1/(2t)}$. Finally, by (2.6) and (2.14), we get

$$T \leqslant \frac{1}{2}x^{1/4} + \mathrm{li}(x^{1/4}) - \mathrm{li}(\sqrt{a}) \leqslant \frac{5}{4}x^{1/4} - \mathrm{li}(\sqrt{a})$$

and (2.16) follows since $\sqrt{a} \geqslant \sqrt{2.11} > 1.452$ so that, from Array (2.3), $\mathrm{li}(\sqrt{a}) > 0$ holds. □

Lemma 2.4. *Under the Riemann Hypothesis, for $x \geqslant 599$, one has*

$$\frac{\theta(x) - x}{\log x} - \frac{9\log^2 x}{10000} \leqslant \mathrm{li}(\theta(x)) - \mathrm{li}(x) \leqslant \frac{\theta(x) - x}{\log x}, \tag{2.17}$$

$$\frac{\psi(x) - x}{\log x} - \frac{9\log^2 x}{10000} \leqslant \mathrm{li}(\psi(x)) - \mathrm{li}(x) \leqslant \frac{\psi(x) - x}{\log x} \tag{2.18}$$

and

$$\frac{\psi(x) - \theta(x)}{\log x} - \frac{9\log^2 x}{10000} \leqslant \mathrm{li}(\psi(x)) - \mathrm{li}(\theta(x)) \leqslant \frac{\psi(x) - \theta(x)}{\log x} + \frac{9\log^2 x}{10000}. \tag{2.19}$$

Proof. Let us suppose that $x \geqslant 599$ holds. From (2.2) and (1.11), we get

$$\frac{\psi(x)}{x} \geqslant \frac{\theta(x)}{x} \geqslant \frac{1}{x}\left(x - \frac{\sqrt{x}\log^2 x}{8\pi}\right) = 1 - \frac{\log^2 x}{8\pi\sqrt{x}} \geqslant 1 - \frac{(\log 599)^2}{8\pi\sqrt{599}} > 0.9335. \tag{2.20}$$

Further, for $h > 1 - x$, Taylor's formula and (2.4) yield

$$\mathrm{li}(x+h) = \mathrm{li}(x) + \frac{h}{\log x} - \frac{h^2}{2\xi\log^2\xi}, \tag{2.21}$$

with $\xi \geqslant \min(x, x+h)$. Let us set $h = \theta(x) - x$; we have $h + x = \theta(x) \geqslant \theta(599) > 1$. From (2.20), we get $\xi \geqslant bx$ with $b = 0.9335$ and

$$\begin{aligned}\xi\log^2\xi &\geqslant bx\log^2(bx) = bx\log^2(x)\left(1 + \frac{\log b}{\log x}\right)^2 \\ &\geqslant bx\log^2(x)\left(1 + \frac{\log b}{\log(599)}\right)^2 \geqslant 0.9135\, x\log^2 x.\end{aligned}$$

From (2.2), it follows that

$$0 \leqslant \frac{h^2}{2\xi\log^2\xi} \leqslant \frac{x\log^4 x}{128\,\pi^2\xi\log^2\xi} \leqslant \frac{\log^2 x}{0.9135 \times 128\,\pi^2} < \frac{9\log^2 x}{10000}$$

which, with (2.21), proves (2.17). In the same way, setting $h = \psi(x) - x$ yields (2.18), and (2.19) follows by subtracting (2.17) from (2.18). □

2.3 *The Riemann ζ function*

We shall use the two explicit formulas valid for $x > 1$

$$\widetilde{\psi}(x) = \psi(x) - \frac{1}{2}\Lambda(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) \tag{2.22}$$

and

$$\widetilde{\Pi}(x) = \Pi(x) - \frac{\Lambda(x)}{2\,\log x} = \mathrm{li}(x) - \sum_\rho \mathrm{li}(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t} \tag{2.23}$$

which can be found in many books in analytic number theory, for instance [5, chap. 4]. To Formula (2.23), we prefer the form described in [6, p. 361 and 362, with $R = 0$] :

$$\widetilde{\Pi}(x) = \mathrm{li}(x) - \sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho - t} dt - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t}, \quad x > 1. \tag{2.24}$$

We also have (cf. [4, p. 67] or [2, p. 272])

$$\sum_\rho \frac{1}{\rho} = 1 + \frac{\gamma_0}{2} - \frac{1}{2}\log\pi - \log 2 = 0.02309570896612103\ldots \tag{2.25}$$

and (cf. (1.5))

$$\sum_\rho \frac{1}{\rho(1-\rho)} = \sum_\rho \left(\frac{1}{\rho} + \frac{1}{1-\rho}\right) = 2\sum_\rho \frac{1}{\rho} = 2 + \gamma_0 - \log(4\pi). \tag{2.26}$$

3 Proof of Theorem 1.1

3.1 *Study of $A_1(x) = \mathrm{li}(\psi(x)) - \Pi(x)$*

Under the Riemann Hypothesis, we write

$$\gamma = \Im\rho \quad \text{i.e.} \quad \rho = \frac{1}{2} + i\gamma.$$

Lemma 3.1. *Under the Riemann Hypothesis, we have*

$$\sum_\rho \frac{1}{|\gamma|^3} \leqslant \frac{1}{300}.$$

Proof. It is possible to get better estimates for the sum $\sum_\rho \frac{1}{|\gamma|^3}$, but, for our purpose, the above upper bound will be enough. By observing that

$$|\rho|^2 = \rho(1-\rho) = \frac{1}{4} + \gamma^2$$

and that the first zero of $\zeta(s)$ is $1/2 + 14.134725\ldots i$ (cf. [4, p. 96] or the extended tables of [8]), we get

$$\sum_\rho \frac{1}{\gamma^2} = \sum_\rho \frac{1 + 1/(4\gamma^2)}{1/4 + \gamma^2} \leqslant \sum_\rho \frac{1 + 1/(4 \times 14.134^2)}{1/4 + \gamma^2} \leqslant \frac{800}{799} \sum_\rho \frac{1}{\rho(1-\rho)}.$$

Further, from (2.26), we get

$$\sum_\rho \frac{1}{|\gamma|^3} \leqslant \frac{1}{14.134} \sum_\rho \frac{1}{\gamma^2} \leqslant \frac{800}{799 \times 14.134} \sum_\rho \frac{1}{\rho(1-\rho)} = 0.00327\ldots$$

which completes the proof of Lemma 3.1. □

Lemma 3.2. *For $x > 1$, under the Riemann Hypothesis, we have*

$$\sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho - t} dt = \sum_\rho \frac{x^\rho}{\rho \log x} + \sum_\rho \frac{x^\rho}{\rho^2 \log^2 x} + K(x)$$

with

$$|K(x)| \leqslant \frac{2}{300} \frac{\sqrt{x}}{\log^3 x}. \tag{3.1}$$

Proof. By partial integration, one has

$$\int_0^\infty \frac{x^{\rho-t}}{\rho-t} dt = \frac{x^\rho}{\rho \log x} + \frac{x^\rho}{\rho^2 \log^2 x} + \frac{2}{\log^2 x} \int_0^\infty \frac{x^{\rho-t}}{(\rho-t)^3} dt$$

and

$$\left| \int_0^\infty \frac{x^{\rho-t}}{(\rho-t)^3} dt \right| \leqslant \frac{1}{|\Im\rho|^3} \int_0^\infty x^{1/2-t} dt = \frac{1}{|\Im\rho|^3} \frac{\sqrt{x}}{\log x}$$

so that we get

$$|K(x)| = \left|\sum_\rho \frac{2}{\log^2 x}\int_0^\infty \frac{x^{\rho-t}}{(\rho-t)^3}dt\right| \leqslant \frac{2\sqrt{x}}{\log^3 x}\sum_\rho \frac{1}{|\Im\rho|^3}$$

and (3.1) follows from Lemma 3.1. □

Proposition 3.3. *Under the Riemann Hypothesis, for $x \geqslant 599$, we have*

$$A_1(x) = \mathrm{li}(\psi(x)) - \Pi(x) = \sum_\rho \frac{x^\rho}{\rho^2 \log^2 x} + J(x)$$

with

$$-0.0009\log^2 x - \frac{2}{300}\frac{\sqrt{x}}{\log^3 x} \leqslant J(x) \leqslant \frac{2}{300}\frac{\sqrt{x}}{\log^3 x} + \log 2. \tag{3.2}$$

Proof. Let us write

$$\mathrm{li}(\psi(x)) = \mathrm{li}(x) + \frac{\psi(x)-x}{\log x} + J_1(x) = \mathrm{li}(x) + \frac{\widetilde{\psi}(x) - x + \Lambda(x)/2}{\log x} + J_1(x)$$

with, from (2.18), for $x \geqslant 599$,

$$-0.0009\log^2 x \leqslant J_1(x) \leqslant 0. \tag{3.3}$$

Therefore, from (2.22) and (2.24), we have

$$\begin{aligned} A_1(x) &= \mathrm{li}(x) + \frac{1}{\log x}\left(-\sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) + \frac{1}{2}\Lambda(x)\right) \\ &+ J_1(x) - \left(\mathrm{li}(x) - \sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho-t}dt + \int_x^\infty \frac{dt}{t(t^2-1)\log t} - \log 2 + \frac{\Lambda(x)}{2\log x}\right) \\ &= \sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho-t}dt - \frac{1}{\log x}\sum_\rho \frac{x^\rho}{\rho} + J_1(x) + J_2(x) + J_3(x) \end{aligned}$$

with

$$J_2(x) = \log 2 - \frac{\log(2\pi)}{\log x} \quad \text{and} \quad J_3(x) = -\frac{\log(1-1/x^2)}{2\log x} - \int_x^\infty \frac{dt}{t(t^2-1)\log t}.$$

Further, from Lemma 3.2, one gets

$$A_1(x) = \sum_\rho \frac{x^\rho}{\rho^2\log^2 x} + J(x) \tag{3.4}$$

with

$$J(x) = K(x) + J_1(x) + J_2(x) + J_3(x) \tag{3.5}$$

and $K(x)$ is as in Lemma 3.2.

It remains to bound $J_2(x) + J_3(x)$. We have

$$J_3(x) = \int_x^\infty \frac{1}{t(t^2-1)}\left(\frac{1}{\log x} - \frac{1}{\log t}\right) dt$$

which, for $x \geqslant 599$, implies

$$0 \leqslant J_3(x) \leqslant \frac{1}{\log x}\int_x^\infty \frac{dt}{t(t^2-1)} = \frac{\log(1+1/(x^2-1))}{2\log x}$$
$$\leqslant \frac{1}{2(x^2-1)\log x} < \frac{\log(2\pi)}{\log x}$$

and $0 < J_2(x) + J_3(x) < \log 2$. Therefore, (3.2) results from (3.1), (3.3), (3.4), and (3.5). □

3.2 *Study of* $A_2(x) = \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x)$

For $y \geqslant 2$, let us set

$$B(y) = \pi(y) - \frac{\theta(y)}{\log y} = \sum_{p \leqslant y}\left(1 - \frac{\log p}{\log y}\right).$$

Note that $B(y)$ is nonnegative. If $q < q'$ are two consecutive primes, $B(y)$ is increasing and continuous on $[q, q')$ and

$$\lim_{y \to q',\, y<q'} B(y) = \pi(q) - \frac{\theta(q)}{\log q'} = \pi(q') - 1 - \frac{\theta(q') - \log q'}{\log q'} = B(q')$$

so that $B(y)$ is continuous and increasing for $y \geqslant 2$. In the two following lemmas, we give estimates of $B(y)$.

Lemma 3.4. *Let y be a real number satisfying $y_0 = 8.3 \leqslant y \leqslant 1.39 \times 10^{17}$. We have*

$$B(y) \leqslant L_1(y) = \mathrm{li}(y) - \frac{y}{\log y} \tag{3.6}$$

while, if $y \geqslant y_1 = 599$, under the Riemann Hypothesis, we have

$$B(y) \leqslant L_1(y) + \frac{\sqrt{y}}{4\pi}. \tag{3.7}$$

Under the Riemann Hypothesis, for $y \geqslant y_2 = 2903$, we have

$$B(y) \geqslant L_1(y) - \frac{\sqrt{y}}{4\pi}. \tag{3.8}$$

Proof. By Stieljes's integral, one has

$$\pi(y) = \int_{2-}^{y} \frac{d[\theta(t)]}{\log t} = \frac{\theta(y)}{\log y} + \int_2^y \frac{\theta(t)}{t \log^2 t} dt. \tag{3.9}$$

Further, we have

$$B(y) = \int_2^y \frac{\theta(t)}{t \log^2 t} dt = \int_2^{y_0} + \int_{y_0}^y \frac{\theta(t)}{t \log^2 t} dt = B(y_0) + \int_{y_0}^y \frac{\theta(t)}{t \log^2 t} dt. \tag{3.10}$$

By (2.1) and (2.6), for $y \leqslant 1.39 \times 10^{17}$, we get

$$\int_{y_0}^y \frac{\theta(t)}{t \log^2 t} dt \leqslant \int_{y_0}^y \frac{1}{\log^2 t} dt = \mathrm{li}(y) - \frac{y}{\log y} - \mathrm{li}(y_0) + \frac{y_0}{\log y_0} = L_1(y) - L_1(y_0),$$

so that (3.10) yields $B(y) \leqslant L_1(y) + B(y_0) - L_1(y_0)$, which proves (3.6), since $B(y_0) - L_1(y_0) = -0.001379\ldots < 0$ (cf. [13].

Replacing y_0 by y_1 in (3.10) yields

$$B(y) = B(y_1) + \int_{y_1}^y \frac{\theta(t)dt}{t \log^2 t} = B(y_1) - L_1(y_1) + L_1(y) + T(y, y_1) \tag{3.11}$$

with $T(y, y_1) = \int_{y_1}^y \frac{\theta(t)-t}{t \log^2 t} dt$ and, from (2.2) ,

$$|T(y, y_1)| \leqslant \int_{y_1}^y \frac{\sqrt{t} \log^2 t}{8\pi t \log^2 t} dt = \frac{\sqrt{y} - \sqrt{y_1}}{4\pi}. \tag{3.12}$$

From (3.11) and (3.12), it follows that

$$B(y) \leqslant L_1(y) + \frac{\sqrt{y}}{4\pi} + B(y_1) - L_1(y_1) - \frac{\sqrt{y_1}}{4\pi}$$

which proves (3.7), since $B(y_1) - L_1(y_1) - \frac{\sqrt{y_1}}{4\pi} = -4.80566\ldots < 0$.

In the same way than the one used to get (3.11), for $y \geqslant y_2$, we obtain

$$B(y) = B(y_2) - L_1(y_2) + L_1(y) + T(y, y_2) \geqslant L_1(y) - \frac{\sqrt{y}}{4\pi} + B(y_2) - L_1(y_2) + \frac{\sqrt{y_2}}{4\pi}$$

and as $B(y_2) - L_1(y_2) + \frac{\sqrt{y_2}}{4\pi} = 0.00671\ldots > 0$, this completes the proof of Lemma 3.4. □

Let us set

$$\varepsilon(y) = \begin{cases} 0 & \text{if } y \leqslant 1.39 \times 10^{17} \\ 1 & \text{if } y > 1.39 \times 10^{17}. \end{cases}$$

It follows from (3.6) and (3.7) that, under the Riemann Hypothesis, one has

$$B(y) \leqslant L_1(y) + \varepsilon(y)\frac{\sqrt{y}}{4\pi} \quad \text{for } y \geqslant 8.3. \tag{3.13}$$

Proposition 3.5. *Under the Riemann Hypothesis, for $x \geqslant 599$, we have*

$$A_2(x) = \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x) = \sum_{k=2}^{\kappa} \frac{1}{k} B(x^{1/k}) + U(x) \tag{3.14}$$

with

$$\kappa := \left\lfloor \frac{\log x}{\log 2} \right\rfloor \quad \textit{and} \quad |U(x)| \leqslant \frac{9 \log^2 x}{10000}. \tag{3.15}$$

Proof. From (2.19), for $x \geqslant 599$, we get

$$\mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) = \frac{\theta(x) - \psi(x)}{\log x} + U(x) \text{ with } |U(x)| \leqslant \frac{9 \log^2 x}{10000}.$$

From the definition of $\psi(x)$ and $\Pi(x)$, this implies

$$A_2(x) = \sum_{k=2}^{\kappa} \left(\frac{\pi(x^{1/k})}{k} - \frac{\theta(x^{1/k})}{\log x} \right) + U(x)$$

which, via the definition of B, proves (3.14). □

It is convenient to introduce the notation

$$\widetilde{F}_2(t) = \begin{cases} 4.05 & \text{if } t \leqslant 381 \\ F_2(t) & \text{if } t > 381 \end{cases} \quad \text{and} \quad \widetilde{F}_1(t) = \begin{cases} 1.785 & \text{if } t \leqslant 95 \\ F_1(t) & \text{if } t > 95 \end{cases} \tag{3.16}$$

so that, from Arrays (2.11) and (2.12), for $t > 1$, $\widetilde{F}_2(t)$ and $\widetilde{F}_1(t)$ are nonincreasing and we have

$$L_2(t) = \frac{t F_2(t)}{\log^3 t} \leqslant \frac{t \widetilde{F}_2(t)}{\log^3 t} \quad \text{and} \quad L_1(t) = \frac{t F_1(t)}{\log^2 t} \leqslant \frac{t \widetilde{F}_1(t)}{\log^2 t}. \tag{3.17}$$

Lemma 3.6. *Let us set $a = 10.4$. For $x > 10^8$, we set $\kappa = \lfloor \frac{\log x}{\log 2} \rfloor$, $\kappa_2 = \lfloor \frac{\log x}{\log a} \rfloor$ and let κ_1 be an integer satisfying $3 \leqslant \kappa_1 < \kappa_2$. Then, under the Riemann Hypothesis, we have*

$$\sum_{k=2}^{\kappa} \frac{B(x^{1/k})}{k} \leqslant \frac{2\sqrt{x}}{\log^2 x} + \frac{4\sqrt{x}}{\log^3 x} \widetilde{F}_2(\sqrt{x}) + \sum_{k=3}^{\kappa_1} \frac{kx^{1/k}}{\log^2 x} \widetilde{F}_1(x^{1/k})$$
$$+ \frac{7.23\, \kappa_1^3 x^{1/\kappa_1}}{\log^3 x} + 2.35 + 0.94 \frac{\sqrt{x}}{\log^5 x}.$$

Proof. For $2 \leqslant k \leqslant \kappa_2$ we have $x^{1/k} \geqslant x^{1/\kappa_2} \geqslant x^{(\log a)/\log x} = a$, and, under the Riemann Hypothesis, it follows from (3.13) that

$$B(x^{1/k}) \leqslant L_1(x^{1/k}) + \varepsilon(x^{1/k}) \frac{x^{1/(2k)}}{4\pi}$$

which implies that

$$\sum_{k=2}^{\kappa} \frac{B(x^{1/k})}{k} \leqslant T_1 + T_2 + T_3 + T_4 + T_5$$

with

$$T_1 = \frac{1}{2} L_1(\sqrt{x}), \quad T_2 = \sum_{k=3}^{\kappa_1} \frac{L_1(x^{1/k})}{k}, \quad T_3 = \sum_{k=\kappa_1+1}^{\kappa_2} \frac{L_1(x^{1/k})}{k},$$

$$T_4 = \sum_{k=\kappa_2+1}^{\kappa} \frac{B(x^{1/k})}{k}, \quad T_5 = \sum_{k=2}^{\kappa_2} \varepsilon(x^{1/k}) \frac{x^{1/(2k)}}{4k\pi}.$$

From the definition of L_1, L_2, F_1, F_2 and from (3.17), one has

$$T_1 = \frac{L_2(\sqrt{x})}{2} + \frac{\sqrt{x}}{2\log^2 \sqrt{x}} = \frac{2\sqrt{x}}{\log^2 x} + 4\frac{\sqrt{x} F_2(\sqrt{x})}{\log^3 x} \leqslant \frac{2\sqrt{x}}{\log^2 x} + 4\frac{\sqrt{x} \widetilde{F}_2(\sqrt{x})}{\log^3 x}$$

and

$$T_2 = \sum_{k=3}^{\kappa_1} \frac{L_1(x^{1/k})}{k} = \sum_{k=3}^{\kappa_1} \frac{kx^{1/k}}{\log^2 x} F_1(x^{1/k}) \leqslant \sum_{k=3}^{\kappa_1} \frac{kx^{1/k}}{\log^2 x} \widetilde{F}_1(x^{1/k}).$$

From Array (2.11), $L_2(10.4)$ is positive, so that, from Lemma 2.2 with $a = 10.4$, we have

$$T_3 \leqslant 1.785\left(4.05\,\frac{\kappa_1^3 x^{1/\kappa_1}}{\log^3 x} - L_2(10.4)\right)$$
$$\leqslant 1.785 \times 4.05\,\frac{\kappa_1^3 x^{1/\kappa_1}}{\log^3 x} \leqslant \frac{7.23\,\kappa_1^3 x^{1/\kappa_1}}{\log^3 x}.$$

For $k \geqslant \kappa_2 + 1 > (\log x)/\log a$, we have $x^{1/k} < a$; since $y \mapsto B(y)$ is nondecreasing, we have $B(x^{1/k}) \leqslant B(a) = B(10.4) = 1.7166\ldots < 1.72$ and

$$T_4 \leqslant 1.72 \sum_{k=\kappa_2+1}^{K} \frac{1}{k} \leqslant 1.72 \int_{\kappa_2}^{K} \frac{dt}{t} \leqslant 1.72 \int_{\frac{\log x}{\log a}-1}^{\frac{\log x}{\log 2}} \frac{dt}{t}$$
$$= 1.72\left(\log\left(\frac{\log x}{\log 2}\right) - \log\left(\frac{\log(x/a)}{\log a}\right)\right)$$
$$= 1.72\left(\log\left(\frac{\log a}{\log 2}\right) + \log\left(\frac{\log x}{\log(x/a)}\right)\right)$$
$$\leqslant 1.72\left(\log\left(\frac{\log a}{\log 2}\right) + \left(\frac{\log x}{\log(x/a)} - 1\right)\right)$$
$$= 1.72\left(\log\left(\frac{\log a}{\log 2}\right) + \frac{\log a}{\log(x/a)}\right)$$
$$\leqslant 1.72\left(\log\left(\frac{\log a}{\log 2}\right) + \frac{\log a}{\log(10^8/a)}\right) = 2.34449\ldots$$

Since $\varepsilon(t)$ is nondecreasing and vanishes for $x \leqslant 10^{17}$, from Lemma 2.3, one gets

$$T_5 = \sum_{k=2}^{\kappa_2} \varepsilon(x^{1/k})\frac{x^{1/(2k)}}{4k\pi} \leqslant \varepsilon(\sqrt{x}) \sum_{k=2}^{\kappa_2} \frac{x^{1/(2k)}}{4k\pi} \leqslant \frac{5}{16\pi}\varepsilon(\sqrt{x})x^{1/4}$$
$$= \frac{5}{16\pi}\varepsilon(\sqrt{x})\frac{\sqrt{x}}{\log^5 x}\,\frac{\log^5 x}{x^{1/4}} < \frac{5}{16\pi}\,\frac{\sqrt{x}}{\log^5 x}\,\frac{\log^5 10^{34}}{10^{34/4}} = 0.93\ldots\frac{\sqrt{x}}{\log^5 x},$$

which completes the proof of Lemma 3.6. □

3.3 *A lower bound for $A(x)$*

Proposition 3.7. *Under the Riemann Hypothesis, for $x \geqslant 9 \times 10^6$, we have*

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{1}{\log x}\left(7.993 - \frac{\log^3 x}{8\pi x^{1/4}} - \frac{18}{10000}\,\frac{\log^5 x}{\sqrt{x}}\right)\right). \quad (3.18)$$

Proof. Since $B(y)$ is nonnegative, from (3.14) and (3.15), we get, for $x \geqslant 599$

$$A_2(x) \geqslant \frac{1}{2} B(\sqrt{x}) - \frac{9 \log^2 x}{10000}.$$

As $x \geqslant 2903^2$, we may apply (3.8) which yields

$$\begin{aligned} A_2(x) &\geqslant \frac{1}{2}\left(L_1(\sqrt{x}) - \frac{x^{1/4}}{4\pi}\right) - \frac{9 \log^2 x}{10000} \\ &= \frac{1}{2}\left(\frac{\sqrt{x}}{\log^2 \sqrt{x}} + L_2(\sqrt{x}) - \frac{x^{1/4}}{4\pi}\right) - \frac{9 \log^2 x}{10000}. \end{aligned}$$

Now, as $x > 29^2$, by (2.10), it follows

$$\begin{aligned} A_2(x) &\geqslant \frac{1}{2}\left(\frac{\sqrt{x}}{\log^2 \sqrt{x}} + \frac{2\sqrt{x}}{\log^3 \sqrt{x}} - \frac{x^{1/4}}{4\pi}\right) - \frac{9 \log^2 x}{10000} \\ &= \frac{\sqrt{x}}{\log^2 x}\left(2 + \frac{8}{\log x} - \frac{\log^2 x}{8\pi x^{1/4}} - \frac{9 \log^4 x}{10000\sqrt{x}}\right). \end{aligned}$$

From Proposition 3.3, one has:

$$A_1(x) \geqslant -\left|\sum_\rho \frac{x^\rho}{\rho^2 \log^2 x}\right| - 0.0009 \log^2 x - \frac{2}{300}\frac{\sqrt{x}}{\log^3 x}$$

so that $A(x) = A_1(x) + A_2(x)$ satisfies

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x}\left(2 - \sum_\rho \frac{1}{|\rho^2|} + \frac{8 - 2/300}{\log x} - \frac{\log^2 x}{8\pi x^{1/4}} - \frac{18 \log^4 x}{10000\sqrt{x}}\right)$$

which, via 1.5, implies (3.18). □

Corollary 3.8. *Under the Riemann Hypothesis, for $x \geqslant 10^8$, we have*

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{5.12}{\log x}\right). \tag{3.19}$$

Proof. From (1.11), the functions $x \mapsto \frac{\log^3 x}{x^{1/4}}$ and $x \mapsto \frac{\log^5 x}{\sqrt{x}}$ are decreasing for $x \geqslant 10^8$ and therefore, we have

$$7.993 - \frac{\log^3 x}{8\pi x^{1/4}} - \frac{18}{10000}\frac{\log^5 x}{\sqrt{x}} \geqslant 7.993 - \frac{\log^3 10^8}{8\sqrt[4]{10^8}\,\pi} - \frac{18}{10000}\frac{\log^5 10^8}{\sqrt{10^8}} = 5.124\ldots$$

(cf. [13]). □

3.4 An upper bound for $A(x)$

Proposition 3.9. *Under the Riemann Hypothesis, for $x \geqslant 10^8$, we have*

$$A(x) \leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{Q(\kappa_1, x)}{\log x}\right) \tag{3.20}$$

where κ_1 is an integer satisfying $3 \leqslant \kappa_1 < \lfloor \frac{\log x}{\log 10.4} \rfloor$ and

$$Q(\kappa_1, x) = 4\widetilde{F}_2(\sqrt{x}) + \frac{2}{300} + \frac{3.05 \log^3 x}{\sqrt{x}} + \sum_{k=3}^{\kappa_1} \frac{k\widetilde{F}_1(x^{1/k}) \log x}{x^{1/2-1/k}} + \frac{7.23\, \kappa_1^3}{x^{1/2-1/\kappa_1}} + \frac{0.94}{\log^2 x} + \frac{9 \log^5 x}{10000\sqrt{x}} \tag{3.21}$$

with $\widetilde{F}_2$ and $\widetilde{F}_1$ defined in (3.16).

Proof. From Proposition 3.3 and (1.5), for $x \geqslant 599$, we have

$$A_1(x) \leqslant \lambda \frac{\sqrt{x}}{\log^2 x} + \frac{2}{300} \frac{\sqrt{x}}{\log^3 x} + 0.7$$

while, from Proposition 3.5, we have

$$A_2(x) \leqslant \sum_{k=2}^{\kappa} \frac{1}{k} B(x^{1/k}) + \frac{9 \log^2 x}{10000}.$$

Therefore, from Lemma 3.6, we get the upper bound (3.20) for $A(x) = A_1(x) + A_2(x)$. □

Corollary 3.10. *Under the Riemann Hypothesis, for $x \geqslant 10^8$, we have*

$$A(x) \leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{25.22}{\log x}\right). \tag{3.22}$$

Proof. We choose $\kappa_1 = 5$ and observe that, from (3.16) and (1.11), all the terms of the right-hand side of (3.21) are positive and nonincreasing for $x \geqslant 10^8$ so that $Q(5, x) \leqslant Q(5, 10^8) = 25.2119\ldots$ (cf. [13]). □

Corollary 3.11. *Under the Riemann Hypothesis, for x tending to infinity, we have*

$$\frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{7.993 + o(1)}{\log x}\right) \leqslant A(x) \leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{8.007 + o(1)}{\log x}\right). \tag{3.23}$$

Proof. The lower bound of (3.23) follows from Proposition 3.7. From Array (2.11), from (2.13) and from (3.16), one sees in (3.21) that $\lim_{x\to\infty} \widetilde{F}_2(\sqrt{x}) = 2$ and $\lim_{x\to\infty} \widetilde{F}_1(x^{1/3}) = 1$ so that (3.21) yields $\lim_{x\to\infty} Q(3,x) = 8 + 2/300$ and the upper bound of (3.23) follows from Proposition 3.9 with $\kappa_1 = 3$.

□

3.5 *Numerical computation*

Let us denote by p^- and p^+ the primes surrounding the prime p.

Proposition 3.12. *For $x < 1.39 \times 10^{17}$, $A(x)$ is nondecreasing. There exists infinitely many primes p for which $A(p) < A(p^-)$ holds.*

Proof. Let us consider a prime p satisfying $3 \leqslant p < 1.39 \times 10^{17}$. From (2.1), one has

$$A(p) - A(p^-) = \mathrm{li}(\theta(p)) - \mathrm{li}(\theta(p^-)) - 1 = -1 + \int_{\theta(p^-)}^{\theta(p)} \frac{dt}{\log t}$$

$$> -1 + \frac{\theta(p) - \theta(p^-)}{\log \theta(p)} = \frac{\log p}{\log \theta(p)} - 1 > 0.$$

From Littlewood (cf. [7] or [5, chap. 5]), we know that there exists $C > 0$ and a sequence of values of x going to infinity such that

$$\theta(x) \geqslant x + C\sqrt{x} \log\log\log x.$$

Let p be the largest prime $\leqslant x$. For x and p large enough, one has

$$\theta(p) = \theta(x) \geqslant x + C\sqrt{x} \log\log\log x > p + \log p$$

and

$$A(p) - A(p^-) < \frac{\log p}{\log \theta(p^-)} - 1 = \frac{\log p}{\log(\theta(p) - \log p)} - 1 < 0$$

which completes the proof of Proposition 3.12. □

Remark. In [9, p. 8], Platt and Trudgian have proved the existence of u satisfying $727 < u < 728$ and $\theta(e^u) - e^u > 10^{152}$. If P is the largest prime $\leqslant e^u$, this implies

$$\theta(P) = \theta(e^u) > e^u + 10^{152} > P + u \geqslant P + \log P$$

and $A(P) < A(P^-) + \dfrac{\log P}{\log(\theta(P) - \log P)} - 1 < A(P^-)$.

Proposition 3.13. *(i) For* $11 \leqslant x \leqslant 1.39 \times 10^{17}$ *we have*

$$A(x) > 0. \tag{3.24}$$

(ii) Under the Riemann Hypothesis, for $x \geqslant 2$ *we have*

$$A(x) \leqslant \frac{\sqrt{x}}{\log x}\left(2 + \lambda + \frac{27.7269\ldots}{\log x}\right) \tag{3.25}$$

with equality for $x = 33647$.
(iii) Under the Riemann Hypothesis, for $x \geqslant 520\,878$ *we have*

$$A(x) \leqslant \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{25.22}{\log x}\right). \tag{3.26}$$

(iv) For $2 \leqslant x \leqslant 10000$ *we have*

$$A(x) \leqslant 5.0643\ldots \frac{\sqrt{x}}{\log^2 x}. \tag{3.27}$$

with equality for $x = 3643$.
(v) Under the Riemann Hypothesis, for $x \geqslant 84.11$ *we have*

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{5.12}{\log x}\right). \tag{3.28}$$

(vi) For $37 \leqslant x < 89$ *we have*

$$A(x) \geqslant \frac{\sqrt{x}}{\log^2 x}(2 - \lambda). \tag{3.29}$$

Proof. First, for $x \geqslant 2$, we define $C(x)$ and $c(x)$ by

$$A(x) = \frac{\sqrt{x}}{\log^2 x}\left(2 + \lambda + \frac{C(x)}{\log x}\right) \quad \text{and} \quad A(x) = \frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{c(x)}{\log x}\right)$$

so that

$$C(x) = (\log x)\left(\frac{A(x)\log^2 x}{\sqrt{x}} - 2 - \lambda\right)$$

and

$$c(x) = (\log x)\left(\frac{A(x)\log^2 x}{\sqrt{x}} - 2 + \lambda\right).$$

(i) (3.24) follows from Proposition 3.12 and $A(11) = 0.1301\ldots$ Note that $A(7) = -0.1541 < 0$ (cf. [13]).

(ii) If $x \geqslant 10^8$, (3.25) follows from Corollary 3.10.

If $2 \leqslant x < 409$, from (1.12), one has $(\log^2 x)/\sqrt{x} \leqslant 16/e^2$ and, from Proposition 3.12, $A(x) \leqslant A(401) \leqslant 2.52$ so that

$$C(x) = (\log x)\left(A(x)\frac{\log^2 x}{\sqrt{x}} - 2 - \lambda\right) \leqslant (\log 409)\left(2.52\,\frac{16}{e^2} - 2 - \lambda\right) < 20.51$$

which proves (3.25).

If $409 \leqslant x < 10^8$, let p be the largest prime $\leqslant x$. As $409 > e^6$ holds, from (1.11), for $x \in [p, p^+)$, the function $x \mapsto (\log x)\left(A(p)\frac{\log^2 x}{\sqrt{x}} - 2 - \lambda\right)$ is decreasing, which implies

$$C(x) \leqslant C(p) \tag{3.30}$$

and, by computation,

$$\max_{409 \leqslant x \leqslant 10^8} C(x) = \max_{409 \leqslant p < 10^8} C(p) = C(33647) = 27.7269\ldots$$

which completes the proof of (3.25).

(iii) For $x \geqslant 10^8$, (3.26) follows from Corollary 3.10.

We compute $p_0 = 520\,867$ the largest prime $< 10^8$ such that $C(p_0) \geqslant 25.22$. For $p_0^+ = 520\,889 \leqslant x < 10^8$, we denote by p the largest prime $\leqslant x$ and, from (3.30), one has $C(x) \leqslant C(p) < 25.22$, which implies (3.26). Then, one calculates

$$\lim_{x \to p_0^+,\, x < p_0^+} C(x) = (\log p_0^+)\left(A(p_0)\frac{\log^2 p_0^+}{\sqrt{p_0^+}} - 2 - \lambda\right) = 25.21964\ldots$$

As the above value is < 25.22, we have to solve the equation $C(t) = 25.22$ for $p_0 \leqslant t < p_0^+$ and find $t = 520\,877.54\ldots$

(iv) For $t \geqslant 1$ the function $t \mapsto (\log^2 t)/\sqrt{t}$ is maximal for $t = e^4 = 54.59\ldots$ where its value is $16/e^2 = 2.16\ldots$ (cf. (1.11) and (1.12)). As $A(x)$ is nondecreasing, for $x < 59$, we have

$$A(x)\frac{\log^2 x}{\sqrt{x}} \leqslant \frac{16}{e^2}A(53) = \frac{16}{e^2}1.155\ldots = 2.501\ldots$$

For $p \geqslant 59$ and $p \leqslant x < p^+$, one has

$$A(x)\frac{\log^2 x}{\sqrt{x}} = A(p)\frac{\log^2 x}{\sqrt{x}} \leqslant A(p)\frac{\log^2 p}{\sqrt{p}}$$

and we compute the maximum of $A(p)\frac{\log^2 p}{\sqrt{p}}$ for $59 \leqslant p < 10000$ which is equal to $5.064\ldots$ for $p = 3643$.

(v) Let us set

$$f(x) = \frac{\sqrt{x}}{\log^2 x}\left(2 - \lambda + \frac{5.12}{\log x}\right).$$

For $x \geqslant 10^8$, $A(x) > f(x)$ follows from Corollary 3.8.

Let p be a prime satisfying $e^6 < 409 \leqslant p < 10^8$. For $p \leqslant x < p^+$, one has $A(x) = A(p)$,

$$c(x) = (\log x)\left(A(p)\frac{\log^2 x}{\sqrt{x}} - 2 + \lambda\right),$$

$$c'(x) = \frac{A(p)(\log^2 x)(6 - \log x) - 2(2-\lambda)\sqrt{x}}{2x^{3/2}} < 0$$

so that $c(x)$ is decreasing and

$$c(x) \geqslant \widetilde{c}(p) \stackrel{def}{=\!=} \lim_{x \to p^-, x<p^+} c(x) = (\log p^+)\left(A(p)\frac{\log^2 p^+}{\sqrt{p^+}} - 2 + \lambda\right).$$

Therefore, for $409 \leqslant x < 10^8$ one has $c(x) \geqslant \min_{409 \leqslant p < 10^8} \widetilde{c}(p)$ and, by computation, one gets

$$\min_{409 \leqslant p < 10^8} \widetilde{c}(p) = \widetilde{c}(409) = 15.3735\ldots$$

which implies $A(x) > f(x)$.

The function f is decreasing on $(1, x_1 = 111.55\ldots]$ and increasing for $x \geqslant x_1$ (cf. [13]). Therefore, for $1 < a < b$, the upper bound of f on the interval $[a, b)$ is $\max(f(a), f(b))$. We have $A(84.1) = A(83) < f(84.1)$ while, for $84.11 \leqslant x < 89$, $A(x) = A(83) > \max(f(84.11), f(89)) \geqslant f(x)$ holds.

For $89 \leqslant p \leqslant 401 = 409^-$, one checks that $A(p) > \max(f(p), f(p^+))$ holds which shows that $A(x) > f(x)$ for $89 \leqslant x < 409$ and completes the proof of (3.28).

(vi) From (1.11), the function $\varphi(t) = (\log^2 t)/\sqrt{t}$ is increasing for $1 \leqslant t \leqslant e^4 = 54.598\ldots$ and decreasing for $t \geqslant e^4$ so that, for $1 < a < b$, the lower bound of φ on the interval $[a, b)$ is $\min(\varphi(a), \varphi(b))$.

Let p be a prime satisfying $11 \leqslant p \leqslant 83$. From (i), one has $A(p) > 0$ and, for $x \in [p, p^+)$,

$$A(x)\frac{\log^2 x}{\sqrt{x}} = A(p)\frac{\log^2 x}{\sqrt{x}} \geqslant A(p)\min(\varphi(p), \varphi(p^+)).$$

To prove (3.29), it remains to check that $A(p)\min(\varphi(p), \varphi(p^+)) > 2 - \lambda$ holds for $37 \leqslant p \leqslant 83$. □

3.6 Proof of Theorem 1.1

Proof. The proof of (1.6) follows from Corollary 3.10 while Corollary 3.8 yields (1.7).

The proof of (1.8) results of Proposition 3.13, (i) and (v).

Inequality (1.9) results of Proposition 3.13, (v) and (vi).

If $x \leqslant 10000$, Inequality (1.10) follows from Proposition 3.13, (iv), while for $x > 10000$, Proposition 3.13, (ii), implies

$$A(x) \leqslant \frac{\sqrt{x}}{\log^2 x}\left(2+\lambda+\frac{27.7269\ldots}{\log x}\right)$$
$$\leqslant \frac{\sqrt{x}}{\log^2 x}\left(2+\lambda+\frac{27.7269\ldots}{\log 10000}\right) = 5.0566\ldots\frac{\sqrt{x}}{\log^2 x}$$

which ends the proof of Theorem 1.1. □

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