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Highly Composite Numbers
by Srinivasa Ramanujan

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Abstract. In 1915, the London Mathematical Society published in its Proceedings a paper of Ramanujan entitled "Highly Composite Numbers". But it was not the whole work on the subject, and in "The lost notebook and other unpublished papers", one can find a manuscript, handwritten by Ramanujan, which is the continuation of the paper published by the London Mathematical Society.

This paper is the typed version of the above mentioned manuscript with some notes, mainly explaining the link between the work of Ramanujan and works published after 1915 on the subject.

A number N is said highly composite if $M < N$ implies $d(M) < d(N)$, where $d(N)$ is the number of divisors of N. In this paper, Ramanujan extends the notion of highly composite number to other arithmetic functions, mainly to $Q_{2k}(N)$ for $1 \leq k \leq 4$ where $Q_{2k}(N)$ is the number of representations of N as a sum of $2k$ squares and $\sigma_{-s}(N)$ where $\sigma_{-s}(N)$ is the sum of the $(-s)$th powers of the divisors of N. Moreover, the maximal orders of these functions are given.

Key words: highly composite number, arithmetical function, maximal order, divisors

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1. Foreword

In 1915, the London Mathematical Society published in its Proceedings a paper of Srinivasa Ramanujan entitled "Highly Composite Numbers". (cf. [16]). In the "Collected Papers" of Ramanujan, this article has number 15, and in the notes (cf. [17], p. 339), it is stated: "The paper, long as it is, is not complete. The London Math. Soc. was in some financial difficulty at the time and Ramanujan suppressed part of what he had written in order to save expenses". This suppressed part had been known to Hardy, who mentioned it in a letter to Watson, in 1930 (cf. [18], p. 391). Most of this suppressed part can be now found in "the lost notebook and other unpublished papers" (cf. [18], p. 280 to 312). An analysis of this book has been done by Rankin, who has written several lines about the pages concerning highly composite numbers (cf. [19], p. 361). Also, some information about this subject has already been published in [12], pp. 238–239 and [13]. Robin (cf. [25]) has given detailed proofs of some of the results dealing with complex variables, and Riemann zeta function, since as usual, Ramanujan sometimes gives formulas which probably were obvious to him, but not to most mathematicians.

The article below is essentially the end of the paper written by Ramanujan which was not published in [16], but can be read in [18]. For convenience, we have kept on the numbering both of paragraphs (which start from 52 to 75) and formulas (from (268) to (408)), so that references to preceding paragraphs or formulas can easily be found in [16]. There is just a small overlap: the last paragraph of [16] is numbered 52, and contains formulas (268) and (269). This last paragraph was probably added by Ramanujan to the first part after he had decided to suppress the second part. However this overlap does not imply any misunderstanding.

There are two gaps in the manuscript of Ramanujan, as presented in "the lost notebook". The first one is just at the beginning, where the definition of $Q_2(n)$ is missing. Probably this definition was sent to the London Math. Soc. in 1915 with the manuscript of "Highly Composite Numbers". It has been reformulated in the same terms as the definition of $\bar{Q}_2(n)$ given in Section 55. The second gap is more difficult to explain: Section 57 is complete and appears on pp. 289 and 290 of [18]. But the lower half of p. 290 is empty, and p. 291 starts with the end of Section 58. We have completed Section 58 by giving the definition of $\sigma_s(N)$, and the proof of formula (301). All these completions are written in italics in the text below. It should be noted that in [18] pp. 295–299 are not handwritten by Ramanujan, and, as observed by Rankin (cf. [19], p. 361) were probably copied by Watson, but that does not create any gap in the text. Pages 282 and 283 of [18] do not belong to number theory, and clearly the text of p. 284 follows p. 281. On the other hand, pp. 309–312 deal with highly composite numbers. With the notation of [16], Section 9, Ramanujan proves in pp. 309–310 that

$$\frac{\log\ p_r}{\log(1+1/r)} = \frac{\log\ p_1}{\log\ 2} + O(r)$$

holds, while on pp. 311–312, he attempts to extend the above formula by replacing p_1 by p_s. More precise results can now be found in [7]. Pages 309–312 do not belong to the paper Highly Composite Numbers and are not included in the paper below.

In the following paper, Ramanujan studies the maximal order of some classical functions, which resemble the number, or the sum, of the divisors of an integer.

In Section 52–54, $Q_2(N)$, the number of representations of N as a sum of two squares is studied, and its maximal order is given under the Riemann hypothesis, or without assuming the Riemann hypothesis. In Section 55–56, a similar work is done for $\bar{Q}_2(N)$ the number of representation of N by the form $m^2 + mn + n^2$. In Section 57, the number of ways of writing N as a product of $(1 + r)$ factors is briefly investigated. Between Section 58 and Section 71, there is a deep study of the maximal order of

$$\sigma_{-s}(N) = \sum_{d \mid N} d^{-s}$$

under the Riemann hypothesis, by introducing generalised superior highly composite numbers. In Section 72–74, $Q_4(N)$, $Q_6(N)$ and $Q_8(N)$ the numbers of representations of N as a sum of 4, 6 *or* 8 squares are studied, and also their maximal orders. In the last paragraph

75, the number of representations of N by some other quadratic forms is considered, but no longer its maximal order. One feels that Ramanujan is ready to leave the subject of highly composite numbers, and to come back to another favourite topic, identities.

The table on p. 150 occurs on p. 280 in [18]. It should be compared with the table of largely composite numbers (p. 151), namely the numbers n such that $m \le n \Rightarrow d(m) \le d(n)$.

Several results obtained by Ramanujan in 1915, but kept unpublished, have been rediscovered and published by other mathematicians. The references for these works are given in the notes at the end of this paper. However, there remain in the paper of Ramanujan, some never published results, for instance, the maximal order of $\bar{Q}_2(N)$ (cf. Section 54) or of $\sigma_{-s}(N)$ (cf. Section 71) whenever $s \neq 1$. (The case $s = 1$ has been studied by Robin, cf. [22]).

A few misprints or mistakes were found in the manuscript of Ramanujan. Finally, it puts one somewhat at ease that even Ramanujan could make mistakes. These mistakes have been corrected in the text, but are also pointed out in the notes.

Hardy did not much like highly composite numbers. In the preface to the "Collected Works" (cf. [17], p. XXXIV) he writes that "The long memoir [16] represents work, perhaps, in a backwater of mathematics," but a few lines later, he does recognize that "it shews very clearly Ramanujan's extraordinary mastery over the algebra of inequalities". One of us can remember Freeman Dyson in Urbana (in 1987) saying that when he was a research student of Hardy, he wanted to do research on highly composite numbers but Hardy dissuaded him as he thought the subject was not sufficiently interesting or important. However, after Ramanujan, several authors have written about them, as can be seen in the survey paper [12]. We think that the manuscript of Ramanujan should be published, since he wrote it with this aim, and we hope that our notes will help readers to a better understanding.

We are indebted to Berndt, and Rankin for much valuable information, to Massias for calculating largely composite numbers and finding the meaning of the table occurring in [18], p. 280 and to Lydia Szyszko for typing this manuscript. We thank also Narosa Publishing House, New Delhi, for granting permission to print in typed form the handwritten manuscript on Highly Composite Numbers which can be found in pages 280–312 of [18].

2. The text of Ramanujan

52. *Let $Q_2(N)$ denote the number of ways in which N can be expressed as $m^2 + n^2$. Let us agree to consider $m^2 + n^2$ as two ways if m and n are unequal and as one way if they are equal or one of* them is zero. Then it can be shown that

$$\begin{aligned}(1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots)^2 \\ = 1 + 4\left(\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \frac{q^7}{1-q^7} + \cdots\right) \\ = 1 + 4\{Q_2(1)q + Q_2(2)q^2 + Q_2(3)q^3 + \cdots\} \qquad (268)\end{aligned}$$

From this it easily follows that

$$\zeta(s)\zeta_1(s) = \frac{Q_2(1)}{1^s} + \frac{Q_2(2)}{2^s} + \frac{Q_2(3)}{3^s} + \cdots, \qquad (269)$$

where

$$\zeta_1(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \cdots \quad .$$

Since

$$\frac{q}{1-q} + \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} + \cdots = d(1)q + d(2)q^2 + d(3)q^3 + \cdots,$$

it follows from (268) that

$$Q_2(N) \leq d(N) \tag{270}$$

for all values of N. Let

$$N = 2^{a_2}.3^{a_3}.5^{a_5} \cdots p^{a_p},$$

where $a_\lambda \geq 0$. Then we see that, if any one of $a_3, a_7, a_{11}, \ldots$, be odd, where 3, 7, 11, ..., are the primes of the form $4n-1$, then

$$Q_2(N) = 0. \tag{271}$$

But, if $a_3, a_7, a_{11}, \ldots$ be even or zero, then

$$Q_2(N) = (1+a_5)(1+a_{13})(1+a_{17})\cdots \tag{272}$$

where 5, 13, 17, ... are the primes of the form $4n+1$. It is clear that (270) is a consequence of (271) and (272).

53. From (272) it is easy to see that, in order that $Q_2(N)$ should be of maximum order, N must be of the form

$$5^{a_5}.13^{a_{13}}.17^{a_{17}} \cdots p^{a_p},$$

where p is a prime of the form $4n+1$, and

$$a_5 \geq a_{13} \geq a_{17} \geq \cdots \geq a_p.$$

Let $\pi_1(x)$ denote the number of primes of the form $4n+1$ which do not exceed x, and let

$$\vartheta_1(x) = \log 5 + \log 13 + \log 17 + \cdots + \log p,$$

where p is the largest prime of the form $4n+1$, not greater than x. Then by arguments similar to those of Section 33 we can show that

$$Q_2(N) \leq N^{\frac{1}{x}} \frac{2^{\pi_1(2^x)}}{e^{\frac{1}{x}\theta_1(2^x)}} \frac{\left(\frac{3}{2}\right)^{\pi_1\left((\frac{3}{2})^x\right)}}{e^{\frac{1}{x}\theta_1\left((\frac{3}{2})^x\right)}} \frac{\left(\frac{4}{3}\right)^{\pi_1\left((\frac{4}{3})^x\right)}}{e^{\frac{1}{x}\theta_1\left((\frac{4}{3})^x\right)}} \cdots \tag{273}$$

for all values of N and x. From this we can show by arguments similar to those of Section 38 that, in order that $Q_2(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta_1(2^x)+\vartheta_1\left((\frac{3}{2})^x\right)+\vartheta_1\left((\frac{4}{3})^x\right)+\cdots}$$

and $Q_2(N)$ of the form

$$2^{\pi_1(2^x)}\left(\frac{3}{2}\right)^{\pi_1\left((\frac{3}{2})^x\right)}\left(\frac{4}{3}\right)^{\pi_1\left((\frac{4}{3})^x\right)}\cdots.$$

Then, without assuming the prime number theorem, we can show that the maximum order of $Q_2(N)$ is

$$2^{\log N\{\frac{1}{\log\log N}+\frac{O(1)}{(\log\log N)^2}\}}. \tag{274}$$

Assuming the prime number theorem we can show that the maximum order of $Q_2(N)$ is

$$2^{\frac{1}{2}Li(2\log N)+O\{\log N e^{-a\sqrt{(\log N)}}\}} \tag{275}$$

where a is a positive constant.

54. We shall now assume the Riemann Hypothesis and its analogue for the function $\zeta_1(s)$. Let ρ_1 be a complex root of $\zeta_1(s)$. Then it can be shown that

$$\sum\frac{1}{\rho_1}=\frac{\gamma-3\log\pi}{2}+\log 2+4\log\Gamma\left(\frac{3}{4}\right),$$

so that

$$\sum\frac{1}{\rho}+\sum\frac{1}{\rho_1}=1+\gamma-2\log\pi+4\log\Gamma\left(\frac{3}{4}\right). \tag{276}$$

It can also be shown that

$$\begin{cases} 2\vartheta_1(x)=x-2\sqrt{x}-\sum x^{\rho}/\rho-\sum x^{\rho_1}/\rho_1+O\left(x^{\frac{1}{3}}\right)\\ 2\pi_1(x)=Li(x)-Li(\sqrt{x})-\sum Li(x^{\rho})-\sum Li(x^{\rho_1})+O\left(x^{\frac{1}{3}}\right)\end{cases} \tag{277}$$

so that

$$\begin{cases} 2\vartheta_1(x)=x+O(\sqrt{x}(\log x)^2)\\ 2\pi_1(x)=Li(x)+O(\sqrt{x}\log x).\end{cases} \tag{278}$$

Now

$$\begin{aligned}2\pi_1(x)=Li(x)&-\frac{1}{\log x}\left(2\sqrt{x}+\sum\frac{x^{\rho}}{\rho}+\sum\frac{x^{\rho_1}}{\rho_1}\right)\\ &-\frac{1}{(\log x)^2}\left(4\sqrt{x}+\sum\frac{x^{\rho}}{\rho^2}+\sum\frac{x^{\rho_1}}{\rho_1^2}\right)+\frac{O(\sqrt{x})}{(\log x)^3}.\end{aligned}$$

But by Taylor's Theorem we have

$$Li\{2\vartheta_1(x)\}=Li(x)-\frac{1}{\log x}\left(2\sqrt{x}+\sum\frac{x^{\rho}}{\rho}+\sum\frac{x^{\rho_1}}{\rho_1}\right)+O\{(\log x)^2\}.$$

Hence

$$2\pi_1(x) = Li\{2\vartheta_1(x)\} - 2R_1(x) + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\} \tag{279}$$

where

$$R_1(x) = \frac{1}{(\log x)^2}\left(2\sqrt{x} + \frac{1}{2}\sum \frac{x^\rho}{\rho^2} + \frac{1}{2}\sum \frac{x^{\rho_1}}{\rho_1^2}\right).$$

It can easily be shown that

$$\sqrt{x}\left(2 + \sum \frac{1}{\rho} + \sum \frac{1}{\rho_1}\right) \geq R_1(x)(\log x)^2 \geq \sqrt{x}\left(2 - \sum \frac{1}{\rho} - \sum \frac{1}{\rho_1}\right)$$

and so from (276) we see that

$$\begin{aligned}\left\{3 + \gamma - 2\log\pi + 4\log\Gamma\left(\frac{3}{4}\right)\right\}\sqrt{x} &\geq R_1(x)(\log x)^2 \\ &\geq \left\{1 - \gamma + 2\log\pi - 4\log\Gamma\left(\frac{3}{4}\right)\right\}\sqrt{x}.\end{aligned} \tag{280}$$

It can easily be verified that

$$\begin{cases} 3 + \gamma - 2\log\pi + 4\log\Gamma\left(\frac{3}{4}\right) = 2.101, \\ 1 - \gamma + 2\log\pi - 4\log\Gamma\left(\frac{3}{4}\right) = 1.899, \end{cases} \tag{281}$$

approximately.

Proceeding as in Section 43 we can show that the maximun order of $Q_2(N)$ is

$$2^{\frac{1}{2}Li(2\log N)+\Phi(N)} \tag{282}$$

where

$$\Phi(N) = \frac{\log\left(\frac{3}{2}\right)}{2\log 2}Li\left\{\frac{3}{2}(\log N)^{\frac{\log(3/2)}{\log 2}}\right\} - \frac{3(\log N)^{\frac{\log(3/2)}{\log 2}}}{4\log(2\log N)} - R_1(2\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.$$

55. Let $\bar{Q}_2(N)$ denote the number of ways in which N can be expressed as $m^2 + mn + n^2$. Let us agree to consider $m^2 + mn + n^2$ as two ways if m and n are unequal, and as one way if they are equal or one of them is zero. Then it can be shown that

$$\begin{aligned}&\frac{1}{2}\left(1 + 2q^{\frac{1}{4}} + 2q^{\frac{4}{4}} + 2q^{\frac{9}{4}} + \cdots\right)\left(1 + 2q^{\frac{3}{4}} + 2q^{\frac{13}{4}} + 2q^{\frac{27}{4}} + \cdots\right) \\ &\quad + \frac{1}{2}\left(1 - 2q^{\frac{1}{4}} + 2q^{\frac{4}{4}} - 2q^{\frac{9}{4}} + \cdots\right)\left(1 - 2q^{\frac{3}{4}} + 2q^{\frac{13}{4}} - 2q^{\frac{27}{4}} + \cdots\right) \\ &\quad = 1 + 6\left(\frac{q}{1-q} - \frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} + \cdots\right) \\ &\quad = 1 + 6\{\bar{Q}_2(1)q + \bar{Q}_2(2)q^2 + \bar{Q}_2(3)q^3 + \cdots\}\end{aligned} \tag{283}$$

where 1, 2, 4, 5, ... are the natural numbers without the multiples of 3. From this it follows that

$$\zeta(s)\zeta_2(s) = 1^{-s}\bar{Q}_2(1) + 2^{-s}\bar{Q}_2(2) + 3^{-s}\bar{Q}_2(3) + \cdots \tag{284}$$

where

$$\zeta_2(s) = 1^{-s} - 2^{-s} + 4^{-s} - 5^{-s} + \cdots$$

It also follows that

$$\bar{Q}_2(N) \le d(N) \tag{285}$$

for all values of N. Let

$$N = 2^{a_2}.3^{a_3}.5^{a_5}\cdots p^{a_p},$$

where $a_\lambda \ge 0$. Then, if any one of $a_2, a_5, a_{11}, \ldots$ be odd, where 2, 5, 11, ... are the primes of the form $3n-1$, then

$$\bar{Q}_2(N) = 0. \tag{286}$$

But, if a_2, a_5, a_{11} be even or zero, then

$$\bar{Q}_2(N) = (1+a_7)(1+a_{13})(1+a_{19})(1+a_{31})\cdots \tag{287}$$

where 7, 13, 19, ... are the primes of the form $6n+1$. Let $\pi_2(x)$ be the number of primes of the form $6n+1$ which do not exceed x, and let

$$\vartheta_2(x) = \log 7 + \log 13 + \log 19 + \cdots + \log p,$$

where p is the largest prime of the form $6n+1$ not greater that x. Then we can show that, in order that $\bar{Q}_2(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta_2(2^x)+\vartheta_2\left((\frac{3}{2})^x\right)+\vartheta_2\left((\frac{4}{3})^x\right)+\cdots}$$

and $\bar{Q}_2(N)$ of the form

$$2^{\pi_2(3^x)}\left(\frac{3}{2}\right)^{\pi_2\left((\frac{3}{2})^x\right)}\left(\frac{4}{3}\right)^{\pi_2\left((\frac{4}{3})^x\right)}\cdots.$$

Without assuming the prime number theorem we can show that the maximum order of $\bar{Q}_2(N)$ is

$$2^{\log N\{\frac{1}{\log\log N}+\frac{O(1)}{(\log\log N)^2}\}}. \tag{288}$$

Assuming the prime number theorem we can show that the maximum order of $\bar{Q}_2(N)$ is

$$2^{\frac{1}{2}Li(2\log N)+O\{\log N e^{-a\sqrt{(\log N)}}\}}. \tag{289}$$

56. We shall now assume the Riemann hypothesis and its analogue for the function $\zeta_2(s)$. Then we can show that

$$2\pi_2(x) = Li\{2\vartheta_2(x)\} - 2R_2(x) + O\{\sqrt{x}/(\log x)^3\} \tag{290}$$

where

$$R_2(x) = \frac{1}{(\log x)^2}\left\{2\sqrt{x} + \frac{1}{2}\sum \frac{x^\rho}{\rho^2} + \frac{1}{2}\sum \frac{x^{\rho_2}}{\rho_2^2}\right\}$$

where ρ_2 is a complex root of $\zeta_2(s)$. It can also be shown that

$$\sum \frac{1}{\rho} + \sum \frac{1}{\rho_2} = 1 + \gamma + \frac{1}{2}\log 3 + 3\log\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \tag{291}$$

and so

$$\left\{3 + \gamma + \frac{1}{2}\log 3 + 3\log\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right\}\sqrt{x} \geq R_2(x)(\log x)^2$$
$$\geq \left\{1 - \gamma - \frac{1}{2}\log 3 - 3\log\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right\}\sqrt{x}. \tag{292}$$

It can easily be verified that

$$\begin{cases} 3 + \gamma + \frac{1}{2}\log 3 + 3\log\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} = 2.080, \\ 1 - \gamma - \frac{1}{2}\log 3 - 3\log\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} = 1.920, \end{cases} \tag{293}$$

approximately. Then we can show that the maximum order of $\bar{Q}_2(N)$ is

$$2^{\frac{1}{2}Li(2\log N)+\Phi(N)} \tag{294}$$

where

$$\Phi(N) = \frac{\log(3/2)}{2\log 2}Li\left\{\frac{3}{2}(\log N)^{\frac{\log(3/2)}{\log 2}}\right\} - \frac{3(\log N)^{\frac{\log(3/2)}{\log 2}}}{4\log(2\log N)} - R_2(2\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.$$

57. Let $d_r(N)$ denote the coefficient of N^{-s} in the expansion of $\{\zeta(s)\}^{1+r}$ as a Dirichlet series. Then since

$$\{\zeta(s)\}^{-1} = (1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{-s})\ldots,$$

it is easy to see that, if

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_n^{a_n},$$

where $p_1, p_2, p_3 \dots$ are any primes, then

$$d_r(N) = \prod_{\nu=1}^{\nu=n} \prod_{\lambda=1}^{\lambda=a_\nu} \left(1 + \frac{r}{\lambda}\right) \tag{295}$$

provided that $r > -1$. It is evident that

$$d_{-1}(N) = 0, \quad d_0(N) = 1, \quad d_1(N) = d(N);$$

and that, if $-1 \leq r \leq 0$, then

$$d_r(N) \leq 1 + r \tag{296}$$

for all values of N. It is also evident that, if N is a prime then

$$d_r(N) = 1 + r$$

for all values of r. It is easy to see from (295) that, if $r > 0$, then $d_r(N)$ is not bounded when N becomes infinite. Now, if r is positive, it can easily be shown that, in order that $d_r(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots},$$

and consequently $d_r(N)$ of the form

$$(1+r)^{\pi(x_1)}\left(1+\frac{r}{2}\right)^{\pi(x_2)}\left(1+\frac{r}{3}\right)^{\pi(x_3)}\dots$$

and proceeding as in Section 46 we can show that N must be of the form

$$e^{\vartheta(1+r)^x+\vartheta(1+\frac{r}{2})^x+\vartheta(1+\frac{r}{3})^x+\cdots} \tag{297}$$

and $d_r(N)$ of the form

$$(1+r)^{\pi\left((1+r)^x\right)}\left(1+\frac{r}{2}\right)^{\pi\left((1+\frac{r}{2})^x\right)}\left(1+\frac{r}{3}\right)^{\pi\left((1+\frac{r}{3})^x\right)}\dots \tag{298}$$

From (297) and (298) we can easily find the maximum order of $d_r(N)$ as in Section 43. It may be interesting to note that numbers of the form (297) which may also be written in the form

$$e^{\vartheta\{x^{\frac{1}{r}\log(1+r)}\}+\vartheta\{x^{\frac{1}{r}\log(1+\frac{r}{2})}\}+\vartheta\{x^{\frac{1}{r}\log(1+\frac{r}{3})}\}+\cdots}$$

approach the form

$$e^{\vartheta(x)+\vartheta(\sqrt{x})+\vartheta(x^{1/3})+\cdots}$$

as $r \to 0$. That is to say, they approach the form of the least common multiple of the natural numbers as $r \to 0$.

58. *Let s be a non negative real number, and let $\sigma_{-s}(N)$ denote the sum of the inverses of the sth powers of the divisors of N. If N denotes*

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_n^{a_n}$$

where $p_1, p_2, p_3, \dots$ are any primes, then

$$\begin{aligned}\sigma_{-s}(N) = &\left(1 + p_1^{-s} + p_1^{-2s} + p_1^{-3s} + \cdots + p_1^{-a_1 s}\right)\\ &\left(1 + p_2^{-s} + p_2^{-2s} + p_2^{-3s} + \cdots + p_2^{-a_2 s}\right)\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ &\left(1 + p_n^{-s} + p_n^{-2s} + p_n^{-3s} + \cdots + p_n^{-a_n s}\right).\end{aligned}$$

For $s = 0$, $\sigma_0(N) = d(N)$ is the number of divisors of N. For $s > 0$, we have:

$$\sigma_{-s}(N) = \left(\frac{1 - p_1^{-(a_1+1)s}}{1 - p_1^{-s}}\right)\left(\frac{1 - p_2^{-(a_2+1)s}}{1 - p_2^{-s}}\right)\cdots\left(\frac{1 - p_n^{-(a_n+1)s}}{1 - p_n^{-s}}\right). \tag{299}$$

Now, from the concavity of the function $\log(1 - e^{-t})$, *we see that*

$$\begin{aligned}&\frac{1}{n}\{\log(1 - e^{-t_1}) + \log(1 - e^{-t_2}) + \cdots + \log(1 - e^{-t_n})\}\\ &\quad \leq \log\left\{1 - \exp\left(- \frac{t_1 + t_2 + \cdots + t_n}{n}\right)\right\}.\end{aligned} \tag{300}$$

Choosing $t_1 = (a_1 + 1)s \log p_1, t_2 = (a_2 + 1)s \log p_2, \dots, t_n = (a_n + 1)s \log p_n$ *in* (300), *formula* (299) *gives*

$$\sigma_{-s}(N) < \frac{\left\{1 - (p_1 p_2 p_3 \cdots p_n N)^{-s/n}\right\}^n}{\left(1 - p_1^{-s}\right)\left(1 - p_2^{-s}\right)\cdots\left(1 - p_n^{-s}\right)}. \tag{301}$$

By arguments similar to those of Section 2 we can show that it is possible to choose the indices $a_1, a_2, a_3, \dots, a_n$ so that

$$\sigma_{-s}(N) = \frac{\{1 - (p_1 p_2 p_3 \cdots p_n N)^{-s/n}\}^n}{\left(1 - p_1^{-s}\right)\left(1 - p_2^{-s}\right)\cdots\left(1 - p_n^{-s}\right)}\{1 - O\{N^{-s/n}(\log N)^{-2/(n-1)}\}\}. \tag{302}$$

There are of course results corresponding to (14) and (15) also.

59. A number N may be said to be a generalised highly composite number if $\sigma_{-s}(N) > \sigma_{-s}(N')$ for all values of N' less than N. We can easily show that, in order that N should be a generalised highly composite number, N must be the form

$$2^{a_2} 3^{a_3} 5^{a_5} \cdots p^{a_p} \tag{303}$$

where

$$a_2 \geq a_3 \geq a_5 \geq \cdots \geq a_p = 1,$$

the exceptional numbers being 36, for the values of s which satisfy the inequality $2^s + 4^s + 8^s > 3^s + 9^s$, and 4 in all cases.

A number N may be said to be a generalised superior highly composite number if there is a positive number ε such that

$$\frac{\sigma_{-s}(N)}{N^{\varepsilon}} \geq \frac{\sigma_{-s}(N')}{(N')^{\varepsilon}} \tag{304}$$

for all values of N' less than N, and

$$\frac{\sigma_{-s}(N)}{N^{\varepsilon}} > \frac{\sigma_{-s}(N')}{(N')^{\varepsilon}} \tag{305}$$

for all values of N' greater than N. It is easily seen that all generalised superior highly composite numbers are generalised highly composite numbers. We shall use the expression

$$2^{a_2} 3^{a_3} 5^{a_5} \cdots p_1^{a_{p_1}}$$

and the expression

$$\begin{array}{lllllllll} & 2 & \cdot\ 3 & \cdot\ 5 & . & 7 & \dots & \dots & p_1 \\ \times & 2 & \cdot\ 3 & \cdot\ 5 & \dots & \dots & p_2 & & \\ \times & 2 & \cdot\ 3 & \cdot\ 5 & \dots & p_3 & & & \\ \times & \dots & & & & & & & \\ \vdots & & & & & & & & \end{array}$$

as the standard forms of a generalised superior highly composite number.

60. Let

$$N' = \frac{N}{\lambda}$$

where $\lambda \leq p_1$. Then from (304) it follows that

$$1 - \lambda^{-s(1+a_\lambda)} \geq (1 - \lambda^{-sa_\lambda})\lambda^{\varepsilon},$$

or

$$\lambda^{\varepsilon} \leq \frac{1 - \lambda^{-s(1+a_\lambda)}}{1 - \lambda^{-sa_\lambda}}. \tag{306}$$

Again let $N' = N\lambda$. Then from (305) we see that

$$1 - \lambda^{-s(1+a_\lambda)} > \left\{1 - \lambda^{-s(2+a_\lambda)}\right\}\lambda^{-\varepsilon}$$

or

$$\lambda^{\varepsilon} > \frac{1-\lambda^{-s(2+a_\lambda)}}{1-\lambda^{-s(1+a_\lambda)}}. \tag{307}$$

Now let us suppose that $\lambda = p_1$, in (306) and $\lambda = P_1$ in (307). Then we see that

$$\frac{\log\left(1+p_1^{-s}\right)}{\log p_1} \geq \varepsilon > \frac{\log\left(1+P_1^{-s}\right)}{\log P_1}. \tag{308}$$

From this it follows that, if

$$0 < \varepsilon \leq \frac{\log(1+2^{-s})}{\log 2},$$

then there is a unique value of p_1 corresponding to each value of ε. It follows from (306) that

$$a_\lambda \leq \frac{\log\left(\frac{\lambda^{\varepsilon}-\lambda^{-s}}{\lambda^{\varepsilon}-1}\right)}{s\log\lambda}, \tag{309}$$

and from (307) that

$$1+a_\lambda > \frac{\log\left(\frac{\lambda^{\varepsilon}-\lambda^{-s}}{\lambda^{\varepsilon}-1}\right)}{s\log\lambda}. \tag{310}$$

From (309) and (310) it is clear that

$$a_\lambda = \left[\frac{\log\left(\frac{\lambda^{\varepsilon}-\lambda^{-s}}{\lambda^{\varepsilon}-1}\right)}{s\log\lambda}\right]. \tag{311}$$

Hence N is of the form

$$2^{\left[\frac{\log\left(\frac{2^{\varepsilon}-2^{-s}}{2^{\varepsilon}-1}\right)}{s\log 2}\right]} 3^{\left[\frac{\log\left(\frac{3^{\varepsilon}-3^{-s}}{3^{\varepsilon}-1}\right)}{s\log 3}\right]} \ldots\ldots\ldots\ldots\ldots\ldots p_1 \tag{312}$$

where p_1 is the prime defined by the inequalities (308).

61. Let us consider the nature of p_r. Putting $\lambda = p_r$ in (306), and remembering that $a_{p_r} \geq r$, we obtain

$$p_r^{\varepsilon} \leq \frac{1-p_r^{-s(1+a_{p_r})}}{1-p_r^{-sa_{p_r}}} \leq \frac{1-p_r^{-s(r+1)}}{1-p_r^{-sr}}. \tag{313}$$

Again, putting $\lambda = P_r$ in (307), and remembering that $a_{P_r} \leq r-1$, we obtain

$$P_r^{\varepsilon} > \frac{1-P_r^{-s(2+a_{P_r})}}{1-P_r^{-s(1+a_{P_r})}} \geq \frac{1-P_r^{-s(r+1)}}{1-P_r^{-sr}}. \tag{314}$$

It follows from (313) and (314) that, if x_r be the value of x satisfying the equation

$$x^{\varepsilon} = \frac{1 - x^{-s(r+1)}}{1 - x^{-sr}} \tag{315}$$

then p_r is the largest prime not greater than x_r. Hence N is of the form

$$e^{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots} \tag{316}$$

where x_r is defined in (315); and $\sigma_{-s}(N)$ is of the form

$$\Pi_1(x_1)\Pi_2(x_2)\Pi_3(x_3)\cdots\Pi_{a_2}(x_{a_2}) \tag{317}$$

where

$$\Pi_r(x) = \frac{1 - 2^{-s(r+1)}}{1 - 2^{-sr}}\,\frac{1 - 3^{-s(r+1)}}{1 - 3^{-sr}}\cdots\frac{1 - p^{-s(r+1)}}{1 - p^{-sr}}.$$

and p is the largest prime not greater than x. It follows from (304) and (305) that

$$\sigma_{-s}(N) \leq N^{\varepsilon}\,\frac{\Pi_1(x_1)}{e^{\varepsilon\vartheta(x_1)}}\,\frac{\Pi_2(x_2)}{e^{\varepsilon\vartheta(x_2)}}\,\frac{\Pi_3(x_3)}{e^{\varepsilon\vartheta(x_3)}}\cdots \tag{318}$$

for all values of N, where $x_1, x_2, x_3, \ldots$ are functions of ε defined by the equation

$$x_r^{\varepsilon} = \frac{1 - x_r^{-s(r+1)}}{1 - x_r^{-sr}}, \tag{319}$$

and $\sigma_{-s}(N)$ is equal to the right hand side of (318) when

$$N = e^{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots}$$

62. In (16) let us suppose that

$$\Phi(x) = \log\frac{1 - x^{-s(r+1)}}{1 - x^{-sr}}.$$

Then we see that

$$\begin{aligned}
\log\Pi_r(x_r) &= \pi(x_r)\log\frac{1 - x_r^{-s(r+1)}}{1 - x_r^{-sr}} - \int\pi(x_r)d\left(\log\frac{1 - x_r^{-s(r+1)}}{1 - x_r^{-sr}}\right)\\
&= \pi(x_r)\log\left(x_r^{\varepsilon}\right) - \int\pi(x_r)d\left(\log x_r^{\varepsilon}\right)\\
&= \varepsilon\pi(x_r)\log x_r - \int\pi(x_r)\log x_r d\varepsilon - \int\frac{\varepsilon\pi(x_r)}{x_r}\,dx_r
\end{aligned}$$

in virtue of (319). Hence

$$\begin{aligned}\log \Pi_r(x_r) - \varepsilon\vartheta(x_r) &= \varepsilon\{\pi(x_r)\log x_r - \vartheta(x_r)\} - \int \pi(x_r)\log x_r d\varepsilon - \int \frac{\varepsilon\pi(x_r)}{x_r}dx_r \\ &= \varepsilon \int \frac{\pi(x_r)}{x_r}dx_r - \int \pi(x_r)\log x_r d\varepsilon - \int \frac{\varepsilon\pi(x_r)}{x_r}dx_r \\ &= \int d\varepsilon \int \frac{\pi(x_r)}{x_r}dx_r - \int \pi(x_r)\log x_r d\varepsilon \\ &= \int \left\{ \int \frac{\pi(x_r)}{x_r}dx_r - \pi(x_r)\log x_r \right\} d\varepsilon \\ &= -\int \vartheta(x_r)\, d\varepsilon. \end{aligned} \tag{320}$$

It follows from (318) and (320) that

$$\sigma_{-s}(N) \leq N^{\varepsilon} e^{-\int \{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots\}d\varepsilon} \tag{321}$$

for all values of N. By arguments similar to those of Section 38 we can show that the right hand side of (321) is a minimum when ε is a function of N defined by the equation

$$N = e^{\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots}. \tag{322}$$

Now let $\sum_{-s}(N)$ be a function of N defined by the equation

$$\sum\nolimits_{-s}(N) = \Pi_1(x_1)\Pi_2(x_2)\Pi_3(x_3)\cdot \tag{323}$$

where ε is a function of N defined by the Eq. (322). Then it follows from (318) that the order of

$$\sigma_{-s}(N) \leq \sum\nolimits_{-s}(N)$$

for all values of N and $\sigma_{-s}(N) = \sum_{-s}(N)$ for all generalised superior highly composite values of N. In other words $\sigma_{-s}(N)$ is of maximum order when N is of the form of a generalised superior highly composite number.

63. We shall now consider some important series which are not only useful in finding the maximum order of $\sigma_{-s}(N)$ but also interesting in themselves. Proceeding as in (16) we can easily show that, if $\Phi'(x)$ be continuous, then

$$\begin{aligned}&\Phi(2)\log 2 + \Phi(3)\log 3 + \Phi(5)\log 5 + \cdots + \Phi(p)\log p \\ &\quad = \Phi(x)\,\theta(x) - \int_2^x \Phi'(t)\theta(t)\,dt \end{aligned} \tag{324}$$

where p is the largest prime not exceeding x. Since $\int\Phi(x)\,dx = x\Phi(x) - \int x\Phi'(x)\,dx$, we have

$$\begin{aligned}\Phi(x)\vartheta(x) - \int \Phi'(x)\vartheta(x)\,dx &= \int \Phi(x)\,dx - \{x-\vartheta(x)\}\Phi(x) \\ &\quad + \int \Phi'(x)\{x-\vartheta(x)\}\,dx. \end{aligned} \tag{325}$$

Remembering that $x-\vartheta(x)=O\{\sqrt{x}(\log x)^2\}$, we have by Taylor's Theorem

$$\int^{\theta(x)}\Phi(t)\,dt=\int\Phi(x)\,dx-\{x-\vartheta(x)\}\Phi(x)+\frac{1}{2}\{x-\vartheta(x)\}^2\Phi'\{x+O(\sqrt{x}(\log x)^2)\}. \tag{326}$$

It follows from (324)–(326) that

$$\begin{aligned}\Phi(2)\log 2+\Phi(3)\log 3+\Phi(5)\log 5+\cdots+\Phi(p)\log p\\ =C+\int^{\theta(x)}\Phi(t)\,dt+\int\Phi'(x)\{x-\vartheta(x)\}\,dx\\ -\frac{1}{2}\{x-\vartheta(x)\}^2\Phi'\{x+O\sqrt{x}(\log x)^2\}\end{aligned} \tag{327}$$

where C is a constant and p is the largest prime not exceeding x.

64. Now let us assume that $\Phi(x)=\frac{1}{x^s-1}$ where $s>0$. Then from (327) we see that, if p is the largest prime not greater than x, then

$$\begin{aligned}\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1}\\ =C+\int^{\theta(x)}\frac{dx}{x^s-1}-s\int\frac{x-\vartheta(x)}{x^{1-s}(x^s-1)^2}\,dx+O\{x^{-s}(\log x)^4\}.\end{aligned} \tag{328}$$

But it is known that

$$x-\theta(x)=\sqrt{x}+x^{\frac{1}{3}}+\sum\frac{x^\rho}{\rho}-\sum\frac{x^{\frac{1}{2}\rho}}{\rho}+O\big(x^{\frac{1}{5}}\big) \tag{329}$$

where ρ is a complex root of $\zeta(s)$. By arguments similar to those of Section 42 we can show that

$$\sum\frac{x^{\frac{1}{2}\rho-s}}{\rho\big(\frac{1}{2}\rho-s\big)}=\int x^{-1-s}\sum\frac{x^{\frac{1}{2}\rho}}{\rho}\,dx.$$

Hence

$$\int\frac{\sum\frac{x^{\frac{1}{2}\rho}}{\rho}}{x^{1-s}(x^s-1)^2}\,dx=\int O\left\{x^{-1-s}\sum\frac{x^{\frac{1}{2}\rho}}{\rho}\right\}dx=O\left\{\sum\frac{x^{\frac{1}{2}\rho-s}}{\rho\big(\frac{1}{2}\rho-s\big)}\right\}=O\big(x^{\frac{1}{4}-s}\big).$$

Similarly

$$\int\frac{\sum\frac{x^\rho}{\rho}}{x^{1-s}(x^s-1)^2}\,dx=\sum\frac{x^{\rho-s}}{\rho(\rho-s)}+O\left(\sum\frac{x^{\rho-2s}}{\rho(\rho-2s)}\right)=\sum\frac{x^{\rho-s}}{\rho(\rho-s)}+O\big(x^{\frac{1}{2}-2s}\big).$$

Hence (328) may be replaced by

$$\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1}$$
$$= C+\int^{\theta(x)}\frac{dt}{t^s-1}-s\int\frac{x^{\frac12}+x^{\frac13}}{x^{1-s}(x^s-1)^2}\,dx$$
$$-s\sum\frac{x^{\rho-s}}{\rho(\rho-s)}+O\left(x^{\frac12-2s}+x^{\frac14-s}\right). \tag{330}$$

It can easily be shown that

$$C=-\frac{\zeta'(s)}{\zeta(s)} \tag{331}$$

when the error term is $o(1)$.

65. Let

$$S_s(x)=-s\sum\frac{x^{\rho-s}}{\rho(\rho-s)}.$$

Then

$$|S_s(x)|\le s\sum\left|\frac{x^{\rho-s}}{\rho(\rho-s)}\right|=s\cdot x^{\frac12-s}\sum\frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}}. \tag{332}$$

If m and n are any two positive numbers, then it is evident that $1/\sqrt{mn}$ lies between $\frac{1}{m}$ and $\frac{1}{n}$.

Hence $\sum\frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}}$ lies between $\chi(1)$ and $\chi(s)$ where

$$\chi(s)=\sum\frac{1}{(\rho-s)(1-\rho-s)}=\sum\frac{1}{\rho(1-\rho)+s^2-s}$$
$$=\frac{1}{1-2s}\left(\sum\frac{1}{\rho-s}+\sum\frac{1}{1-\rho-s}\right)=\sum\frac{\frac{1}{s-\rho}}{s-1/2}. \tag{333}$$

We can show as in Section 41 that

$$\sum\frac{1}{s-\rho}=\frac{2s-1}{s^2-s}-\frac12\log\pi+\frac12\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+\frac{\zeta'(s)}{\zeta(s)}. \tag{334}$$

Hence

$$\chi(s)=\frac{2}{s^2-s}+\frac{1}{2s-1}\left\{\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+2\frac{\zeta'(s)}{\zeta(s)}-\log\pi\right\} \tag{335}$$

so that

$$\chi(0)=\chi(1)=2+\gamma-\log 4\pi. \tag{336}$$

By elementary algebra, it can easily be shown that if m_r and n_r be not negative and G_r be the geometric mean between m_r and n_r then

$$G_1 + G_2 + G_3 + \cdots < \sqrt{\{m_1 + m_2 + m_3 + \cdots\}\{n_1 + n_2 + \cdots\}} \tag{337}$$

unless $\frac{m_1}{n_1} = \frac{m_2}{n_2} = \frac{m_3}{n_3} = \cdots$

From this it follows that

$$\sum \frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}} < \sqrt{\{\chi(1)\chi(s)\}}. \tag{338}$$

The following method leads to still closer approximation. It is easy to see that if m and n are positive, then $1/\sqrt{mn}$ is the geometric mean between

$$\frac{1}{3m} + \frac{8}{3(m+3n)} \quad \text{and} \quad \frac{1}{3n} + \frac{8}{3(3m+n)} \tag{339}$$

and so $\frac{1}{\sqrt{mn}}$ lies between both. Hence

$$\begin{aligned} &\sum \frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}} \quad \text{lies between} \\ &\frac{1}{3}\sum \frac{1}{\rho(1-\rho)} + \frac{2}{3}\sum \frac{1}{\rho(1-\rho) + \frac{3}{4}(s^2-s)} \quad \text{and} \\ &\frac{1}{3}\sum \frac{1}{(\rho-s)(1-\rho-s)} + \frac{2}{3}\sum \frac{1}{\rho(1-\rho) + \frac{1}{4}(s^2-s)} \end{aligned} \tag{340}$$

and is also less than the geometric mean[1] between these two in virtue of (337)

$$\sum \frac{1}{\rho(1-\rho) + \frac{1}{4}(s^2-s)} = \chi\left\{\frac{1+\sqrt{(1-s+s^2)}}{2}\right\} \quad \text{and}$$

$$\sum \frac{1}{\rho(1-\rho) + \frac{3}{4}(s^2-s)} = \chi\left\{\frac{1+\sqrt{(1-3s+3s^2)}}{2}\right\}$$

1.

$$\frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}} = \frac{1}{\rho(1-\rho)} - \frac{1}{2}\frac{s^2-s}{\rho(1-\rho)} + \frac{3}{8}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^2 - \frac{10}{32}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^3 + \cdots$$

$$\frac{1}{3}\frac{1}{\rho(1-\rho)} + \frac{2}{3}\frac{1}{\rho(1-\rho) + \frac{3}{4}(s^2-s)} = \frac{1}{\rho(1-\rho)} - \frac{1}{2}\frac{s^2-s}{\rho(1-\rho)} + \frac{3}{8}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^2 - \frac{9}{32}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^3 + \cdots$$

$$\frac{1}{3}\frac{1}{\rho(1-\rho)+s^2-s} + \frac{2}{3}\frac{1}{\rho(1-\rho) + \frac{1}{4}(s^2-s)} = \frac{1}{\rho(1-\rho)} - \frac{1}{2}\frac{s^2-s}{\rho(1-\rho)} + \frac{3}{8}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^2 - \frac{11}{32}\left\{\frac{s^2-s}{\rho(1-\rho)}\right\}^3 + \cdots$$

Since the first value of $\rho(1-\rho)$ is about 200 we see that the geometric mean is a much closer approximation than either.

Hence

$$\sum \frac{1}{\sqrt{\{\rho(1-\rho)(\rho-s)(1-\rho-s)\}}} \quad \text{lies between}$$
$$\frac{1}{3}\chi(1) + \frac{2}{3}\chi\left\{\frac{1+\sqrt{(1-3s+3s^2)}}{2}\right\} \quad \text{and}$$
$$\frac{1}{3}\chi(s) + \frac{2}{3}\chi\left\{\frac{1+\sqrt{(1-s+s^2)}}{2}\right\} \tag{341}$$

and is also less than the geometric mean between these two.

66. In this and the following few sections it is always understood that p is the largest prime not greater than x. It can easily be shown that

$$\int^{\theta(x)} \frac{dt}{t^s-1} - s\int \frac{x^{\frac{1}{2}}+x^{\frac{1}{3}}}{x^{1-s}(x^s-1)^2}\,dx = \frac{\{\theta(x)\}^{1-s}}{1-s} + \frac{\{\theta(x)\}^{1-2s}}{1-2s}$$
$$+ \frac{x^{1-3s}}{1-3s} + \frac{x^{1-4s}}{1-4s} + \cdots + \frac{x^{1-ns}}{1-ns}$$
$$- \frac{2sx^{\frac{1}{2}-s}}{1-2s} - \frac{3sx^{\frac{1}{3}-s}}{1-3s} - \frac{4sx^{\frac{1}{2}-2s}}{1-4s} + O\left(x^{\frac{1}{2}-2s}\right) \tag{342}$$

where $n = [2 + \frac{1}{2s}]$.

It follows from (330) and (342) that if $s > 0$, then

$$\frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1}$$
$$= -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{\{\vartheta(x)\}^{1-2s}}{1-2s} + \frac{x^{1-3s}}{1-3s} + \frac{x^{1-4s}}{1-4s} + \cdots + \frac{x^{1-ns}}{1-ns}$$
$$- \frac{2sx^{\frac{1}{2}-s}}{1-2s} - \frac{3sx^{\frac{1}{3}-s}}{1-3s} - \frac{4sx^{\frac{1}{2}-2s}}{1-4s} + S_s(x) + O\left(x^{\frac{1}{2}-2s} + x^{\frac{1}{4}-s}\right) \tag{343}$$

where $n = [2 + \frac{1}{2s}]$.

When $s = 1, \frac{1}{2}, \frac{1}{3}$ or $\frac{1}{4}$ we must take the limit of the right hand side when s approaches $1, \frac{1}{2}, \frac{1}{3}$ *or* $\frac{1}{4}$. We shall consider the following cases:

Case I. $0 < s < \frac{1}{4}$

$$\frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1}$$
$$= \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{\{\vartheta(x)\}^{1-2s}}{1-2s} + \frac{x^{1-3s}}{1-3s} + \frac{x^{1-4s}}{1-4s} + \cdots$$
$$+ \frac{x^{1-ns}}{1-ns} - \frac{2sx^{\frac{1}{2}-s}}{1-2s} - \frac{3sx^{\frac{1}{3}-s}}{1-3s} + S_s(x) + O\left(x^{\frac{1}{2}-2s}\right), \tag{344}$$

where $n = [2 + \frac{1}{2s}]$.

Case II. $s = \frac{1}{4}$

$$\frac{\log 2}{2^{\frac{1}{4}-1}} + \frac{\log 3}{3^{\frac{1}{4}-1}} + \frac{\log 5}{5^{\frac{1}{4}-1}} + \cdots + \frac{\log p}{p^{\frac{1}{4}-1}}$$
$$= \frac{4}{3}\{\vartheta(x)\}^{\frac{3}{4}} + 2\sqrt{\{\vartheta(x)\}} + 3x^{\frac{1}{4}} - 3x^{\frac{1}{12}} + \frac{1}{2}\log x + S_{\frac{1}{4}}(x) + O(1). \quad (345)$$

Case III. $s > \frac{1}{4}$

$$\frac{\log 2}{2^s - 1} + \frac{\log 3}{3^s - 1} + \frac{\log 5}{5^s - 1} + \cdots + \frac{\log p}{p^s - 1}$$
$$= -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{x^{1-2s} - 2sx^{\frac{1}{2}-s}}{1-2s} + \frac{x^{1-3s} - 3sx^{\frac{1}{3}-s}}{1-3s} + S_s(x) + O\left(x^{\frac{1}{4}-s}\right). \quad (346)$$

67. Making $s \to 1$ in (346), and remembering that

$$\lim_{s\to 1}\left\{\frac{v^{1-s}}{1-s} - \frac{\zeta'(s)}{\zeta(s)}\right\} = \log v - \gamma$$

where γ is the Eulerian constant, we have

$$\frac{\log 2}{2-1} + \frac{\log 3}{3-1} + \frac{\log 5}{5-1} + \cdots + \frac{\log p}{p-1}$$
$$= \log\vartheta(x) - \gamma + 2x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{2}{3}} + S_1(x) + O\left(x^{-\frac{3}{4}}\right). \quad (347)$$

From (332) we know that

$$\sqrt{x}\ |S_1(x)| \le 2 + \gamma - \log(4\pi) = .046\cdots \quad (348)$$

approximately, for all positive values of x.

When $s > 1$, (346) reduces to

$$\frac{\log 2}{2^s - 1} + \frac{\log 3}{3^s - 1} + \frac{\log 5}{5^s - 1} + \cdots + \frac{\log p}{p^s - 1}$$
$$= -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{2sx^{\frac{1}{2}-s}}{2s-1} + \frac{3sx^{\frac{1}{3}-s}}{3s-1} + S_s(x) + O\left(x^{\frac{1}{4}-s}\right) \quad (349)$$

Writing $O(x^{\frac{1}{2}-s})$ for $S_s(x)$ in (343), we see that, if $s > 0$, then

$$\frac{\log 2}{2^s - 1} + \frac{\log 3}{3^s - 1} + \frac{\log 5}{5^s - 1} + \cdots + \frac{\log p}{p^s - 1}$$
$$= -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{x^{1-2s}}{1-2s} + \frac{x^{1-3s}}{1-3s} + \cdots$$
$$+ \frac{x^{1-ns}}{1-ns} - \frac{2sx^{\frac{1}{2}-s}}{1-2s} + O\left(x^{\frac{1}{2}-s}\right) \quad (350)$$

when $n = [1 + \frac{1}{2s}]$.

Now the following three cases arise:

Case I. $0 < s < \frac{1}{2}$

$$\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1} = \frac{\{\vartheta(x)\}^{1-s}}{1-s}+\frac{x^{1-2s}}{1-2s}+\frac{x^{1-3s}}{1-3s}+\cdots+\frac{x^{1-ns}}{1-ns}+O\left(x^{\frac{1}{2}-s}\right) \tag{351}$$

where $n = [1+\frac{1}{2s}]$.
Case II. $s = \frac{1}{2}$

$$\frac{\log 2}{\sqrt{2}-1}+\frac{\log 3}{\sqrt{3}-1}+\frac{\log 5}{\sqrt{5}-1}+\cdots+\frac{\log p}{\sqrt{p}-1} = 2\sqrt{\{\vartheta(x)\}}+\frac{1}{2}\log x+O(1). \tag{352}$$

Case III. $s > \frac{1}{2}$

$$\frac{\log 2}{2^s-1}+\frac{\log 3}{3^s-1}+\frac{\log 5}{5^s-1}+\cdots+\frac{\log p}{p^s-1} = -\frac{\zeta'(s)}{\zeta(s)}+\frac{\{\vartheta(x)\}^{1-s}}{1-s}+O\left(x^{\frac{1}{2}-s}\right). \tag{353}$$

68. We shall now consider the product

$$(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{-s}).$$

It can easily be shown that

$$\int \frac{x^{a+bs}}{a+bs}\,ds = \frac{1}{b}Li(x^{a+bs}) \tag{354}$$

where $Li(x)$ is the principal value of $\int_0^x \frac{dt}{\log t}$; and that

$$\int S_s(x)\,ds = -\frac{S_s(x)}{\log x}+O\left\{\frac{x^{\frac{1}{2}-s}}{(\log x)^2}\right\}. \tag{355}$$

Now remembering (354) and (355) and integrating (343) with respect to s, we see that if $s > 0$, then

$$\begin{aligned}\log\{(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{-s})\} &= -\log\,|\zeta(s)|\, - Li\{\vartheta(x)\}^{1-s} - \frac{1}{2}Li(x^{1-2s}) - \frac{1}{3}Li(x^{1-3s}) - \cdots \\ &\quad - \frac{1}{n}Li(x^{1-ns}) + \frac{1}{2}Li\left(x^{\frac{1}{2}-s}\right) - \frac{x^{\frac{1}{2}-s}+S_s(x)}{\log x} + O\left\{\frac{x^{\frac{1}{2}-s}}{(\log x)^2}\right\}\end{aligned} \tag{356}$$

where $n = [1+\frac{1}{2s}]$.

Now the following three cases arise.

Case I. $0 < s < \frac{1}{2}$

$$\begin{aligned}
&\log\{(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{s})\} \\
&= -Li\{\vartheta(x)\}^{1-s} - \frac{1}{2}Li(x^{1-2s}) - \frac{1}{3}Li(x^{1-3s}) - \cdots \\
&\quad - \frac{1}{n}Li(x^{1-ns}) + \frac{2sx^{\frac{1}{2}-s}}{(1-2s)\log x} - \frac{S_s(x)}{\log x} + O\left\{x^{\frac{1}{2}-s}/(\log x)^2\right\} \qquad (357)
\end{aligned}$$

where $n = [1 + \frac{1}{2s}]$. Making $s \to \frac{1}{2}$ in (356) and remembering that

$$\lim_{h\to 0}\{Li(1+h) - \log\ |h|\ \} = \gamma \qquad (358)$$

where γ is the Eulerian constant, we have
Case II. $s = \frac{1}{2}$

$$\begin{aligned}
&\frac{1}{\left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{5}}\right)\cdots\left(1-\frac{1}{\sqrt{p}}\right)} \\
&= -\sqrt{2}\zeta\left(\frac{1}{2}\right)\exp\left\{Li\left(\sqrt{\theta(x)}\right) + \frac{1+S_{\frac{1}{2}}(x)}{\log x} + \frac{O(1)}{(\log x)^2}\right\}. \qquad (359)
\end{aligned}$$

It may be observed that

$$-(\sqrt{2}-1)\zeta\left(\frac{1}{2}\right) = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots. \qquad (360)$$

Case III. $s > \frac{1}{2}$

$$\begin{aligned}
&\frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{-s})} \\
&= |\zeta(s)|\exp\left[Li\{(\theta(x))^{1-s}\} + \frac{2sx^{\frac{1}{2}-s}}{(2s-1)\log x} + \frac{S_s(x)}{\log x} + O\left\{\frac{x^{\frac{1}{2}-s}}{(\log x)^2}\right\}\right]. \qquad (361)
\end{aligned}$$

Remembering (358) and making $s \to 1$ in (361) we obtain

$$\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{p}\right)} = e^{\gamma}\left\{\log\vartheta(x) + \frac{2}{\sqrt{x}} + S_1(x) + \frac{O(1)}{\sqrt{x}\log x}\right\}. \qquad (362)$$

It follows from this and (347) that

$$\begin{aligned}
&\frac{e^{-\gamma}}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{p}\right)} \\
&= \gamma + \frac{\log 2}{2-1} + \frac{\log 3}{3-1} + \cdots + \frac{\log p}{p-1} + O\left(\frac{1}{\sqrt{p}\log p}\right). \qquad (363)
\end{aligned}$$

69. We shall consider the order of x_r. Putting $r = 1$ in (319) we have

$$\varepsilon = \frac{\log\left(1 + x_1^{-s}\right)}{\log x_1};$$

and so

$$x_r^{\frac{\log(1+x_1^{-s})}{\log x_1}} = \frac{1 - x_r^{-s(r+1)}}{1 - x_r^{-sr}}. \tag{364}$$

Let

$$x_r = x_1^{t_r/r}.$$

Then we have

$$\left(1 + x_1^{-s}\right)^{t_r/r} = \frac{1 - x_1^{-st_r(1+\frac{1}{r})}}{1 - x_1^{-st_r}}.$$

From this we can easily deduce that

$$t_r = 1 + \frac{\log r}{s \log x_1} + O\left\{\frac{1}{(\log x_1)^2}\right\}.$$

Hence

$$x_r = x_1^{1/r}\left\{r^{1/(rs)} + O\left(\frac{1}{\log x_1}\right)\right\}; \tag{365}$$

and so

$$x_r \sim \left(r^{1/s} x_1\right)^{1/r}. \tag{366}$$

Putting $\lambda = 2$ in (311) we see that the greatest possible value of r is

$$a_2 = \frac{\log \frac{1}{\varepsilon}}{s \log 2} + O(1) = \frac{\log x_1}{\log 2} + \frac{\log\log x_1}{s \log 2} + O(1). \tag{367}$$

Again

$$\log N = \vartheta(x_1) + \vartheta(x_2) + \vartheta(x_3) + \cdots = \vartheta(x_1) + x_2 + O\left(x_1^{1/3}\right) \tag{368}$$

in virtue of (366). It follows from Section 68 and the definition of $\Pi_r(x)$, that, if sr and $s(r+1)$ are not equal to 1, then

$$\Pi_r(x) = \left|\frac{\zeta(sr)}{\zeta\{s(r+1)\}}\right| e^{O(x^{1-sr})};$$

and consequently

$$\Pi_r(x_r) = \left|\frac{\zeta(sr)}{\zeta\{s(r+1)\}}\right| e^{O\left(x_1^{\frac{1}{r}-s}\right)} \tag{369}$$

in virtue of (366). But if sr or $s(r+1)$ is unity, it can easily be shown that

$$\Pi_{r-1}(x_{r-1})\Pi_r(x_r)\Pi_{r+1}(x_{r+1}) = \left|\frac{\zeta\{s(r-1)\}}{\zeta\{s(r+2)\}}\right| e^{O(x_1^{\frac{1}{r-1}-s})}. \tag{370}$$

70. We shall now consider the order of $\sum_{-s}(N)$ i.e., the maximum order of $\sigma_{-s}(N)$. It follows from (317) that if $3s \neq 1$, then

$$\sum\nolimits_{-s}(N) = \Pi_1(x_1)\Pi_2(x_2)\ |\zeta(3s)| e^{O(x_1^{\frac{1}{3}-s})} \tag{371}$$

in virtue of (367), (369) and (370). But if $3s = 1$, we can easily show, by using (362), that

$$\sum\nolimits_{-s}(N) = \Pi_1(x_1)\Pi_2(x_2) e^{O(\log\log x_1)}. \tag{372}$$

It follows from Section 68 that

$$\begin{aligned}\log \Pi_1(x_1) = \log\ \left|\frac{\zeta(s)}{\zeta(2s)}\right| &+ Li\{\theta(x_1)\}^{1-s} - \frac{1}{2}Li\{\vartheta(x_1)\}^{1-2s}\\ &+ \frac{1}{3}Li\{\vartheta(x_1)\}^{1-3s} - \cdots - \frac{(-1)^n}{n}Li\{\vartheta(x_1)\}^{1-ns}\\ &- \frac{1}{2}Li\left(x_1^{\frac{1}{2}-s}\right) + \frac{x_1^{\frac{1}{2}-s} + S_s(x_1)}{\log x_1} + O\left\{\frac{x_1^{\frac{1}{2}-s}}{(\log x_1)^2}\right\}\end{aligned} \tag{373}$$

where $n = [1 + \frac{1}{2s}]$; and also that, if $3s \neq 1$, then,

$$\log \Pi_2(x_2) = \log\ \left|\frac{\zeta(2s)}{\zeta(3s)}\right| + Li\left(x_2^{1-2s}\right) + O\left\{\frac{x_1^{\frac{1}{2}-s}}{(\log x_1)^2}\right\}; \tag{374}$$

and when $3s = 1$

$$\log \Pi_2(x_2) = Li\left(x_2^{1-2s}\right) + O\left\{\frac{x_1^{\frac{1}{2}-s}}{(\log x_1)^2}\right\}. \tag{375}$$

It follows from (371)–(375) that

$$\begin{aligned}\log \sum\nolimits_{-s}(N) = \log\ |\zeta(s)| &+ Li\{\vartheta(x_1)\}^{1-s} - \frac{1}{2}Li\{\vartheta(x_1)\}^{1-2s}\\ &+ \frac{1}{3}Li\{\vartheta(x_1)\}^{1-3s} \cdots - \frac{(-1)^n}{n}Li\{\vartheta(x_1)\}^{1-ns}\\ &- \frac{1}{2}Li\left(x_1^{\frac{1}{2}-s}\right) + Li\left(x_2^{1-2s}\right) + \frac{x_1^{\frac{1}{2}-s} + S_s(x_1)}{\log x_1} + O\left\{\frac{x_1^{\frac{1}{2}-s}}{(\log x_1)^2}\right\}\end{aligned} \tag{376}$$

where $n = [1 + \frac{1}{2s}]$. But from (368) it is clear that, if $m > 0$ then

$$\begin{aligned} Li\{\vartheta(x_1)\}^{1-ms} &= Li\Big\{\log N - x_2 + O\big(x_1^{1/3}\big)\Big\}^{1-ms} \\ &= Li\Big\{(\log N)^{1-ms} - (1-ms)x_2(\log N)^{-ms} + O\big(x_1^{\frac{1}{3}-ms}\big)\Big\} \\ &= Li(\log N)^{1-ms} - \frac{x_2(\log N)^{-ms}}{\log\log N} + O\big(x_1^{\frac{1}{3}-ms}\big). \end{aligned}$$

By arguments similar to those of Section 42 we can show that

$$S_s(x_1) = S_s\Big\{\log N + O\big(\sqrt{x_1}(\log x_1)^2\big)\Big\} = S_s(\log N) + O\big\{x_1^{-s}(\log x_1)^4\big\}.$$

Hence

$$\begin{aligned} \log \sum\nolimits_{-s}(N) = \log\ |\zeta(s)|\ &+ Li(\log N)^{1-s} - \frac{1}{2}Li(\log N)^{1-2s} + \frac{1}{3}Li(\log N)^{1-3s} \\ &- \cdots - \frac{(-1)^n}{n}Li(\log N)^{1-ns} - \frac{1}{2}Li(\log N)^{\frac{1}{2}-s} \\ &+ \frac{(\log N)^{\frac{1}{2}-s} + S_s(\log N)}{\log\log N} + Li\big(x_2^{1-2s}\big) \\ &- \frac{x_2(\log N)^{-s}}{\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\} \end{aligned} \tag{377}$$

where $n = [1 + \frac{1}{2s}]$ and

$$x_2 = 2^{1/(2s)}\sqrt{x_1} + O\left(\frac{\sqrt{x_1}}{\log x_1}\right) = 2^{1/(2s)}\sqrt{(\log N)} + O\left\{\frac{\sqrt{(\log N)}}{\log\log N}\right\} \tag{378}$$

in virtue of (365).

71. Let us consider the order of $\sum_{-s}(N)$ in the following three cases.

Case I. $0 < s < \frac{1}{2}$.

Here we have

$$\begin{aligned} Li(\log N)^{\frac{1}{2}-s} &= \frac{(\log N)^{\frac{1}{2}-s}}{\big(\frac{1}{2}-s\big)\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\}. \\ Li\big(x_2^{1-2s}\big) &= \frac{x_2^{1-2s}}{(1-2s)\log x_2} + O\left\{\frac{x_2^{1-2s}}{(\log x_2)^2}\right\} \\ &= \frac{2^{1/(2s)}(\log N)^{\frac{1}{2}-s}}{(1-2s)\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\}. \\ \frac{x_2(\log N)^{-s}}{\log\log N} &= \frac{2^{1/(2s)}(\log N)^{\frac{1}{2}-s}}{\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\}. \end{aligned}$$

It follows from these and (377) that

$$\log \sum\nolimits_{-s}(N) = Li(\log N)^{1-s} - \frac{1}{2}Li(\log N)^{1-2s} + \frac{1}{3}Li(\log N)^{1-3s}$$
$$- \cdots - \frac{(-1)^n}{n}Li(\log N)^{1-ns}$$
$$+ \frac{2s(2^{1/(2s)}-1)(\log N)^{\frac{1}{2}-s}}{(1-2s)\log\log N} + \frac{S_s(\log N)}{\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\} \quad (379)$$

where $n = [1 + \frac{1}{2s}]$. Remembering (358) and (378) and making $s \to \frac{1}{2}$ in (377) we have Case II. $s = \frac{1}{2}$.

$$\sum\nolimits_{-\frac{1}{2}}(N) = \frac{-\sqrt{2}}{2}\zeta\left(\frac{1}{2}\right)\exp\left\{Li\sqrt{(\log N)} + \frac{2\log 2 - 1 + S_{\frac{1}{2}}(\log N)}{\log\log N} + \frac{O(1)}{(\log\log N)^2}\right\} \quad (380)$$

Case III. $s > \frac{1}{2}$.

$$\sum\nolimits_{-s}(N) = |\zeta(s)|\exp\left\{Li(\log N)^{1-s} - \frac{2s(2^{1/(2s)}-1)}{2s-1}\ \frac{(\log N)^{\frac{1}{2}-s}}{\log\log N}\right\}$$
$$+ \frac{S_s(\log N)}{\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\}. \quad (381)$$

Now making $s \to 1$ in this we have

$$\sum\nolimits_{-1}(N) = e^{\gamma}\left\{\log\log N - \frac{2(\sqrt{2}-1)}{\sqrt{(\log N)}} + S_1(\log N) + \frac{O(1)}{\sqrt{(\log N)}\log\log N}\right\}. \quad (382)$$

Hence

$$\underline{\mathrm{Lim}}\left\{\sum\nolimits_{-1}(N) - e^{\gamma}\log\log N\right\}\sqrt{(\log N)} \geq -e^{\gamma}(2\sqrt{2} + \gamma - \log 4\pi) = -1.558$$

approximately and

$$\overline{\mathrm{Lim}}\left\{\sum\nolimits_{-1}(N) - e^{\gamma}\log\log N\right\}\sqrt{(\log N)} \leq -e^{\gamma}(2\sqrt{2} - 4 - \gamma + \log 4\pi) = -1.393$$

approximately.

The maximum order of $\sigma_s(N)$ is easily obtained by multiplying the values of $\sum_{-s}(N)$ by N^s. It may be interesting to see that $x_r \to x_1^{1/r}$ as $s \to \infty$; and ultimately N assumes the form

$$e^{\vartheta(x_1)+\vartheta(\sqrt{x_1})+\vartheta(x_1^{1/3})+\cdots}$$

that is to say the form of a generalised superior highly composite number approaches that of the least common multiple of the natural numbers when s becomes infinitely large.

The maximum order of $\sigma_{-s}(N)$ without assuming the prime number theorem is obtained by changing $\log N$ to $\log Ne^{O(1)}$ in all the preceding results. In particular

$$\sum\nolimits_{-1}(N) = e^{\gamma}\{\log\log N + O(1)\}. \tag{383}$$

72. Let

$$(1 + 2q + 2q^4 + 2q^9 + \cdots)^4 = 1 + 8\{Q_4(1)q + Q_4(2)q^2 + Q_4(3)q^3 + \cdots\}.$$

Then, by means of elliptic functions, we can show that

$$\begin{aligned} &Q_4(1)q + Q_4(2)q^2 + Q_4(3)q^3 + \cdots \\ &\quad = \frac{q}{1-q} + \frac{2q^2}{1+q^2} + \frac{3q^3}{1-q^3} + \frac{4q^4}{1+q^4} + \cdots \\ &\quad = \frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \frac{4q^4}{1+q^4} + \cdots \\ &\qquad - \left(\frac{4q^4}{1-q^4} + \frac{8q^8}{1-q^8} + \frac{12q^{12}}{1-q^{12}} + \cdots\right). \end{aligned} \tag{384}$$

But

$$\frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \cdots = \sigma_1(1)q + \sigma_1(2)q^2 + \sigma_1(3)q^3 + \cdots .$$

It follows that

$$Q_4(N) \leq \sigma_1(N) \tag{385}$$

for all values of N. It also follows from (384) that

$$(1 - 4^{1-s})\zeta(s)\zeta(s-1) = 1^{-s}Q_4(1) + 2^{-s}Q_4(2) + 3^{-s}Q_4(3) + \cdots . \tag{386}$$

Let

$$N = 2^{a_2}3^{a_3}5^{a_5}\cdots p^{a_p}$$

where $a_\lambda \geq 0$. Then, the coefficient of q^N in

$$\frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \cdots$$

is

$$N\frac{1-2^{-a_2-1}}{1-2^{-1}}\frac{1-3^{-a_3-1}}{1-3^{-1}}\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}};$$

and that in

$$\frac{4q^4}{1-q^4} + \frac{8q^8}{1-q^8} + \frac{12q^{12}}{1-q^{12}} + \cdots$$

is 0 when N is not a multiple of 4 and

$$N\frac{1-2^{-a_2-1}}{1-2^{-1}}\;\frac{1-3^{-a_3-1}}{1-3^{-1}}\;\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}}$$

when N is a multiple of 4. From this and (384) it follows that, if N is not a multiple of 4, then

$$Q_4(N) = N\,\frac{1-2^{-a_2-1}}{1-2^{-1}}\;\frac{1-3^{-a_3-1}}{1-3^{-1}}\;\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}}; \tag{387}$$

and if N is a multiple of 4, then

$$Q_4(N) = 3N\,\frac{1-2^{-a_2-1}}{1-2^{-1}}\;\frac{1-3^{a_3-1}}{1-3^{-1}}\;\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}}. \tag{388}$$

It is easy to see from (387) and (388) that, in order that $Q_4(N)$ should be of maximum order, a_2 must be 1. From (382) we see that the maximum order of $Q_4(N)$ is

$$\begin{aligned}&\frac{3}{4}e^{\gamma}\left\{\log\log N - \frac{2(\sqrt{2}-1)}{\sqrt{(\log N)}} + S_1(\log N) + \frac{O(1)}{\sqrt{(\log N)}\log\log N}\right\} \qquad (389)\\ &\quad = \frac{3}{4}e^{\gamma}\left\{\log\log N + \frac{O(1)}{\sqrt{(\log N)}}\right\}.\end{aligned}$$

It may be observed that, if N is not a multiple of 4, then

$$Q_4(N) = \sigma_1(N);$$

and if N is a multiple of 4, then

$$Q_4(N) = \frac{3\sigma_1(N)}{2^{a_2+1}-1}.$$

73. Let

$$(1+2q+2q^4+2q^9+\cdots)^6 = 1 + 12\{Q_6(1)q + Q_6(2)q^2 + Q_6(3)q^3 + \cdots\}.$$

Then, by means of elliptic functions, we can show that

$$\begin{aligned}&Q_6(1)q + Q_6(2)q^2 + Q_6(3)q^3 + \cdots\\ &\quad = \frac{4}{3}\left(\frac{1^2q}{1+q^2} + \frac{2^2q^2}{1+q^4} + \frac{3^2q^3}{1+q^6} + \cdots\right) - \frac{1}{3}\left(\frac{1^2q}{1-q} - \frac{3^2q^3}{1-q^3} + \frac{5^2q^5}{1-q^5} - \cdots\right).\end{aligned} \tag{390}$$

But

$$\begin{aligned}&\frac{5}{3}\{\sigma_2(1)q + \sigma_2(2)q^2 + \sigma_2(3)q^3 + \cdots\}\\ &\quad = \frac{4}{3}\left\{\frac{1^2q}{1-q} + \frac{2^2q^2}{1-q^2} + \frac{3^2q^3}{1-q^3} + \cdots\right\} + \frac{1}{3}\left\{\frac{1^2q}{1-q} + \frac{2^2q^2}{1-q^2} + \frac{3^2q^3}{1-q^3} + \cdots\right\}.\end{aligned}$$

It follows that

$$Q_6(N) \leq \frac{5\sigma_2(N) - 2}{3} \tag{391}$$

for all values of N. It also follows from (390) that

$$\frac{4}{3}\zeta(s-2)\zeta_1(s) - \frac{1}{3}\zeta(s)\zeta_1(s-2) = 1^{-s}Q_6(1) + 2^{-s}Q_6(2) + 3^{-s}Q_6(3) + \cdots . \tag{392}$$

Let

$$N = 2^{a_2}3^{a_3}5^{a_5}\cdots p^{a_p},$$

where $a_\lambda \geq 0$. Then from (390) we can show, as in the previous section, that if $2^{-a_2}N$ be of the form $4n+1$, then

$$Q_6(N) = N^2 \frac{1-(2^2)^{-a_2-1}}{1-2^{-2}} \, \frac{1-(-3^2)^{-a_3-1}}{1+3^{-2}} \, \frac{1-(5^2)^{-a_5-1}}{1-5^{-2}} \cdots \frac{1-\left\{(-1)^{\frac{p-1}{2}}p^2\right\}^{-a_p-1}}{1-(-1)^{\frac{p-1}{2}}p^{-2}}; \tag{393}$$

and if $2^{-a_2}N$ be of the form $4n-1$, then

$$Q_6(N) = N^2 \frac{1+(2^2)^{-a_2-1}}{1-2^{-2}} \, \frac{1-(-3^2)^{-a_3-1}}{1+3^{-2}} \, \frac{1-(5^2)^{-a_5-1}}{1-5^{-2}} \cdots \frac{1-\left\{(-1)^{\frac{p-1}{2}}p^2\right\}^{-a_p-1}}{1-(-1)^{\frac{p-1}{2}}p^{-2}}. \tag{394}$$

It follows from (393) and (394) that, in order that $Q_6(N)$ should be of maximum order, $2^{-a_2}N$ must be of the form $4n-1$ and $a_2, a_3, a_7, a_{11}, \ldots$ must be 0; 3, 7, 11, ... being primes of the form $4n-1$. But all these cannot be satisfied at the same time since $2^{-a_2}N$ cannot be of the form $4n-1$, when $a_3, a_7, a_{11}, \ldots$ are all zeros. So let us retain a single prime of the form $4n-1$ in the end, that is to say, the largest prime of the form $4n-1$ not exceeding p. Thus we see that, in order that $Q_6(N)$ should be of maximum order, N must be of the form

$$5^{a_5}.13^{a_{13}}.17^{a_{17}}\cdots p^{a_p}.p'$$

where p is a prime of the form $4n+1$ and p' is the prime of the form $4n-1$ next above or below p; and consequently

$$Q_6(N) = \frac{5}{3}N^2 \frac{1-5^{-2(a_5+1)}}{1-5^{-2}} \, \frac{1-13^{-2(a_{13}+1)}}{1-13^{-2}} \cdots \frac{1-p^{-2(a_p+1)}}{1-p^{-2}} \, \{1-(p')^{-2}\}.$$

From this we can show that the maximum order of $Q_6(N)$ is

$$\frac{5N^2 e^{\frac{1}{2}Li\left(\frac{1}{2\log N}\right)+O\left\{\frac{\log\log N}{\log N\sqrt{(\log N)}}\right\}}}{3\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{13^2}\right)\left(1-\frac{1}{17^2}\right)\left(1-\frac{1}{29^2}\right)\cdots} = \frac{5N^2\left\{1+\frac{1}{2}Li\left(\frac{1}{2\log N}\right)+\frac{O(\log\log N)}{\log N\sqrt{(\log N)}}\right\}}{3\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{13^2}\right)\left(1-\frac{1}{17^2}\right)\left(1-\frac{1}{29^2}\right)\cdots} \tag{395}$$

where 5, 13, 17, ... are the primes of the form $4n+1$

74. Let

$$(1+2q+2q^4+2q^9+\cdots)^8$$
$$= 1+16\{Q_8(1)q+Q_8(2)q^2+Q_8(3)q^3+\cdots\}.$$

Then, by means of elliptic functions, we can show that

$$Q_8(1)q+Q_8(2)q^2+Q_8(3)q^3+\cdots$$
$$= \frac{1^3q}{1+q}+\frac{2^3q^2}{1-q^2}+\frac{3^3q^3}{1+q^3}+\frac{4^3q^4}{1-q^4}+\cdots. \tag{396}$$

But

$$\sigma_3(1)q+\sigma_3(2)q^2+\sigma_3(3)q^3+\cdots$$
$$= \frac{1^3q}{1-q}+\frac{2^3q^2}{1-q^2}+\frac{3^3q^3}{1-q^3}+\cdots.$$

It follows that

$$Q_8(N) \le \sigma_3(N) \tag{397}$$

for all values of N. It can also be shown from (396) that

$$(1-2^{1-s}+4^{2-s})\zeta(s)\zeta(s-3) = Q_8(1)1^{-s}+Q_8(2)2^{-s}+Q_8(3)3^{-s}+\cdots. \tag{398}$$

Let

$$N = 2^{a_2}.3^{a_3}.5^{a_5}\cdots p^{a_p},$$

where $a_\lambda \ge 0$. Then from (396) we can easily show that, if N is odd, then

$$Q_8(N) = N^3\frac{1-2^{-3(a_2+1)}}{1-2^{-3}}\frac{1-3^{-3(a_3+1)}}{1-3^{-3}}\cdots\frac{1-p^{-3(a_p+1)}}{1-p^{-3}}; \tag{399}$$

and if N is even then

$$Q_8(N) = N^3\frac{1-15.2^{-3(a_2+1)}}{1-2^{-3}}\frac{1-3^{-3(a_3+1)}}{1-3^{-3}}\cdots\frac{1-p^{-3(a_p+1)}}{1-p^{-3}}. \tag{400}$$

Hence the maximum order of $Q_8(N)$ is

$$\zeta(3)N^3 e^{Li(\log N)^{-2}+O\left(\frac{(\log N)^{-5/2}}{\log\log N}\right)} = \zeta(3)N^3\left\{1 + Li(\log N)^{-2} + O\left(\frac{(\log N)^{-5/2}}{\log\log N}\right)\right\}$$

or more precisely

$$\zeta(3)N^3\left\{1 + Li(\log N)^{-2} - \frac{6(2^{1/6}-1)(\log N)^{-5/2}}{5\log\log N} + \frac{S_3(\log N)}{\log\log N} + O\left(\frac{(\log N)^{-5/2}}{(\log\log N)^2}\right)\right\}. \tag{401}$$

75. There are of course results corresponding to those of Sections 72–74 for the various powers of $\bar{Q}$ where

$$\bar{Q} = 1 + 6\left(\frac{q}{1-q} - \frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} + \cdots\right).$$

Thus for example

$$(\bar{Q})^2 = 1 + 12\left(\frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{4q^4}{1-q^4} + \frac{5q^5}{1-q^5} + \cdots\right), \tag{402}$$

$$(\bar{Q})^3 = 1 - q\left(\frac{1^2q}{1-q} - \frac{2^2q^2}{1-q^2} + \frac{4^2q^4}{1-q^4} - \frac{5^2q^5}{1-q^5} + \cdots\right) + 27\left(\frac{1^2q}{1+q+q^2} + \frac{2^2q^2}{1+q^2+q^4} + \frac{3^3q^3}{1+q^3+q^6} + \cdots\right), \tag{403}$$

$$(\bar{Q})^4 = 1 + 24\left(\frac{1^3q}{1-q} + \frac{2^3q^2}{1-q^2} + \frac{3^3q^3}{1-q^3} + \cdots\right) + 8\left(\frac{3^3q^3}{1-q^3} + \frac{6^3q^6}{1-q^6} + \frac{9^3q^9}{1-q^9} + \cdots\right). \tag{404}$$

The number of ways in which a number can be expressed in the forms m^2+2n^2, $k^2+l^2+2m^2+2n^2$, m^2+3n^2, and $k^2+l^2+3m^2+3n^2$ can be found from the following formulae.

$$(1+2q+2q^4+2q^9+\cdots)(1+2q^2+2q^8+2q^{18}+\cdots) = 1 + 2\left(\frac{q}{1-q} + \frac{q^3}{1-q^3} - \frac{q^5}{1-q^5} - \frac{q^7}{1-q^7} + \cdots\right), \tag{405}$$

$$(1+2q+2q^4+2q^9+\cdots)^2(1+2q^2+2q^8+2q^{18}+\cdots)^2 = 1 + 4\left(\frac{q}{1-q^2} + \frac{2q^2}{1-q^4} + \frac{3q^3}{1-q^6} + \frac{4q^4}{1-q^8} + \cdots\right), \tag{406}$$

$$(1+2q+2q^4+2q^9+\cdots)(1+2q^3+2q^{12}+2q^{27}+\cdots) = 1 + 2\left(\frac{q}{1-q} - \frac{q^2}{1+q^2} + \frac{q^4}{1+q^4} - \frac{q^5}{1-q^5} + \frac{q^7}{1-q^7} - \cdots\right), \tag{407}$$

$$(1+2q+2q^4+2q^9+\cdots)^2(1+2q^3+2q^{12}+2q^{27}+\cdots)^2$$
$$= 1+4\left(\frac{q}{1+q}+\frac{2q^2}{1-q^2}+\frac{4q^4}{1-q^4}+\frac{5q^5}{1+q^5}+\frac{7q^7}{1+q^7}+\cdots\right) \qquad (408)$$

where 1, 2, 4, 5 ... are the natural numbers without the multiples of 3.

Notes

52. The definition of $Q_2(N)$ given in italics is missing in [18]. It has been formulated in the same terms as the definition of $\bar{Q}_2(N)$ given in Section 55. For $N \neq 0$, $4Q_2(N)$ is the number of pairs $(x, y) \epsilon Z^2$ such that $x^2 + y^2 = N$.

Formula (269) links together Dirichlet's series and Lambert's series (see [5], p. 258).

53. Effective upper bounds for $Q_2(N)$ can be found in [21], p. 50 for instance:

$$\log Q_2(N) \leq \frac{(\log 2)(\log N)}{\log\log N}\left(1+\frac{1-\log 2}{\log\log N}+\frac{2.40104}{(\log\log N)^2}\right).$$

The maximal order of $Q_2(N)$ is studied in [8], but not so deeply as here. See also [12], pp. 218–219.

54. For a proof of (276), see [25], p. 22. In (276), we remind the reader that ρ is a zero of the Riemann zeta-function. Formula (279) has been rediscovered and extended to all arithmetical progressions [23].

56. For a proof of (291), see [25], p. 22. In the definition of $R_2(x)$, between formulas (290) and (291), and in the definition of $\Phi(N)$, after formula (294), three misprints in [18] have been corrected, namely $\sum \frac{x^\rho}{\rho^2}$ and $\sum \frac{x^{\rho_2}}{\rho_2^2}$ have been written instead of $\sum \frac{x^\rho}{\rho}$ and $\sum \frac{x^{\rho_2}}{\rho_2}$, and $R_2(2\log N)$ instead of $R_2(\log N)$.

57. Effective upper bounds for $d_2(N)$ can be found in [21], p. 51, for instance:

$$\log d_2(N) \leq \frac{(\log 3)(\log N)}{\log\log N}\left(1+\frac{1}{\log\log N}+\frac{5.5546}{(\log\log N)^2}\right).$$

For a more general study of $d_k(n)$, when k and n go to infinity, see [3] and [14].

58. The words in italics do not occur in [18] where the definition of $\sigma_{-s}(N)$ and the proof of (301) were missing. It is not clear why Ramanujan considered $\sigma_{-s}(N)$ only with $s \geq 0$. Of course he knew that

$$\sigma_s(N) = N^s \sigma_{-s}(N),$$

(cf. for instance Section 71, after formula (382)), but for $s > 0$ the generalised highly composite numbers for $\sigma_s(N)$ are quite different, and for instance property (303) does not hold for them.

59. It would be better to call these numbers s-generalised highly composite numbers, because their definition depends on s. For $s = 1$, these numbers have been called superabundant by Alaoglu and Erdös (cf. [1, 4]) and the generalised superior highly composite numbers have been called colossally abundant. The solution of $2^s + 4^s + 8^s = 3^s + 9^s$ is approximately 1.6741.

60–61. For $s = 1$, the results of these sections are in [1] and [4].

62. The references given here, formula (16) and Section 38 are from [16]. For a geometrical interpretation of $\sum_{-s}(N)$, see [12], p. 230. Consider the piecewise linear function $u \mapsto f(u)$ such that for all generalised superior highly composite numbers N, $f(\log N) = \log \sigma_{-s}(N)$, then for all N,

$$\sum\nolimits_{-s}(N) = \exp(f(\log N)).$$

Infinite integrals mean in fact definite integrals. For instance, in formula (320), $\int \frac{\varepsilon\pi(x_r)}{x_r}\, dx_r$ should be read $\int_2^{x_r} \frac{\varepsilon\pi(t)}{t}\, dt$.

64. Formula (329) is proved in [25] p. 29 from the classical explicit formula in prime number theory.

65. There is a misprint in the last term of formula (340) in [18], but, may be it is only a mistake of copying, since the next formula is correct. This section belongs to the part of the manuscript which is not handwritten by Ramanujan in [18].

*2	*7560	942480	49008960
3	9240	982800	54774720
*4	*10080	997920	56548800
*6	12600	1053360	60540480
8	13860	*1081080	*61261200
10	*15120	1330560	64864800
*12	18480	1413720	68468400
18	*20160	*1441440	*73513440
20	*25200	1663200	82162080
*24	*27720	1801800	86486400
30	30240	1884960	91891800
*36	32760	1965600	98017920
*48	36960	2106720	99459360
*60	37800	*2162160	102702600
72	40320	2827440	107442720
84	41580	*2882880	108108000
90	42840	3326400	109549440
96	43680	*3603600	*110270160
108	*45360	*4324320	*122522400
*120	*50400	5405400	136936800
168	*55440	5654880	*147026880
*180	65520	5765760	164324160
*240	75600	6126120	*183783600
336	*83160	6320160	205405200
*360	98280	*6486480	220540320
420	*110880	*7207200	232792560
480	131040	*8648640	*245044800
504	138600	*10810800	273873600
540	150840	12252240	*294053760
600	163800	12972960	328648320
630	*166320	13693680	349188840
660	196560	14137200	*367567200
672	*221760	*14414400	410810400
*720	262080	*17297280	465585120
*840	*277200	18378360	490089600
1080	327600	20540520	497296800
*1260	*332640	*21621600	514594080
1440	360360	24504480	537213600
*1680	393120	27387360	547747200
2160	443520	28274400	*551350800
*2520	471240	28828800	616215600
3360	480480	30270240	*698377680
3780	*498960	30630600	*735134400
3960	*554400	31600800	821620800
4320	655200	*32432400	931170240
4620	*665280	*36756720	994593600
4680	*720720	41081040	1029188160
*5040	831600	*43243200	1074427200

Table of largely composite numbers.

n	*d*		*n*	*d*		*n*	*d*	
1	1		7560	64	$2^3.3^3.5.7$	942480	240	$2^4.3^2.5.7.11.17$
*2	2	2	9240	64	$2^3.3.5.7.11$	982800	240	$2^4.3^3.5^2.7.13$
3	2	3	10080	72	$2^5.3^2.5.7$	997920	240	$2^5.3^4.5.7.11$
4	3	2^2	12600	72	$2^3.3^2.5^2.7$	1053360	240	$2^4.3^2.5.7.11.19$
*6	4	2.3	13860	72	$2^2.3^2.5.7.11$	1081080	256	$2^3.3^3.5.7.11.13$
8	4	2^3	15120	80	$2^4.3^3.5.7$	1330560	256	$2^7.3^3.5.7.11$
10	4	2.5	18480	80	$2^4.3.5.7.11$	1413720	256	$2^3.3^3.5.7.11.17$
*12	6	$2^2.3$	20160	84	$2^6.3^2.5.7$	*1441440	288	$2^5.3^2.5.7.11.13$
18	6	2.3^2	25200	90	$2^4.3^2.5^2.7$	1663200	288	$2^5.3^3.5^2.7.11$
20	6	$2^2.5$	27720	96	$2^3.3^2.5.7.11$	1801800	288	$2^3.3^2.5^2.7.11.13$
24	8	$2^3.3$	30240	96	$2^5.3^3.5.7$	1884960	288	$2^5.3^2.5.7.11.17$
30	8	2.3.5	32760	96	$2^3.3^2.5.7.13$	1965600	288	$2^5.3^3.5^2.7.13$
36	9	$2^2.3^2$	36960	96	$2^5.3.5.7.11$	2106720	288	$2^5.3^2.5.7.11.19$
48	10	$2^4.3$	37800	96	$2^3.3^3.5^2.7$	2162160	320	$2^4.3^3.5.7.11.13$
*60	12	$2^2.3.5$	40320	96	$2^7.3^2.5.7$	2827440	320	$2^4.3^3.5.7.11.17$
72	12	$2^3.3^2$	41580	96	$2^2.3^3.5.7.11$	2882880	336	$2^6.3^2.5.7.11.13$
84	12	$2^2.3.7$	42840	96	$2^3.3^2.5.7.17$	3326400	336	$2^6.3^3.5^2.7.11$
90	12	$2.3^2.5$	43680	96	$2^5.3.5.7.13$	3603600	360	$2^4.3^2.5^2.7.11.13$
96	12	$2^5.3$	45360	100	$2^4.3^4.5.7$	*4324320	384	$2^5.3^3.5.7.11.13$
108	12	$2^2.3^3$	50400	108	$2^5.3^2.5^2.7$	5405400	384	$2^3.3^3.5^2.7.11.13$
*120	16	$2^3.3.5$	*55440	120	$2^4.3^2.5.7.11$	5654880	384	$2^5.3^3.5.7.11.17$
168	16	$2^3.3.7$	65520	120	$2^4.3^2.5.7.13$	5765760	384	$2^7.3^2.5.7.11.13$
180	18	$2^2.3^2.5$	75600	120	$2^4.3^3.5^2.7$	6126120	384	$2^3.3^2.5.7.11.13.17$
240	20	$2^4.3.5$	83160	128	$2^3.3^3.5.7.11$	6320160	384	$2^5.3^3.5.7.11.19$
336	20	$2^4.3.7$	98280	128	$2^3.3^3.5.7.13$	6486480	400	$2^4.3^4.5.7.11.13$
*360	24	$2^3.3^2.5$	110880	144	$2^5.3^2.5.7.11$	7207200	432	$2^5.3^2.5^2.7.11.13$
420	24	$2^2.3.5.7$	131040	144	$2^5.3^2.5.7.13$	8648640	448	$2^6.3^3.5.7.11.13$
480	24	$2^5.3.5$	138600	144	$2^3.3^2.5^2.7.11$	10810800	480	$2^4.3^3.5^2.7.11.13$
504	24	$2^3.3^2.7$	151200	144	$2^5.3^3.5^2.7$	12252240	480	$2^4.3^2.5.7.11.13.17$
540	24	$2^2.3^3.5$	163800	144	$2^3.3^2.5^2.7.13$	12972960	480	$2^5.3^4.5.7.11.13$
600	24	$2^3.3.5^2$	166320	160	$2^4.3^3.5.7.11$	13693680	480	$2^4.3^2.5.7.11.13.19$
630	24	$2.3^2.5.7$	196560	160	$2^4.3^3.5.7.13$	14137200	480	$2^4.3^3.5^2.7.11.17$
660	24	$2^2.3.5.11$	221760	168	$2^6.3^2.5.7.11$	14414400	504	$2^6.3^2.5^2.7.11.13$
672	24	$2^5.3.7$	262080	168	$2^6.3^2.5.7.13$	17297280	512	$2^7.3^3.5.7.11.13$
720	30	$2^4.3^2.5$	277200	180	$2^4.3^2.5^2.7.11$	18378360	512	$2^3.3^3.5.7.11.13.17$
840	32	$2^3.3.5.7$	327600	180	$2^4.3^2.5^2.7.13$	20540520	512	$2^3.3^3.5.7.11.13.19$
1080	32	$2^3.3^3.5$	332640	192	$2^5.3^3.5.7.11$	*21621600	576	$2^5.3^3.5^2.7.11.13$
1260	36	$2^2.3^2.5.7$	360360	192	$2^3.3^2.5.7.11.13$	24504480	576	$2^5.3^2.5.7.11.13.17$
1440	36	$2^5.3^2.5$	393120	192	$2^5.3^3.5.7.13$	27387360	576	$2^5.3^2.5.7.11.13.19$
1680	40	$2^4.3.5.7$	415800	192	$2^3.3^3.5^2.7.11$	28274400	576	$2^5.3^3.5^2.7.11.17$
2160	40	$2^4.3^3.5$	443520	192	$2^7.3^2.5.7.11$	28828800	576	$2^7.3^2.5^2.7.11.13$
*2520	48	$2^3.3^2.5.7$	471240	192	$2^3.3^2.5.7.11.17$	30270240	576	$2^5.3^3.5.7^2.11.13$
3360	48	$2^5.3.5.7$	480480	192	$2^5.3.5.7.11.13$	30630600	576	$2^3.3^2.5^2.7.11.13.17$
3780	48	$2^2.3^3.5.7$	491400	192	$2^3.3^3.5^2.7.13$	31600800	576	$2^5.3^3.5^2.7.11.19$
3960	48	$2^3.3^2.5.11$	498960	200	$2^4.3^4.5.7.11$	32432400	600	$2^4.3^4.5^2.7.11.13$
4200	48	$2^3.3.5^2.7$	554400	216	$2^5.3^2.5^2.7.11$	36756720	640	$2^4.3^3.5.7.11.13.17$
4320	48	$2^5.3^3.5$	655200	216	$2^5.3^2.5^2.7.13$	41081040	640	$2^4.3^3.5.7.11.13.19$
4620	48	$2^2.3.5.7.11$	665280	224	$2^6.3^3.5.7.11$	43243200	672	$2^6.3^3.5^2.7.11.13$
4680	48	$2^3.3^2.5.13$	*720720	240	$2^4.3^2.5.7.11.13$	49008960	672	$2^6.3^2.5.7.11.13.17$
*5040	60	$2^4.3^2.5.7$	831600	240	$2^4.3^3.5^2.7.11$	54774720	672	$2^6.3^2.5.7.11.13.19$

n	d		n	d	
56548800	672	$2^6.3^3.5^2.7.11.17$	232792560	960	$2^4.3^2.5.7.11.13.17.19$
60540480	672	$2^6.3^3.5.7^2.11.13$	245044800	1008	$2^6.3^2.5^2.7.11.13.17$
61261200	720	$2^4.3^2.5^2.7.11.13.17$	273873600	1008	$2^6.3^2.5^2.7.11.13.19$
64864800	720	$2^5.3^4.5^2.7.11.13$	294053760	1024	$2^7.3^3.5.7.11.13.17$
68468400	720	$2^4.3^2.5^2.7.11.13.19$	328648320	1024	$2^7.3^3.5.7.11.13.19$
73513440	768	$2^5.3^3.5.7.11.13.17$	349188840	1024	$2^3.3^3.5.7.11.13.17.19$
82162080	768	$2^5.3^3.5.7.11.13.19$	*367567200	1152	$2^5.3^3.5^2.7.11.13.17$
86486400	768	$2^7.3^3.5^2.7.11.13$	410810400	1152	$2^5.3^3.5^2.7.11.13.19$
91891800	768	$2^3.3^3.5^2.7.11.13.17$	465585120	1152	$2^5.3^2.5.7.11.13.17.19$
98017920	768	$2^7.3^2.5.7.11.13.17$	490089600	1152	$2^7.3^2.5^2.7.11.13.17$
99459360	768	$2^5.3^3.5.7.11.13.23$	497296800	1152	$2^5.3^3.5^2.7.11.13.23$
102702600	768	$2^3.3^3.5^2.7.11.13.19$	514594080	1152	$2^5.3^3.5.7^2.11.13.17$
107442720	768	$2^5.3^3.5.7.11.17.19$	537213600	1152	$2^5.3^3.5^2.7.11.17.19$
108108000	768	$2^5.3^3.5^3.7.11.13$	547747200	1152	$2^7.3^2.5^2.7.11.13.19$
109549440	768	$2^7.3^2.5.7.11.13.19$	551350800	1200	$2^4.3^4.5^2.7.11.13.17$
110270160	800	$2^4.3^4.5.7.11.13.17$	616215600	1200	$2^4.3^4.5^2.7.11.13.19$
122522400	864	$2^5.3^2.5^2.7.11.13.17$	698377680	1280	$2^4.3^3.5.7.11.13.17.19$
136936800	864	$2^5.3^2.5^2.7.11.13.19$	735134400	1344	$2^6.3^3.5^2.7.11.13.17$
147026880	896	$2^6.3^3.5.7.11.13.17$	821620800	1344	$2^6.3^3.5^2.7.11.13.19$
164324160	896	$2^6.3^3.5.7.11.13.19$	931170240	1344	$2^6.3^2.5.7.11.13.17.19$
183783600	960	$2^4.3^3.5^2.7.11.13.17$	994593600	1344	$2^6.3^3.5^2.7.11.13.23$
205405200	960	$2^4.3^3.5^2.7.11.13.19$	1029188160	1344	$2^6.3^3.5.7^2.11.13.17$
220540320	960	$2^5.3^4.5.7.11.13.17$	1074427200	1344	$2^6.3^3.5^2.7.11.17.19$

This table has been built to explain the table handwritten by S. Ramanujan which is displayed on p. 150. An integer n is said largely composite if $m \le n \Rightarrow d(m) \le d(n)$. The numbers marked with one asterisk are superior highly composite numbers.

Notes (Continued)

The approximations given for $1/\sqrt{mn}$ comes from the Padé approximant of $\sqrt{t}$ in the neighborhood of $t = 1$: $\frac{3t+1}{t+3} = 1/(\frac{1}{3} + \frac{8}{3(3t+1)})$.

68. There are two formulas (362) in [18], p. 299. Formula (362) can be found in [11]. As observed by Birch (cf. [2], p. 74), there is some similarity between the calculation of Section 63 to Section 68, and those appearing in [18], pp. 228–232. In formulas (356) and (357) $Li\{\theta(x)\}^{1-s}$ should be read $Li(\{\theta(x)\}^{1-s})$, the same for $Li\sqrt{\log N}$ in (380) and for several other formulas.

71. There is a wrong sign in formula (379) of [18], and also in formulas (381) and (382). The two inequalities following formula (382) were also wrong. In formula (380), the right coefficient in the right hand side is $-\frac{\sqrt{2}}{2}\zeta(1/2)$ instead of $-\sqrt{2}\zeta(1/2)$ in [18]. It follows from (382) that under the Riemann hypothesis, and for n_0 large enough,

$$n > n_0 \Rightarrow \sigma(n)/n \le e^{\gamma} \log\log n.$$

It has been shown in [22] that the above relation with $n_0 = 5040$ is equivalent to the Riemann hypothesis.

72. Formula (384) is due to Jacobi. For a proof see [5] p. 311. See also [6], pp. 132–160. In formula (389) of [18], the sign of the second term in the curly bracket was wrong.

73. Formula (390) is proved in [15], p. 198 (90.3). It is true that if

$$N = 5^{a_5} 13^{a_{13}} 17^{a_{17}} \cdots p^{a_p} p'$$

with $p' \sim p$, then $Q_6(N)$ will have the maximal order (395). But, if we define a superior champion for Q_6, that is to say an N which maximises $Q_6(N)N^{-2-\varepsilon}$ for an $\varepsilon > 0$, it will be of the above form, with $p' \sim p\sqrt{\frac{\log p}{2}}$. In (395), the error term was written $O(\frac{1}{(\log N)^{3/2} \log\log N})$ in [18], cf. [25].

74. Formula (396) is proved in [15], p. 198 (90.4). In formula (401) the sign of the third term in the curly bracket was wrong in [18]. In [18], the right hand side of (398) was written as the left hand side of (396).

Table, p. 150: This table calculated by Ramanujan occurs on p. 280 in [18]. It should be compared to the table of largely composite numbers, p. 151–152. The entry 150840 is not a largely composite number:

$$150840 = 2^3.3^2.5.419 \quad \text{and} \quad d(150840) = 48$$

while the four numbers 4200, 151200, 415800, 491400 are largely composite and do not appear in the table of Ramanujan. Largely composite numbers are studied in [9].

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