Bi-Hamiltonian nature of the equation $u_{tx} = u_{xy}\, u_y - u_{yy}\, u_x$

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Abstract

We study non-linear integrable partial differential equations naturally arising as bi-Hamiltonian Euler equations related to the looped cotangent Virasoro algebra. This infinite-dimensional Lie algebra (constructed in [16]) is a generalization of the classical Virasoro algebra to the case of two space variables. Two main examples of integrable equations we obtain are quite well known. We show that the relation between these two equations is similar to that between the Korteweg-de Vries and Camassa-Holm equations.

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1 Introduction

The differential equation

$$u_{tx} = u_{xy}\, u_y - u_{yy}\, u_x, \tag{1.1}$$

where $u = u(t, x, y)$ and where u_x, u_y, etc. are the partial derivatives, is a nice example of a non-linear integrable model. This equation is quite well known and appears in the Martínez Alonzo-Shabat "universal hierarchy" (see [13], formula (8)).

The main purpose of this note is to show that this equation (coupled together with another differential equation, see formula (3.6) below) naturally appears as a bi-Hamiltonian vector field (in particular, the variable t plays the rôle of time while x, y are space variables). More precisely, we will show that this equation is an Euler equation on the space dual to the "looped cotangent Virasoro algebra" introduced in [16]. This, in particular, implies its integrability in the (weak algebraic) sense of existence of a hierarchy of first integrals in involution.

The bi-Hamiltonian approach to the same Lie algebra has already been considered in [16] and led to another non-linear differential equation:

$$f_t = f_x\, \partial_x^{-1} f_y - f_y\, u + c\, \partial_x^{-1}\, f_{yy},$$

which can be rewritten without non-local terms:

$$u_{tx} = u_{xx}\, u_y - u_{xy}\, u_x + c\, u_{yy}, \tag{1.2}$$

after the substitution $f = u_x$. Here $c \in \mathbb{R}$ is an arbitrary constant (the "central charge"). Note that this equation is also a quite well known integrable system (see [5, 6] and also [4]) that appears both in differential geometry and hydrodynamic.

Equations (1.2) and (1.1) look alike but they are not equivalent to each other. We will show that the relation between these equations is similar to that between the classical Korteweg-de Vries equation (KdV) and the Camassa-Holm equation (CH). Recall that both KdV and CH are bi-Hamiltonian systems on the dual of the Virasoro algebra, see [12, 3, 11, 14, 10]. An interesting "tri-Hamiltonian" viewpoint was suggested in [15], in order to establish a certain duality between KdV and CH. Equations (1.2) and (1.1) are dual in the same sense.

This paper fits into the general framework due to V.I. Arnold, see [1]. Non-linear partial equations are viewed as Euler equations on the dual of a Lie algebra (for instance, the Lie algebra of vector fields). This approach explains the geometric meaning of the equations: every Euler equation describes geodesics of some left-invariant metric on the corresponding group (of diffeomorphisms).

The bi-Hamiltonian Euler equations are of special interest. Most of the known bi-Hamiltonian non-linear partial differential equations (KdV,CH, etc.) are of dimension $1+1$ (i.e., contain only one space variable). Equations (1.1) and (1.2) provide with examples of such equations in the $(2+1)$-dimensional case.

2 The bi-Hamiltonian formalism on the dual of a Lie algebra

In this section we recall the general construction of pairs of compatible Poisson structures on the space dual to a Lie algebra. We also give the standard construction of bi-Hamiltonian vector fields on this space, due to F. Magri [12].

Let $\mathfrak{a}$ be a (finite-dimensional) Lie algebra, the canonical Lie-Poisson(-Berezin-Kirillov-Kostant) bracket on $\mathfrak{a}^*$ is given by

$$\{F,G\}(m) = \langle [d_mF, d_mG], m\rangle, \tag{2.1}$$

where $m \in \mathfrak{a}^*$ and where d_mF and d_mG are the differentials of F and G at m understood as elements of $\mathfrak{a}$, namely $dF_m \in (\mathfrak{a}^*)^* \cong \mathfrak{a}$. This Poisson structure is linear, i.e., the space of linear functions equipped with the bracket (2.1) is a Lie subalgebra of $C^\infty(\mathfrak{a}^*)$ (isomorphic to $\mathfrak{a}$).

Given a skew-symmetric bilinear form $\omega : \mathfrak{a}\wedge\mathfrak{a} \to \mathbb{R}$, one defines another Poisson structure on $\mathfrak{a}$:

$$\{F,G\}_\omega(m) = \omega(d_mF, d_mG). \tag{2.2}$$

This structure is with constant coefficients, i.e., the bracket of two linear functions is a constant function on $\mathfrak{a}^*$.

Two Poisson structures are called compatible (or a Poisson pair) if their linear combination is again a Poisson structure. The following simple fact is well known (see, e.g., [2], Section 5.2).

Proposition 2.1. *The Poisson structures (2.1) and (2.2) are compatible if an only if ω is a 2-cocycle on $\mathfrak{a}$.*

The simplest example of a constant Poisson structure (2.2) corresponds to the case where the 2-cocycle ω is trivial (i.e., a coboundary). Every such structure is of the following form. Fix a point $m_0 \in \mathfrak{a}^*$ and set

$$\omega(x,y) = \langle m_0,\, [x,y]\rangle. \tag{2.3}$$

It worth noticing that one can understand this particular case of constant Poisson structure on $\mathfrak{a}^*$ as the most general one. Indeed, it suffices to replace $\mathfrak{a}$ by its central extension.

Every function H on $\mathfrak{a}^*$ defines two vector fields that we denote X_H and X_H^ω on $\mathfrak{a}^*$: the first one is Hamiltonian with respect to the linear structure (2.1) and is given by

$$X_H(m) = \mathrm{ad}^*_{d_mH}\, m, \tag{2.4}$$

while the vector field X_H^ω is Hamiltonian with respect to the constant bracket (2.2). In the particular case (2.3), one has explicitly

$$X_H^\omega(m) = \mathrm{ad}^*_{d_m H}\, m_0. \tag{2.5}$$

Given two compatible Poisson structures, a vector field which is Hamiltonian with respect to the both structures is called bi-Hamiltonian. The usual way to construct bi-Hamiltonian vector fields on $\mathfrak{a}^*$ is as follows. Consider the following 1-parameter family of Poisson structures

$$\{\,,\,\}_\lambda = \{\,,\,\}_\omega - \lambda\,\{\,,\,\},$$

(parameterized by $\lambda \in \mathbb{R}$). Assume that H is a Casimir function of this bracket, i.e., one has

$$\{H,\, F\}_\lambda = 0, \qquad \text{for all} \quad F \in C^\infty(\mathfrak{a}^*).$$

Assume also that H is written in a form of a series

$$H = H_0 + \lambda\, H_1 + \lambda^2\, H_2 + \cdots \tag{2.6}$$

One immediately obtains the following facts:

1. the function H_0 is a Casimir function of $\{\,,\,\}_\omega$;
2. the Hamiltonian vector field corresponding to H_k are bi-Hamiltonian, namely

   $$X_{H_k} = X^\omega_{H_{k+1}},$$

   for all k;
3. all the functions H_k are in involution with respect to the both Poisson structures, indeed, for $k \leq \ell$ one has

   $$\{H_k,\, H_\ell\} = \{H_{k+1},\, H_\ell\}_\omega = \{H_{k+1},\, H_{\ell-1}\} = \cdots = 0,$$

   and therefore are first integrals of every vector field X_{H_k}.

Let us summarize the method. To construct an integrable hierarchy, one chooses a function H_0 which is a Casimir function of the constant Poisson structure $\{\,,\,\}_\omega$; one then considers its Hamiltonian vector field, X_{H_0}, with respect to the Lie-Poisson structure. This vector field is again Hamiltonian with respect to the constant Poisson structure, with some Hamiltonian function H_1, so that one has: $X_{H_0} = X^\omega_{H_1}$. One then iterates the procedure to find H_2, H_3, etc.

3 The looped cotangent Virasoro algebra and its dual

In this section we recall the definition [16] of the looped cotangent Virasoro algebra. We also describe its coadjoint representation.

Let us start with the definition of the classical Virasoro algebra. Consider the Lie algebra, $\mathrm{Vect}(S^1)$, of vector fields on the circle: $f(x)\,\frac{\partial}{\partial x}$ where $f \in C^\infty(S^1)$ and x is a coordinate on S^1, we assume $x \sim x + 2\pi$. To simplify the formulæ, we will identify $\mathrm{Vect}(S^1)$ with $C^\infty(S^1)$; the Lie bracket in $\mathrm{Vect}(S^1)$ is then given by

$$[f, g] = f\, g_x - f_x\, g.$$

The Virasoro algebra, Vir, is a (unique up to isomorphism) one-dimensional central extension of $\mathrm{Vect}(S^1)$. It is defined on the space $\mathrm{Vect}(S^1) \oplus \mathbb{R}$, the commutator being given by

$$[(f,\,\alpha),\,(g,\,\beta)] = \left(f\,g_x - f_x\,g,\, \int_{S^1} f\,g_{xxx}\,dx \right). \tag{3.1}$$

Note that the constants α and β do not enter the right hand side of the above formula since they belong to the center of Vir.

The Virasoro algebra was found by Gelfand and Fuchs [8], the constant term in the right hand side of (3.1) is called the Gelfand-Fuchs cocycle. This Lie algebra plays an important rôle in mathematical physics, essentially because of the applications of its representations to conformal field theory, but also because of its applications to integrable systems.

The dual space, $\mathrm{Vect}(S^1)^*$, is the space of distributions. One often considers only a subspace, $\mathrm{Vect}(S^1)^*_{\mathrm{reg}}$, called the "regular dual" (cf. [9]). As a vector space, this regular dual is, again, isomorphic to $C^\infty(S^1)$, the pairing $\langle .,. \rangle : \mathrm{Vect}(S^1) \otimes C^\infty(S^1) \to \mathbb{R}$ being give by

$$\left\langle f(x)\,\frac{\partial}{\partial x},\, a(x) \right\rangle := \int_{S^1} f(x)\,a(x)\,dx.$$

The regular dual to the Virasoro algebra is $\mathrm{Vir}^*_{\mathrm{reg}} = C^\infty(S^1) \oplus \mathbb{R}$; the coadjoint action of Vir on its regular dual is:

$$\mathrm{ad}^*_{(f,\alpha)}(a,\,c) = (f\,a_x + 2\,f_x\,a + c\,f_{xxx},\,0)\,.$$

This formula easily follows from (3.1) and the definition of ad^*, see [9]. Note that the constant c is preserved by the action, it is therefore a parameter called the central charge.

Remark 3.1. The Virasoro algebra is, indeed, exceptional. The reason is that the Lie algebras of vector fields on a manifold of dimension ≥ 2, has no central extensions, cf. [7]. The problem of generalization of the Virasoro algebra is an interesting subject studied by many authors.

The looped cotangent Virasoro algebra [16] is a generalization of Vir in the case of two variables. We consider the 2-torus $\mathbb{T}^2$ and define a Lie algebra structure on the space

$$\mathfrak{g} = C^\infty(\mathbb{T}^2) \oplus C^\infty(\mathbb{T}^2) \oplus \mathbb{R}^2.$$

given by the commutator

$$\left[\begin{pmatrix} f \\ a \\ (\alpha,\alpha') \end{pmatrix}, \begin{pmatrix} g \\ b \\ (\beta,\beta') \end{pmatrix} \right] = \begin{pmatrix} f\,g_x - f_x\,g \\ f\,b_x + 2\,f_x\,b - g\,a_x - 2\,g_x\,a \\ \left(\int_{S^1\times S^1} f\,g_{xxx}\,dxdy\,,\, \int_{S^1\times S^1} (f\,b_y - g\,a_y)\,dxdy \right) \end{pmatrix} \tag{3.2}$$

where (x,y) are the usual coordinates on $\mathbb{T}^2$ and where f,g,a,b are smooth functions in x,y; the constants $\alpha,\alpha',\beta,\beta' \in \mathbb{R}$ are elements of the center. Note that, unlike the Virasoro algebra, the center of $\mathfrak{g}$ is two-dimensional.

Remark 3.2. One notices that the quotient-algebra $\mathfrak{g}/\mathbb{R}^2$ (by the center) is the loop algebra with coefficients in the semidirect sum $\mathrm{Vect}(S^1) \ltimes \mathrm{Vect}(S^1)^*_{\mathrm{reg}}$. The dependence in y-variable in this quotient-algebra is somehow trivial. The second 2-cocycle in (3.2), however, makes this dependence in y non-trivial. Note also that this cocycle is rather similar to the Kac-Moody cocycle.

We will need the coadjoint representation of $\mathfrak{g}$ and the notion of regular dual space. Consider the pairing $\langle .,. \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$

$$\left\langle \begin{pmatrix} f \\ a \\ (\alpha_1, \alpha_2) \end{pmatrix}, \begin{pmatrix} g \\ b \\ (\alpha_1, \alpha_2) \end{pmatrix} \right\rangle = \int_{S^1 \times S^1} (f\, b + g\, a)\, dxdy + \alpha_1 \beta_1 + \alpha_2 \beta_2,$$

that identifies $\mathfrak{g}$ with a part of its dual space: $\mathfrak{g} \hookrightarrow \mathfrak{g}^*$, we call this subspace the regular dual space of $\mathfrak{g}$ and denote it by $\mathfrak{g}^*_{\text{reg}}$. The coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*_{\text{reg}}$ can be easily calculated:

$$\widehat{\mathrm{ad}}^*_{\begin{pmatrix} f \\ a \\ (\alpha_1, \alpha_2) \end{pmatrix}} \begin{pmatrix} g \\ b \\ (c_1, c_2) \end{pmatrix} = \begin{pmatrix} f\, g_x - f_x\, g + c_2\, f_y \\ f\, b_x + 2\, f_x\, b - a_x\, g - 2\, a\, g_x + c_1\, f_{xxx} + c_2\, a_y \\ (0, 0) \end{pmatrix}. \tag{3.3}$$

Note that the center $\mathbb{R}^2 \subset \mathfrak{g}$ acts trivially.

The Lie algebra $\mathfrak{g}$ is infinite-dimensional. In order to define the brackets (2.1) and (2.2) in this case, we consider only the space of so-called pseudodifferential polynomials on $\mathfrak{g}^*_{\text{reg}}$:

$$H(f,\, a) = \int_{S^1 \times S^1} h\left(f,\, a,\, f_x,\, a_x,\, f_y,\, a_y,\, \partial_x^{-1} f,\, \partial_x^{-1} a,\, \partial_y^{-1} f,\, \partial_y^{-1} a,\, f_{xy},\, a_{xy}, \ldots\right) dxdy,$$

where h is a polynomial and $f, a, f_x, a_x, f_y, a_y, \partial_x^{-1} f, \ldots$ are understood as independent variables.

The differential $d_m H$ is replaced by the standard variational derivative:

$$d_{(f,\, a)} H := (\delta_a H,\, \delta_f H)$$

understood as element of $\mathfrak{g}/\mathbb{R}^2$. The Lie-Poisson structure (2.1) then makes sense on $\mathfrak{g}^*_{\text{reg}}$ and the Hamiltonian vector fields are again given by (2.4).

Example 3.3. Recall that the Euler-Lagrange equation provides an explicit formula for variational derivatives. For instance, one has

$$\begin{aligned} \delta_a H \;&=\; h_a - \partial_x\left(h_{a_x}\right) - \partial_y\left(h_{a_y}\right) - \partial_x^{-1}\left(h_{\partial_x^{-1} a}\right) - \partial_y^{-1}\left(h_{\partial_y^{-1} a}\right) \\ &\quad + (\partial_x)^2\left(h_{a_{xx}}\right) + \partial_x \partial_y\left(h_{a_{xy}}\right) + (\partial_y)^2\left(h_{a_{yy}}\right) \pm \cdots \end{aligned}$$

where, as usual, h_u means the partial derivative $\frac{\partial h}{\partial a}$, similarly $h_{a_x} = \frac{\partial h}{\partial a_x}$, etc..

One of course should be careful with the definition of the non-local operators ∂_x^{-1} and ∂_y^{-1}. We use the expression

$$(\partial_x^{-1} f)(x, y) = \int_0^x f(\xi, y)\, d\xi - \int_0^{2\pi} f(x, y)\, dx,$$

and similarly for ∂_y^{-1}.

We refer to [3] for further details on Hamiltonian formalism on infinite-dimensional (functional) Lie algebras.

3.1 Calculating the bi-Hamiltonian equations

Let us fix the following point of $\mathfrak{g}^*_{\text{reg}}$:

$$m_0 = (f(x),\, a(x),\, c_1,\, c_2)_0 = (1,\, 1,\, 0,\, c)\,, \tag{3.4}$$

with arbitrary $c \in \mathbb{R}$, and consider the constant Poisson structure (2.2) corresponding to the coboundary (2.3). The Hamiltonian vector field X^{ω}_{H} with the Hamiltonian H is then given by

$$\begin{aligned} f_t &= -\left(\delta_a H\right)_x + c\left(\delta_a H\right)_y \\ a_t &= 2\left(\delta_a H\right)_x - \left(\delta_f H\right)_x + c\left(\delta_f H\right)_y\,. \end{aligned}$$

The limit case $c \to \infty$ corresponds to the following structure

$$\begin{aligned} f_t &= \left(\delta_a H\right)_y \\ a_t &= \left(\delta_f H\right)_y\,. \end{aligned} \tag{3.5}$$

We are ready to formulate our main result.

Theorem 3.4. *The following system on* $\mathfrak{g}^*_{\text{reg}}$

$$\begin{aligned} u_{tx} &= u_{xy}\, u_y - u_{yy}\, u_x \\ v_{tx} &= 2\left(u_{yy}\, v_x - u_{xy}\, v_y\right) + u_y\, v_{xy} - u_x\, v_{yy} \\ &\quad -2\left(u_{yy}\, u_x + 2\, u_{xy}\, u_y\right) \end{aligned} \tag{3.6}$$

is bi-Hamiltonian with respect to the standard Lie-Poisson structure on $\mathfrak{g}_{\text{reg}}$*, together with (3.5), where* $f = u_y$ *and* $a = v_y$*.*

Proof. The simplest class of Casimir functions of this constant Poisson structure are linear combinations of the functionals $\int f\, dxdy$ and $\int u\, dxdy$. We will choose the Casimir function

$$H_0(f,\, a) = \int_{S^1\times S^1} (a - f)\; dxdy.$$

The Hamiltonian vector field, X_{H_0}, with respect to the Lie-Poisson structure defines the following vector field

$$\begin{aligned} f_t &= f_x \\ a_t &= 2\, f_x + a_x. \end{aligned} \tag{3.7}$$

Indeed, one obviously has $(\delta_a H_0,\, \delta_f H_0) = (-1,\, 1)$ (understood as an element of $\mathfrak{g}/\mathbb{R}^2$) and one then applies the definition (2.4).

One thus looks for a function $H_1(f,\, a)$ on $\mathfrak{g}^*_{\text{reg}}$ such that its Hamiltonian vector field with respect to the constant Poisson structure satisfies

$$X^{\omega}_{H_1} = X_{H_0},$$

which leads to the following system of equation on the variational derivatives $\delta_f H_1$ and $\delta_u H_1$:

$$\begin{aligned} -\left(\delta_a H_1\right)_x + c\left(\delta_a H_1\right)_y &= f_x \\ 2\left(\delta_a H_1\right)_x - \left(\delta_f H_1\right)_x + c\left(\delta_f H_1\right)_y &= 2\, f_x + a_x. \end{aligned}$$

Introducing the first-order differential operator

$$\Lambda = -\partial_x + c\,\partial_y,$$

one shows by a simple straightforward calculation that following function:

$$H_1(f,\,a) = \int_{S^1\times S^1} \left(\Lambda^{-1}(f_x)\,a + \Lambda^{-1}(f_x)\,f - \Lambda^{-2}(f_{xx})\,f\right) dxdy \tag{3.8}$$

is a solution of the above system.

The Hamiltonian vector field X_{H_1} is then as follows

$$\begin{aligned} f_t &= \Lambda^{-1}(f_x)\,f_x - \Lambda^{-1}(f_{xx})\,f + c_2\,\Lambda^{-1}(f_{xy}) \\ a_t &= \Lambda^{-1}(f_x)\,a_x + 2\,\Lambda^{-1}(f_{xx})\,a - \Lambda^{-1}(a_{xx})\,f - 2\,\Lambda^{-1}(a_x)\,f_x \\ &\quad -2\left(\Lambda^{-1}(f_{xx}) - \Lambda^{-2}(f_{xxx})\right)f - 4\left(\Lambda^{-1}(f_x) - \Lambda^{-2}(f_{xx})\right)f_x \\ &\quad c_1\,\Lambda^{-1}(f_{xxxx}) + c_2\left(\Lambda^{-1}(a_{xy}) + 2\,\Lambda^{-1}(f_{xy}) - 2\,\Lambda^{-2}(f_{xxy})\right) \end{aligned}$$

In the same way as in [14], we substitute to this equation $f = \Lambda(u)$ and $a = \Lambda(v)$ and rewrite it in the following form:

$$\begin{aligned} -u_{tx} + c\,u_{ty} &= c\,(u_{xy}\,u_x - u_{xx}\,u_y) + c_2\,u_{xy} \\ -v_{tx} + c\,v_{ty} &= c\,(2\,u_{xx}\,v_y - 2\,u_{xy}\,v_x + u_x\,v_{xy} - u_y\,v_{xx}) \\ &\quad 2\left(u_{xx} - \Lambda^{-1}(u_{xxx})\right)(u_x - c\,u_y) + 4\left(u_x - \Lambda^{-1}(u_{xx})\right)(u_{xx} - c\,u_{xy}) \\ &\quad c_1\,u_{xxxx} + c_2\left(v_{xy} + 2\,u_{xy} - 2\,\Lambda^{-1}(u_{xxy})\right). \end{aligned} \tag{3.9}$$

It is very easy to check that, in the limit case $c \to \infty$, this system coincides with (3.6) with exchanged notation for the variables $(x, y) \leftrightarrow (y, x)$. □

Theorem 3.4 implies the existence of an infinite series of first integrals in involution for the equation (1.1), as well as of an infinite hierarchy of commuting flows, see [16], Section 5.5.

Remark 3.5. 1) The special case $c = 0$ in (3.4) was considered in the details in [16]. This case is related to the equation (1.2).

2) One can also choose a non-zero value of the first central charge c_1 in (3.4). This will, however, only change the second equation in (3.9).

3) Consider the first equation in (3.9). The term $c_2\,u_{xy}$ can be removed by the transformation $u \mapsto u - \frac{c_2}{c}\,x$. Furthermore, the coordinate transformation $(x, y) \to (x, y + c\,x)$ leads to the following family:

$$u_{tx} = c\,(u_{xy}\,u_y - u_{yy}\,u_x) + u_{xx}\,u_y - u_{xy}\,u_x$$

depending on c as parameter. This family gives one an interpolation between the equations (1.1) and (1.2), but with zero central charge.

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